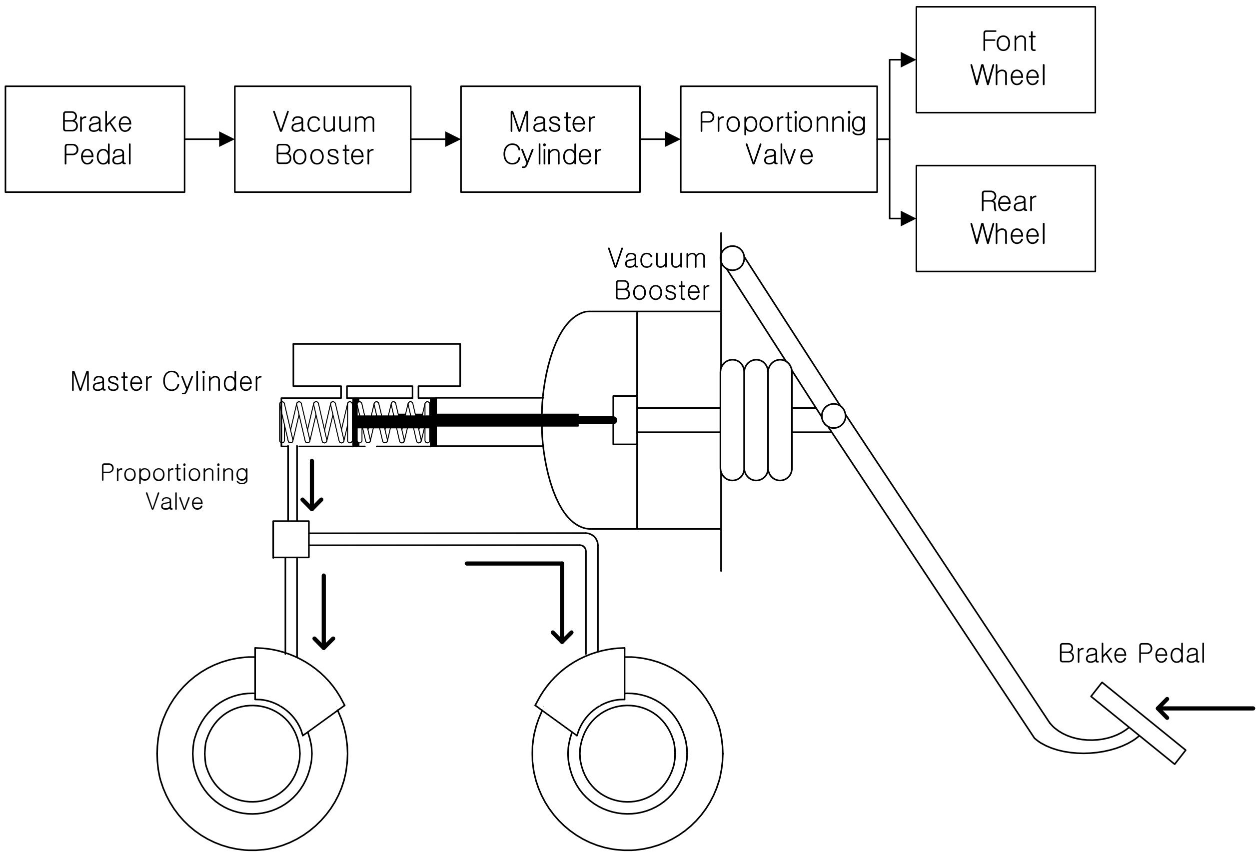
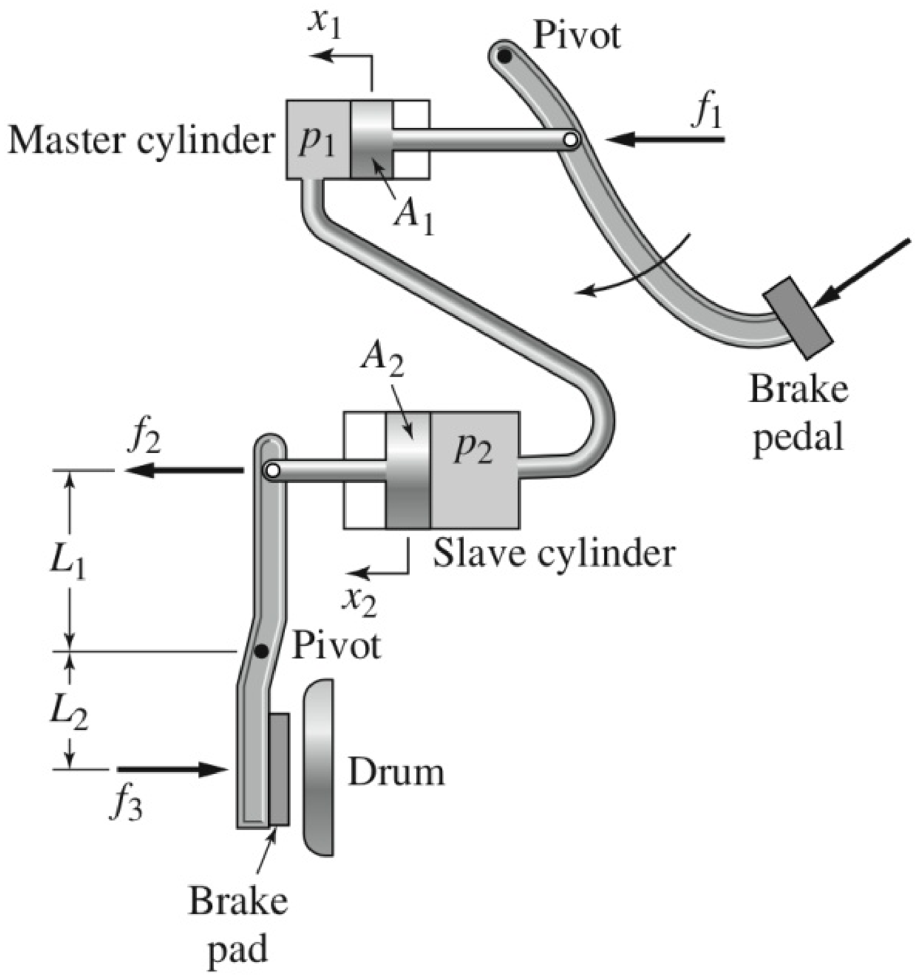
Fluid Systems I

Brake systems



Fundamental structure of a hydraulic brake

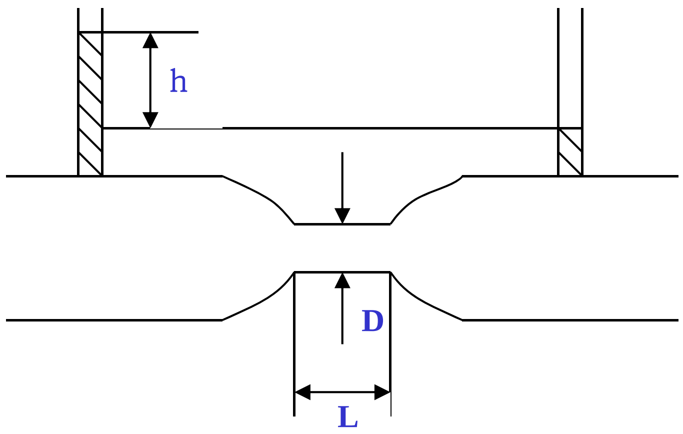
Conservation of Mass, Force and Pressure



Fluid Resistance

Electrical systems, resistance is defined as

$$i = \frac{v}{R}, \quad R = \frac{v}{i}$$



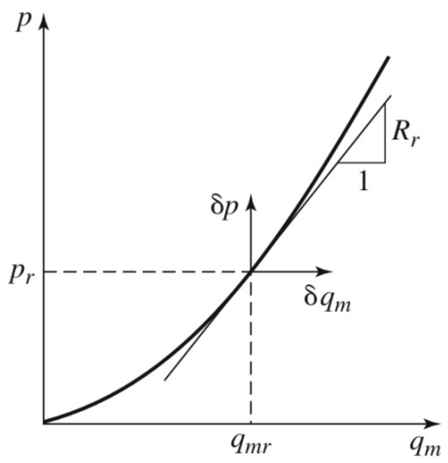
For fluid systems, R is defined as

$$q_m = \frac{P}{R}, \quad R = \frac{P}{q_m}$$

Mass flow rate,
pressure difference

Fluid Resistance

$$R = \frac{change\ in\ pressure}{change\ in\ flow\ rate} = \left.\frac{dp}{dq_m}\right|_{q=q_{mr}}$$



Fluid Capacitance

Electrical systems,

$$\frac{dv}{dt} = \frac{1}{C} i$$

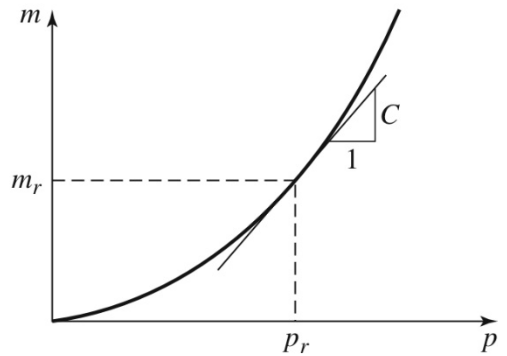
Fluid systems,

$$\frac{dp}{dt} = \frac{1}{C} q_m, \qquad dp = \frac{1}{C} q_m dt, \qquad C = \frac{q_m dt}{dp}$$

$$C = \frac{change\ in\ stored\ mass}{change\ pressure} = \frac{dm}{dp}$$

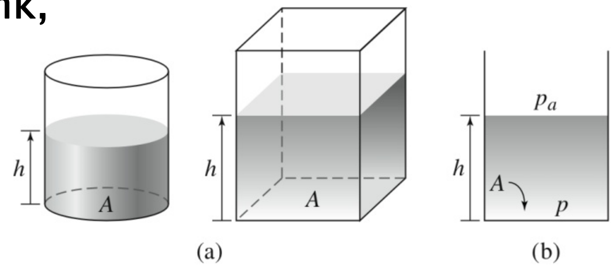
Fluid Capacitance

$$C = \frac{change\ in\ stored\ mass}{change\ pressure} = \frac{dm}{dp}$$



General fluid capacitance

For a tank,



(a) (b)

$$C = \rho A \frac{dh}{dp} = \rho A \frac{1}{\rho g} = \frac{A}{g}$$

$$\frac{dm}{dt} = C\frac{dp}{dt} = q_{mi} - q_{mo}$$

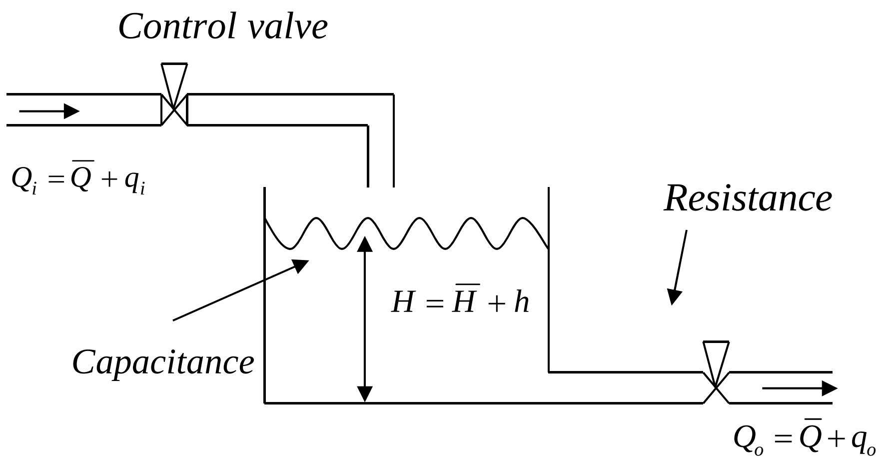
$$C\frac{dp}{dt} = \rho A \frac{dh}{dt} = q_{mi} - q_{mo}$$

For liquid level system, C is defined as

$$C = \frac{change\ in\ liquid\ stored}{change\ in\ head} \frac{m^3}{m} = \rho A$$

use h(pressure head) , instead of p.

Liquid Level Systems



$$R = \frac{\text{change in level difference}}{\text{change in flow rate}} \quad \frac{m}{m^3/s}$$

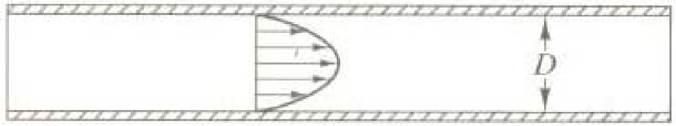
$$C = \frac{\text{change in liquid stored}}{\text{change in head}} \quad \frac{m^3}{m}$$

$$\text{Steady state}: \quad Q_i = \bar{Q} = Q_0 = \frac{\bar{H}}{R}$$

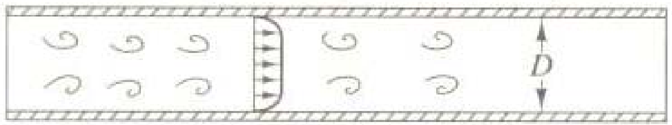
$$\text{consider}, \quad Q_o = \frac{H}{R}, \qquad Q_i - Q_o = C\frac{dH}{dt}$$

Basic Concepts

Laminar flow : When the viscous forces are dominant (slow flow, low Re) they are sufficient enough to keep all the fluid particles in line, then the flow is laminar.



Turbulent flow : It can be interpreted that when the inertial forces dominate over the viscous forces (when the fluid is flowing faster and Re is larger) then the flow is turbulent.



Reynolds number (Re) : The ratio of inertial forces to viscous forces.

It is used to determine whether a flow will be laminar or turbulent.

$$\mathrm{Re} = \frac{\rho v D}{\mu}$$

μ : *the dynamic viscousity of the fluid*

ρ : *density of the fluid*

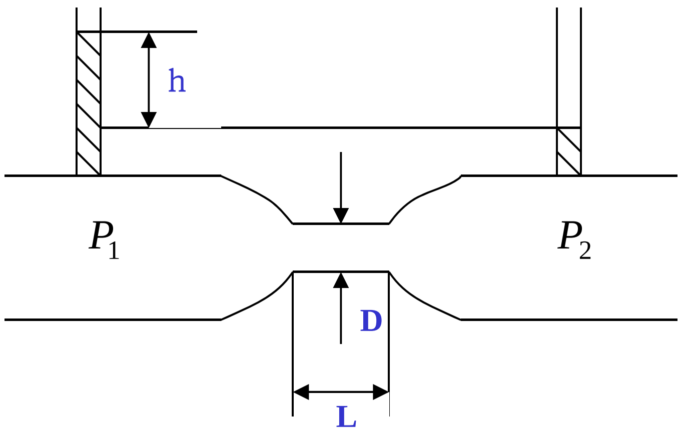
D : *diameter*

$\mathrm{Re} < 2000$: always laminar

$\mathrm{Re} > 4000$: always turbulent

Laminar Flow

Cylindrical pipe



$$P_1 - P_2 = \rho g h, \quad Q\frac{128\nu L}{g\pi D^4} = h$$

$\nu : viscosity,$

$L : length\ of\ \ pipe$

$D : diameter\ of\ \ pipe$

$$Q = \frac{h}{R} = K_l\, h, \qquad R = \frac{128\nu L}{g\pi D^4} \quad [s/m^2]$$

$Q\ \ : steady - state\ \ flow\ rate$

$K_l : constant$

$h\ \ : steady - state\ \ head$

Laminar Flow

Liquid level dynamics :

$$C\frac{dH}{dt} = Q_i - Q_o$$

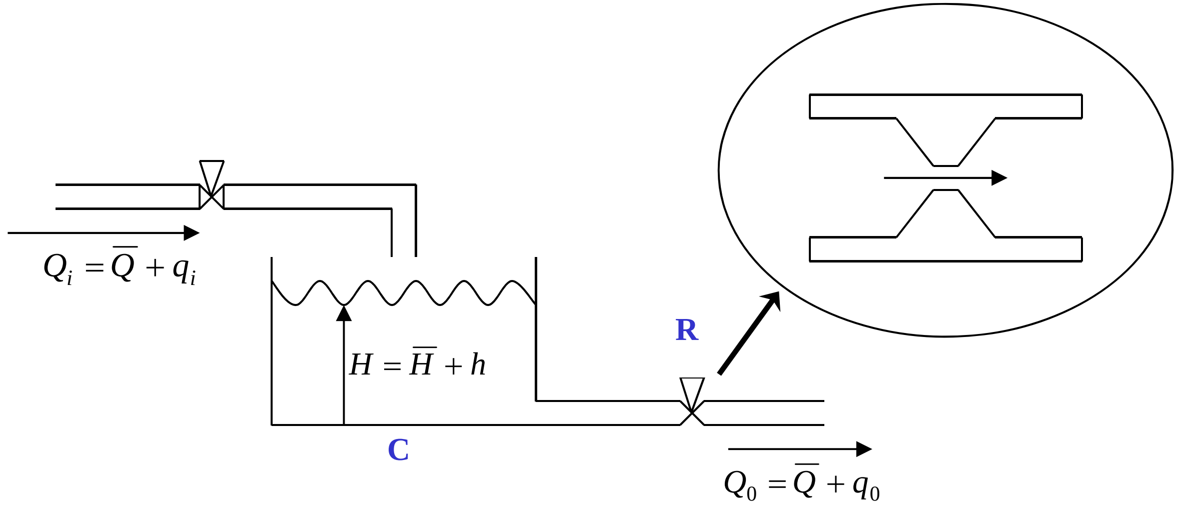
$$= \bar{Q} + q_i - (\bar{Q} + q_o) = q_i - q_o$$

$$\frac{dH}{dt} = \frac{d}{dt}(\bar{H} + h) = \frac{dh}{dt}$$

$$Q_o = \bar{Q} + q_o = \frac{\bar{H} + h}{R}, \qquad q_o = \frac{h}{R}$$

$$\rightarrow \quad \frac{dh}{dt} = -\frac{1}{CR}h + \frac{1}{C}q_i$$

Turbulent Flow



$$Q = C_d \cdot a \cdot \sqrt{\frac{2}{\rho}(P_1 - P_2)}$$

$\rho : density, \quad a : area, \quad C_d : discharge\ coefficient$

$$steady\ state: \ Q_i = \overline{Q} = Q_0 = K_t \sqrt{\overline{H}}$$

Turbulent Flow

Liquid level dynamics

$$C\frac{dH}{dt} = Q_i - Q_0 = \overline{Q} + q_i - K\sqrt{H}$$

$$\frac{dH}{dt} = \frac{1}{C}Q_i - \frac{1}{C}K\sqrt{H} = f(Q_i, H)$$

$$f(Q_i, H) = f(\overline{Q}_i, \overline{H}) + \left.\frac{\partial f}{\partial Q_i}\right|_{\overline{Q},\overline{H}} (Q_i - \overline{Q}) + \frac{1}{2!}\frac{\partial^2 f}{\partial Q_i^2}(Q_i - \overline{Q})^2 + \cdots$$

$$\cdots + \left.\frac{\partial f}{\partial H}\right|_{\overline{Q},\overline{H}} (H - \overline{H}) + \frac{1}{2!}\frac{\partial^2 f}{\partial H^2}(H - \overline{H})^2 + \cdots$$

$$= \frac{1}{C}\overline{Q} - \frac{1}{C}K\sqrt{\overline{H}} + \left(\frac{1}{C}q_i - \frac{K}{C \cdot 2\sqrt{\overline{H}}}h\right) + \ high \ \ order \ \ term$$

Turbulent Flow

$$\frac{dH}{dt} = \frac{d\bar{H}}{dt} + \frac{dh}{dt}$$

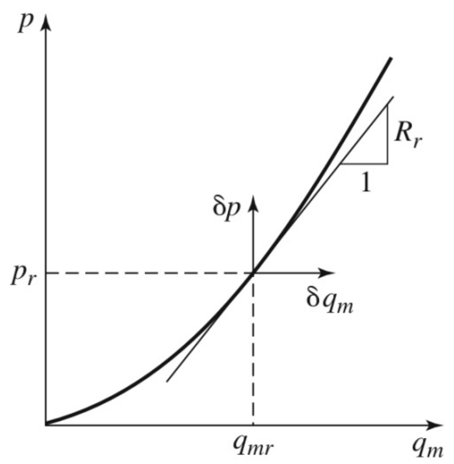
$$\Rightarrow \qquad \frac{dh}{dt} = \frac{1}{C} q_i - \frac{K}{C \cdot 2\sqrt{\bar{\bar{H}}}} h$$

$$q_o = \frac{h}{R}, \qquad R = \frac{2\sqrt{\bar{\bar{H}}}}{K} = \frac{2\bar{H}}{\bar{\bar{Q}}}, \qquad \left(\bar{Q} = K\sqrt{\bar{\bar{H}}}\right)$$

$$\therefore \begin{cases} \dfrac{dh}{dt} = -\dfrac{1}{CR} h + \dfrac{1}{C} q_i \\ q_o = \dfrac{h}{R} \end{cases} \qquad \textit{for small } q_i$$

Fluid Resistance(laminar flow vs turbulent flow)

$$R = \frac{change\ in\ pressure}{change\ in\ flow\ rate} = \left.\frac{dp}{dq_m}\right|_{q=q_{mr}}$$



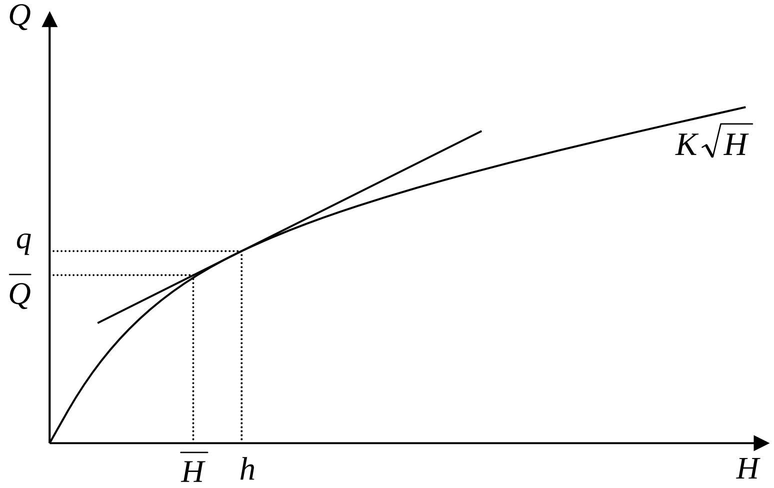
$$q_m = \frac{p}{R}$$ For laminar flow

$$q_m = \sqrt{\frac{p}{R_l}}$$ For turbulent flow

For liquid level system, R is defined as

$$R = \frac{change\ in\ level\ difference}{change\ in\ flow\ rate} \frac{m}{m^3/s} = \frac{dh}{dq_m}$$

Linearization



$$Q = \overline{Q} + q = K\sqrt{\overline{H}} + (\quad)\cdot h$$

$$y = f(x), \qquad \overline{y} = f(\overline{x}), \qquad x = \overline{x} + \Delta x$$

$$y = f(\overline{x}) + \left.\frac{\partial f}{\partial x}\right|_{\overline{x}} \cdot \Delta x + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{\overline{x}} \cdot \Delta x^2 + \cdots$$

$$\approx \overline{y} + \left.\frac{\partial f}{\partial x}\right|_{\overline{x}} \cdot \Delta x$$

$$y - \overline{y} = \left.\frac{\partial f}{\partial x}\right|_{\overline{x}} (x - \overline{x}) = K(x - \overline{x})$$

$$y = f(x_1, x_2) \approx f(\overline{x_1}, \overline{x_2}) + \left.\frac{\partial f}{\partial x_1}\right|_{\overline{x}_1, \overline{x}_2} (x_1 - \overline{x_2}) + \left.\frac{\partial f}{\partial x_2}\right|_{\overline{x}_1, \overline{x}_2} (x_2 - \overline{x_2})$$

$$= \overline{y} + K_1(x_1 - \overline{x_1}) + K_2(x_2 - \overline{x_2})$$

$$\therefore\ y - \overline{y} = K_1(x_1 - \overline{x_1}) + K_2(x_2 - \overline{x_2})$$

Summary of Liquid Level Systems

Laminar flow

$$\frac{dh}{dt} = -\frac{1}{CR}h + \frac{1}{C}q_i$$

$$\bar{Q} = K \cdot \bar{H} \qquad R = \frac{\bar{H}}{\bar{Q}}$$

$$R = \frac{128\nu L}{g\pi D^4}$$

$$q_o = \frac{h}{R}$$

Turbulent flow

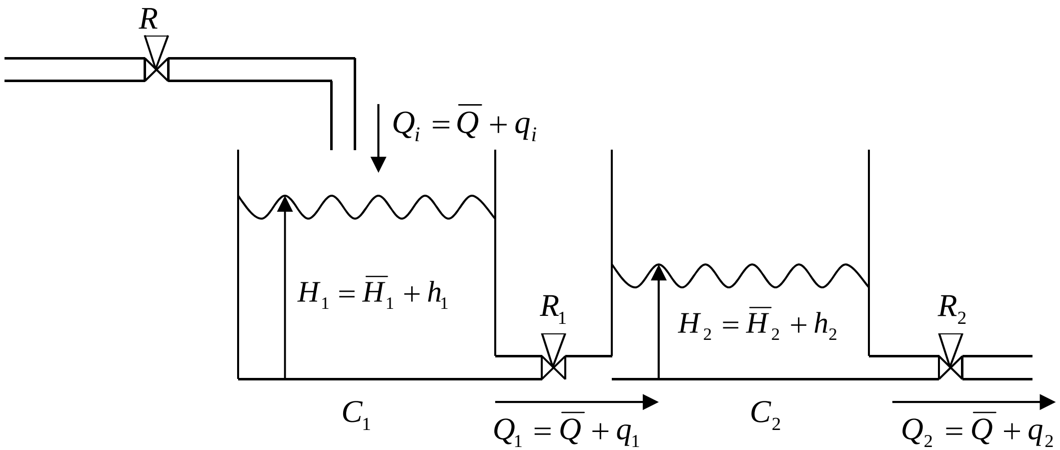
$$\frac{dh}{dt} = -\frac{1}{CR}h + \frac{1}{C}q_i$$

$$q_o = \frac{h}{R}$$

$$R = \frac{2\bar{H}}{\bar{Q}} = \frac{2\sqrt{\bar{H}}}{K} = \frac{2\sqrt{\bar{H}}}{C_d a\sqrt{2g}}$$

$$\left(\because \bar{Q} = K\sqrt{\bar{H}} = C_d a\sqrt{\frac{2}{\rho}\rho g \bar{H}} = C_d a\sqrt{2g}\sqrt{\bar{H}} \right)$$

Liquid Level Systems with Interaction



Steady state :

$$Q_i = \bar{Q} = Q_1 = Q_2$$

$$= \frac{\bar{H}_1 - \bar{H}_2}{R_1} = \frac{\bar{H}_2}{R_2}$$

Liquid level dynamics :

$$C_1 \frac{dH_1}{dt} = \bar{Q} + q_i - \left(\bar{Q} + q_1\right) \rightarrow C_1 \frac{dh_1}{dt} = q_i - q_1$$

$$Q_1 = \bar{Q} + q_1 = \frac{1}{R_1}\left(\bar{H}_1 + h_1 - (\bar{H}_2 + h_2)\right) \rightarrow q_1 = \frac{h_1 - h_2}{R_1}$$

$$C_2 \frac{dH_2}{dt} = \bar{Q} + q_1 - \left(\bar{Q} + q_2\right) \rightarrow C_2 \frac{dh_2}{dt} = q_1 - q_2$$

$$Q_2 = \bar{Q} + q_2 = \frac{\bar{H}_2 + h_2}{R_2} \rightarrow q_2 = \frac{h_2}{R_2}$$

Liquid Level Systems with Interaction

$$\therefore \begin{cases} \dfrac{dh_1}{dt} = -\dfrac{1}{C_1}\left(\dfrac{h_1 - h_2}{R_1}\right) + \dfrac{1}{C_1} q_i \\ \\ \dfrac{dh_2}{dt} = \dfrac{1}{C_2}\left(\dfrac{h_1 - h_2}{R_1}\right) - \dfrac{1}{C_2}\dfrac{h_2}{R_2} \end{cases}$$

$$Transfer\ Function = \frac{Q_2(S)}{Q_i(S)} = \frac{1}{R_1 C_1 R_2 C_2 S^2 + \left(R_1 C_1 + R_2 C_2 + R_2 C_1\right) S + 1}$$