Lecture 10-1

Linear Systems Analysis in the Time Domain

- Transient Response -

System Poles and Zeros

Consider a system with T.F. $$G(s) = \frac{N(s)}{D(s)}$$

Factor the numerator and denominator polynomials $$G(s) = K\frac{(s-z_1)(s-z_2).....(s-z_m)}{(s-p_1)(s-p_2)......(s-p_n)}$$

where $\boldsymbol{p}_1, \boldsymbol{p}_2, ...\boldsymbol{p}_n$: Roots of D(s), **system poles**

$\boldsymbol{z}_1, \boldsymbol{z}_2, ...\boldsymbol{z}_m$: Roots of N(s), **system zeroes**

Note that because the coefficient of N(s) and D(s) are real, (modeling parameters), the system poles must be either

i) Purely real, or

ii) Appear as complex conjugates

$$\boldsymbol{p}_i \textbf{ or } \boldsymbol{z}_i = \sigma_i + \boldsymbol{j}\omega_i$$

System Poles and Zeros completely characterize the transfer function (therefore the system itself) except for an overall gain of constant *K:*

$$G(s) = K\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}$$

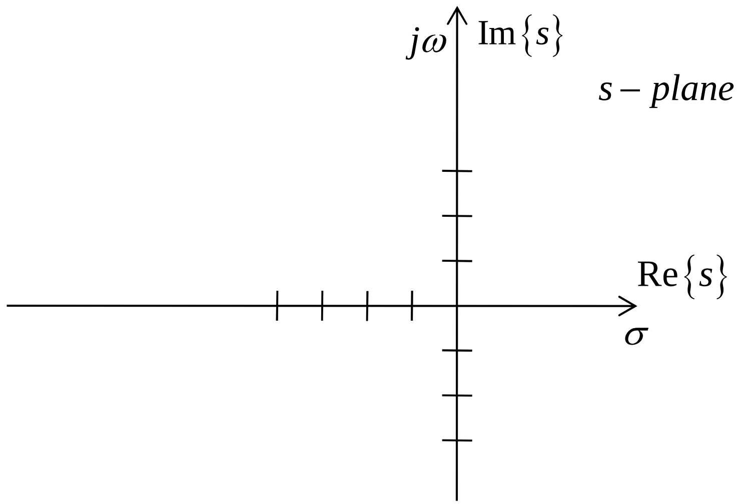
Pole Zero Plot

The value of system poles and zeros are shown graphically on the complex s-plane .

Ex)

$$G(s) = \frac{5s^2 + 10s}{s^3 + 5s^2 + 11s + 5} = \frac{5s(s+10)}{(s+3)(s^2+2s+5)} = \frac{5s(s+10)}{(s+3)(s+(1+j2))(s+(1-j2))}$$

zeros at s=0, s=-2   poles at s=-3, -1+j2,   s=-1-j2



You can use Sys=zpk(zeros, poles, gain) in matlab.

Pole Zero Plot

The characteristic equation of the system:

$$D(s) = (s - p_1)(s - p_2)......(s - p_n)$$

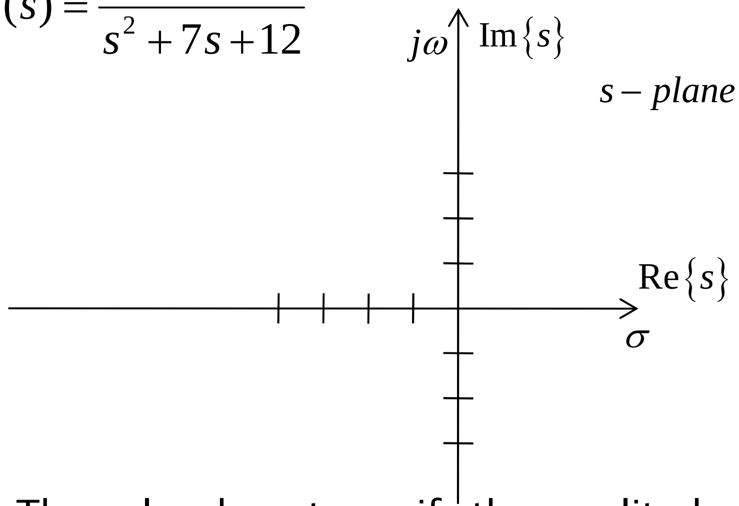
Poles are the system eigenvalues.

Form of the homogeneous solution:

$$y_h(t) = \sum_{i=1}^{n} C_i e^{p_i t}$$

Ex)

$$G(s) = \frac{12}{s^2 + 7s + 12}$$



$$y_1(t) = C_1 e^{-3t} \quad y_2(t) = C_2 e^{-4t}$$

Note: The poles do not specify the amplitude. It just indicates the natural response components.

Complex Poles and Zeros

In general, $s = \sigma + j\omega$

$$y(t) = \ldots + C_i e^{(\sigma_i + j\omega_i)t} + C_{i+1} e^{(\sigma_i - j\omega_i)t} \ldots.$$

$$C_i e^{(\sigma_i + j\omega_i)t} + C_{i+1} e^{(\sigma_i - j\omega_i)t} = (a + jb)e^{\sigma_i t} e^{j\omega_i t} + (a - jb)e^{\sigma_i t} e^{-j\omega_i t}$$

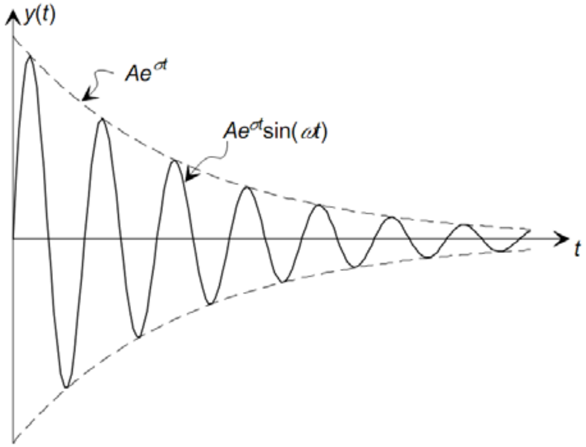
$$= ae^{\sigma_i t}(e^{j\omega_i t} + e^{-j\omega_i t}) + jbe^{\sigma_i t}(e^{j\omega_i t} - e^{-j\omega_i t})$$

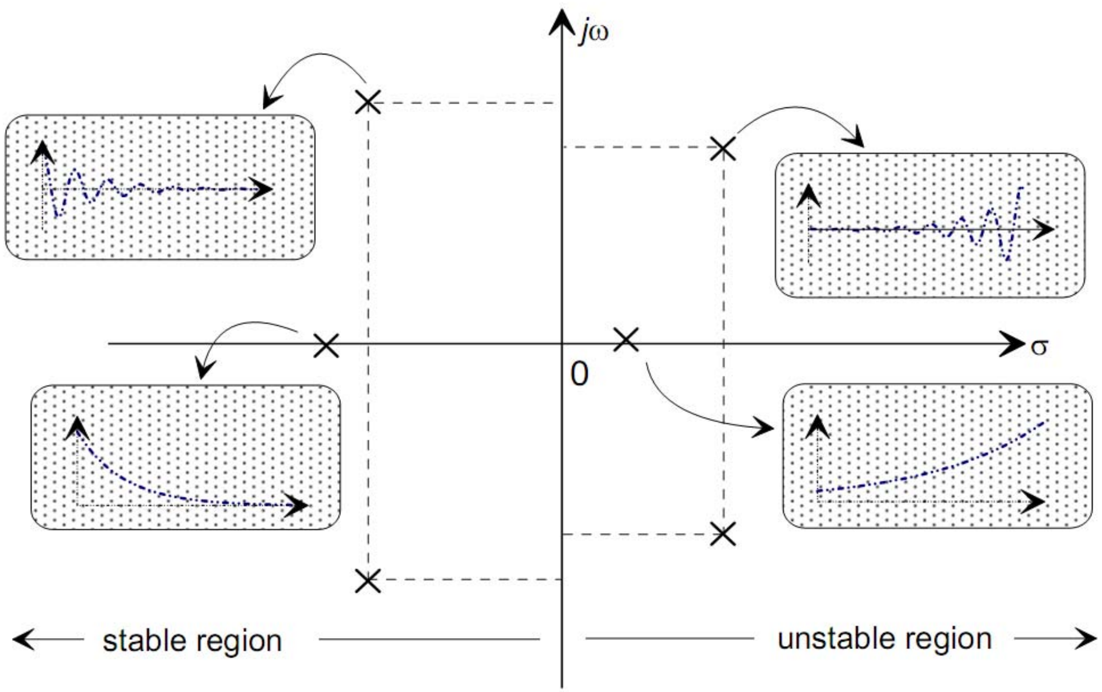
$$y_{i,\, i+1}(t) = 2ae^{\sigma_i t}\cos(\omega_i t) - 2ae^{\sigma_i t}\sin(\omega_i t)$$

$$= 2\sqrt{a^2 + b^2}\, e^{\sigma_i t}\left(\frac{a}{\sqrt{a^2 + b^2}}\cos(\omega_i t) - \frac{b}{\sqrt{a^2 + b^2}}\sin(\omega_i t)\right)$$

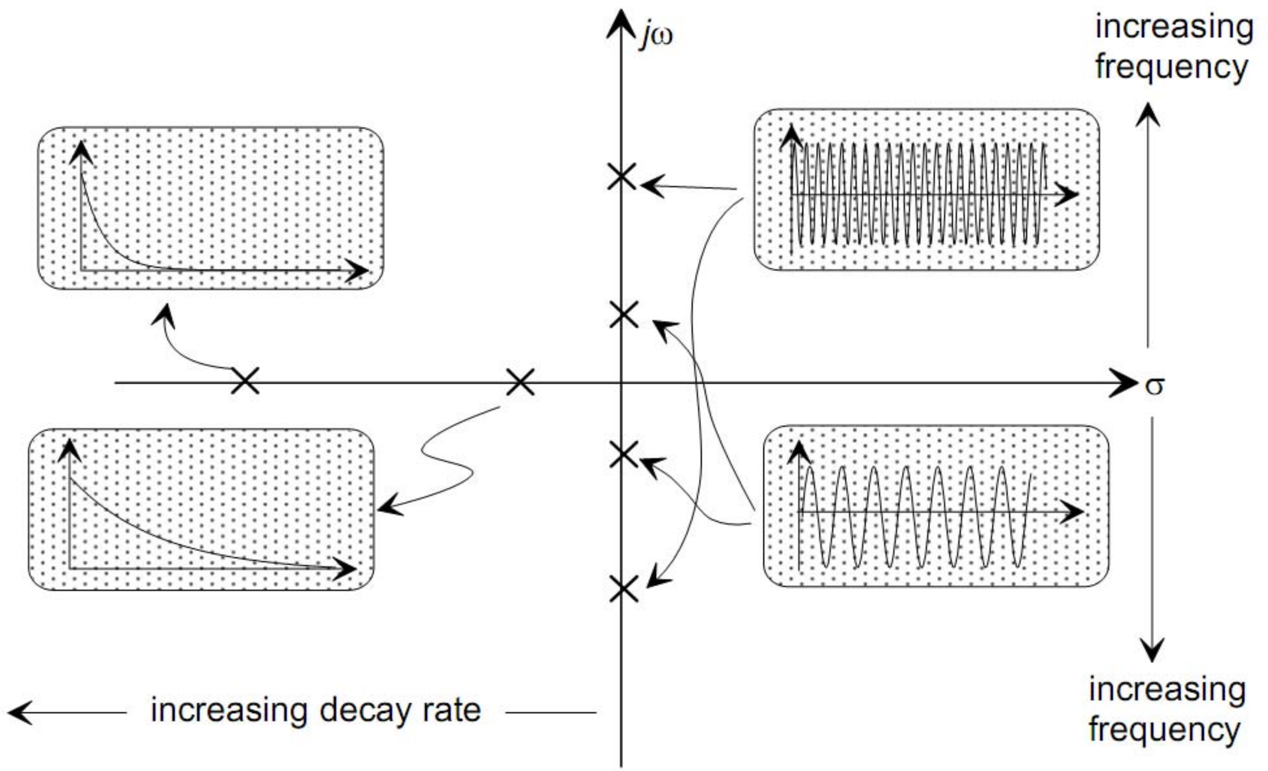
$$= A_i e^{\sigma_i t}\sin(\omega_i t + \phi_i)$$

$$A_i = 2\sqrt{a^2 + b^2} \quad \phi_i = \tan^{-1}\left(\frac{a}{b}\right)$$





1) Poles in the left-half plane → decays with time
2) Poles in the right-half plane → grow with time
3) Pole on the imaginary axis → purely oscillatory
4) Pole at the origin → constant



5) the oscillatory frequency and decay rate is determined by the distance of the poles from the origin.

6)The rate of decay/growth is determined by the real part of the pole, and poles deep in the lhp generate rapidly decaying components

7) For complex conjugate pole pairs, the oscillatory frequency is determined by the imaginary part of the pole pair.

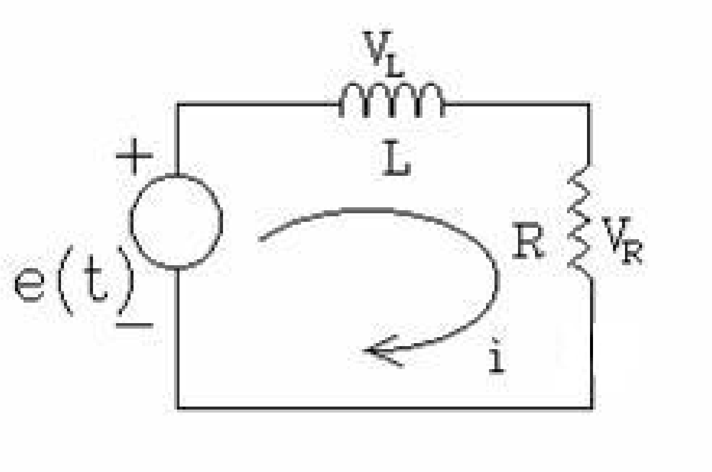
System Stability

A system is defined to be unstable if its response from any finite initial conditions increases without bound.

$$y_h(t) = \sum_{i=1}^{n} C_i e^{p_i t}$$

1) System is unstable if any pole has a positive real part
2) For a system to be stabel, all poles must lie in the lhp.
3) System with poles on the imaginary axis is defined to be marginally stable.

First Order Systems



$$v_L = L\frac{di}{dt}, \qquad v_R = Ri,$$

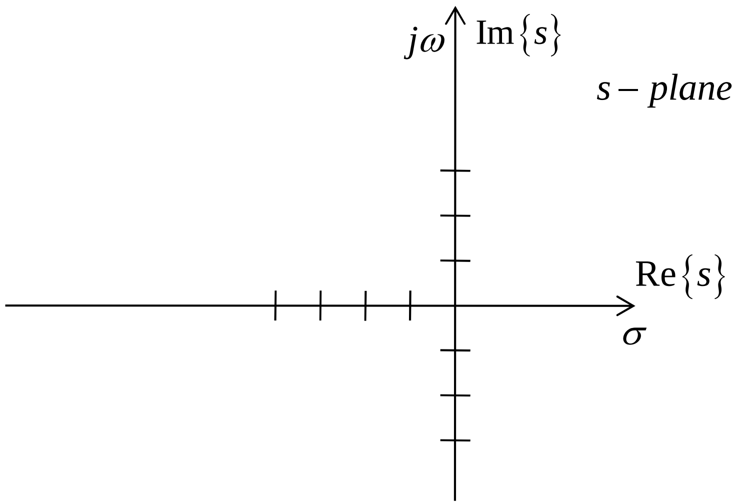
$$L\frac{di}{dt} + Ri = e(t)$$

$$x = i, \;\; \dot{x} = -\frac{R}{L}x + \frac{1}{L}e(t)$$

$$\frac{X(s)}{U(s)} = \frac{I(s)}{E(s)} = \frac{1}{Ls+R} = \frac{1}{R}\cdot\frac{1}{\frac{L}{R}s+1}$$

$$u(t) = e(t) = 1, \qquad i(0) = 0$$

$$i(t) = \frac{1}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$



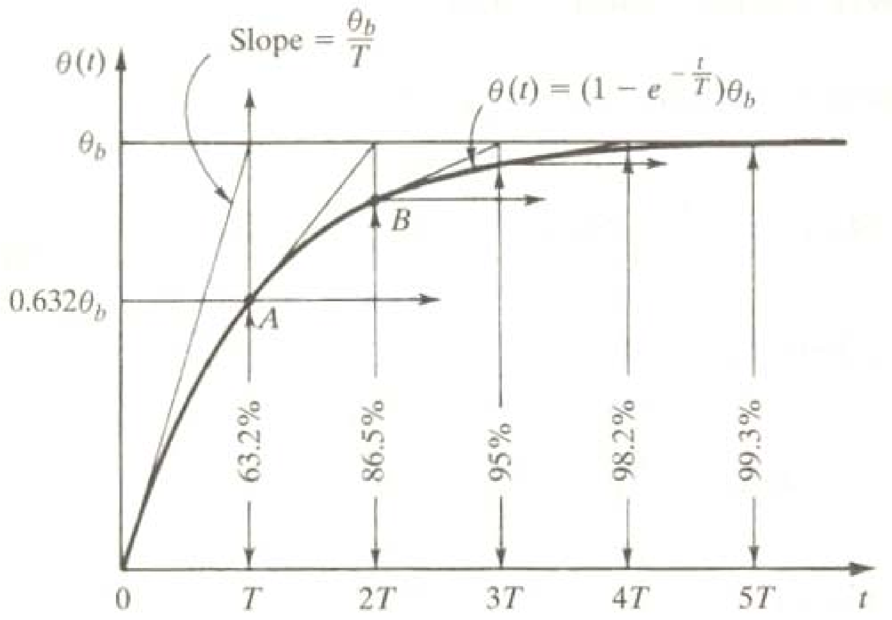
First Order Systems

$$\frac{Y(s)}{U(s)} = \frac{1}{Ts+1}, \qquad U(s) = \frac{1}{s}, \qquad u(t) = 1$$

T : *time constant*

$$Y(s) = \frac{1}{Ts+1}\frac{1}{s} = \frac{1}{s} - T \cdot \frac{1}{Ts+1}$$

$$\therefore \quad y(t) = 1 - e^{-\frac{1}{T}t}, \qquad \dot{y}(t) = \frac{1}{T} e^{-\frac{1}{T}t}$$



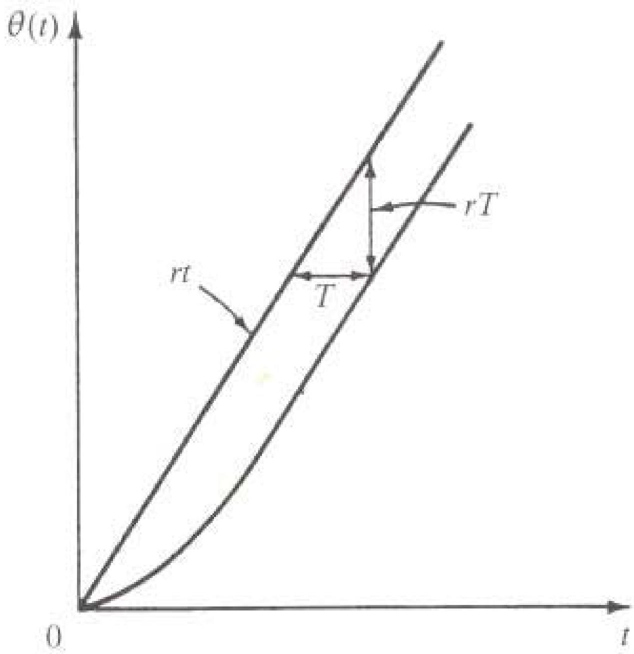
1) Settling Time: The time taken for the response to reach 98% of its final value

$$T_s = 4T$$

2) Rise Time: Commonly taken as time taken for the step response to rise from 10% to 90% of the steady-state response to a step input.

$$T_R = 2.2T$$

First Order Systems



$$R(t) = rt$$

$$R(s) = r \cdot \frac{1}{s^2}$$

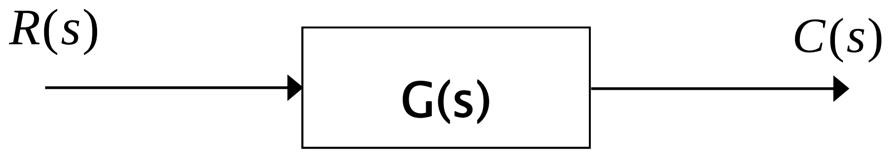
$$Y(s) = \frac{1}{Ts+1} \cdot r \cdot \frac{1}{s^2} = r\left(\frac{1}{s^2} - \frac{T}{s} + \frac{T}{s+(1/T)}\right)$$

$$y(t) = r(t - T + Te^{-\frac{t}{T}})$$

$$e(t) = R(t) - y(t) = rT(1 - e^{-\frac{t}{T}})$$

$$e(\infty) = rT$$

Second Order Systems



$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$R(s) = \frac{1}{s} \ (step\ input), \qquad C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0, \qquad s = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\,\omega_n$$

Second Order Systems

Underdamped case $s = -\zeta\omega_n \pm \sqrt{1-\zeta^2}\,\omega_n i\,, \qquad (\omega_d = \omega_n\sqrt{1-\zeta^2})$

$0 < \zeta < 1$

$$C(s) = \frac{\omega_n^2}{(s+\zeta\varpi_n)^2 + \omega_n^2(1-\zeta^2)} \cdot \frac{1}{s} = \frac{1}{s} - \frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

$$= \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d{}^2}$$

$$\therefore C(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\omega_d t = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \eta)$$

$$\eta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

Critically damped case

$\zeta = 1$

$$R(s) = \frac{1}{s}, \qquad C(s) = \frac{\omega_n{}^2}{(s+\zeta\omega_n)^2 s}$$

$$c(t) = 1 - e^{-\omega_n t}(1+\omega_n t)$$

Second Order Systems

Overdamped case $\zeta > 1$

$$C(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})s}$$

$$c(t) = 1 + \frac{1}{2\sqrt{\zeta^2 - 1}(\zeta + \sqrt{\zeta^2 - 1})} e^{-(\zeta + \sqrt{\zeta^2 - 1})\omega_n t} - \frac{1}{2\sqrt{\zeta^2 - 1}(\zeta - \sqrt{\zeta^2 - 1})} e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$$

$$= 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \left(\frac{e^{-\left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t}}{\left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n} - \frac{e^{-\left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}}{\left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n} \right)$$

Approximation (After the faster term disappeared)

$$\frac{C(s)}{R(s)} = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}$$

$$\therefore c(t) = 1 - e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$$

Damping ratio and Pole placement

i) $\zeta > 1$: poles are real and distinct

ii) $\zeta = 1$: poles are real and coincident

iii) $0 < \zeta < 1$: pole are complex conjugates

iv) $\zeta = 0$: The pole are purely imaginar