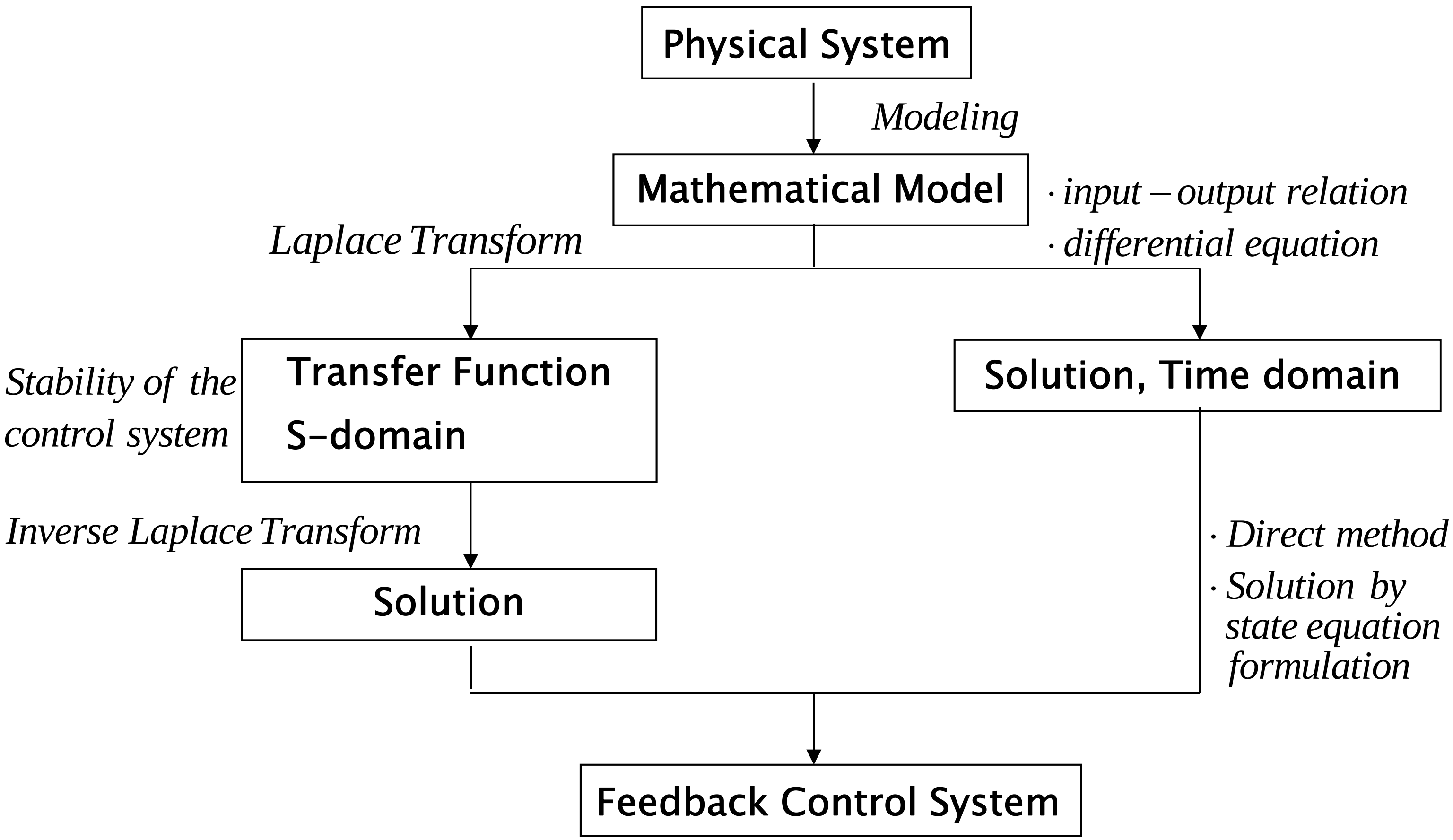
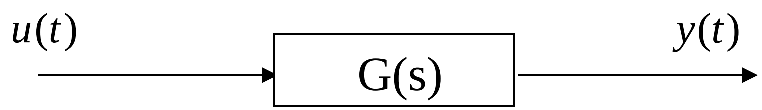
Frequency Domain Analysis I


Physical System
Modeling
Mathematical Model
· input – output relation
· differential equation
Laplace Transform
Transfer Function
S-domain
Stability of the
control system
Solution, Time domain
Inverse Laplace Transform
Solution
· Direct method
· Solution by
state equation
formulation
Feedback Control System

Concept of Frequency Response



$$u(t) = P \sin \omega t$$

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_n)}{(s+s_1)(s+s_2)\cdots(s+s_n)}$$

$$Y(s) = G(s) \cdot u(s)\,, \qquad u(s) = P \cdot \frac{\omega}{s^2+\omega^2}$$

$$Y(s) = G(s) \cdot \frac{P\omega}{s^2+\omega^2}$$

$$= \frac{a}{s+j\omega} + \frac{\bar{a}}{s-j\omega} + \frac{b_1}{s+s_1} + \frac{b_2}{s+s_2} + \cdots + \frac{b_n}{s+s_n}$$

$$y(t) = \boxed{a e^{-j\omega t} + \bar{a} e^{j\omega t} + b_1 e^{-s_1 t} + b_2 e^{-s_2 t} + \cdots + b_n e^{-s_n t}}$$

Frequency Response

$$y(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t}$$

$$a = G(s)\cdot\frac{P\omega}{s^2+\omega^2}(s+j\omega)\Big|_{s=-j\omega} = -\frac{P}{2j}G(-j\omega)$$

$$\bar{a} = G(s)\cdot\frac{P\omega}{s^2+\omega^2}(s-j\omega)\Big|_{s=j\omega} = \frac{P}{2j}G(j\omega)$$

$$G(j\omega) = G_x + jG_y$$

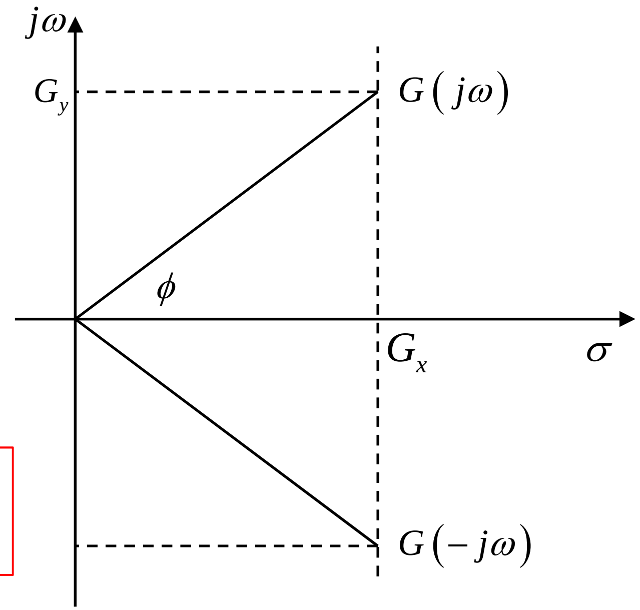
$$= |G(j\omega)|\cos\phi + j|G(j\omega)|\sin\phi$$

$$= |G(j\omega)|(\cos\phi + j\sin\phi) = |G(j\omega)|e^{j\phi}$$

Similarly, $G(-j\omega) = |G(-j\omega)|e^{-j\phi}$

$$\Rightarrow \quad a = -\frac{P}{2j}G(-j\omega) = -\frac{P}{2j}|G(j\omega)|e^{-j\phi}$$

$$\bar{a} = \frac{P}{2j}G(j\omega) = \frac{P}{2j}|G(j\omega)|e^{j\phi}$$



Frequency Response

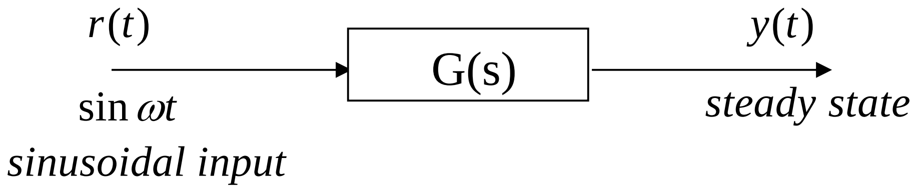
$$\therefore\ y(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t}$$

$$= -\frac{P}{2j}\left|G\left(j\omega\right)\right|e^{-j\phi}e^{-j\omega t} + \frac{P}{2j}\left|G\left(j\omega\right)\right|e^{j\phi}e^{j\omega t}$$

$$= \left|G\left(j\omega\right)\right|\frac{P}{2j}\left(e^{j\left(\omega t+\phi\right)} - e^{-j\left(\omega t+\phi\right)}\right)$$

$$= \boxed{\left|G\left(j\omega\right)\right|P\sin\left(\omega t+\phi\right)}$$

Frequency Response of First Order Systems



Consider, $\dfrac{Y(s)}{R(s)} = G(s) = \dfrac{1}{Ts+1}$

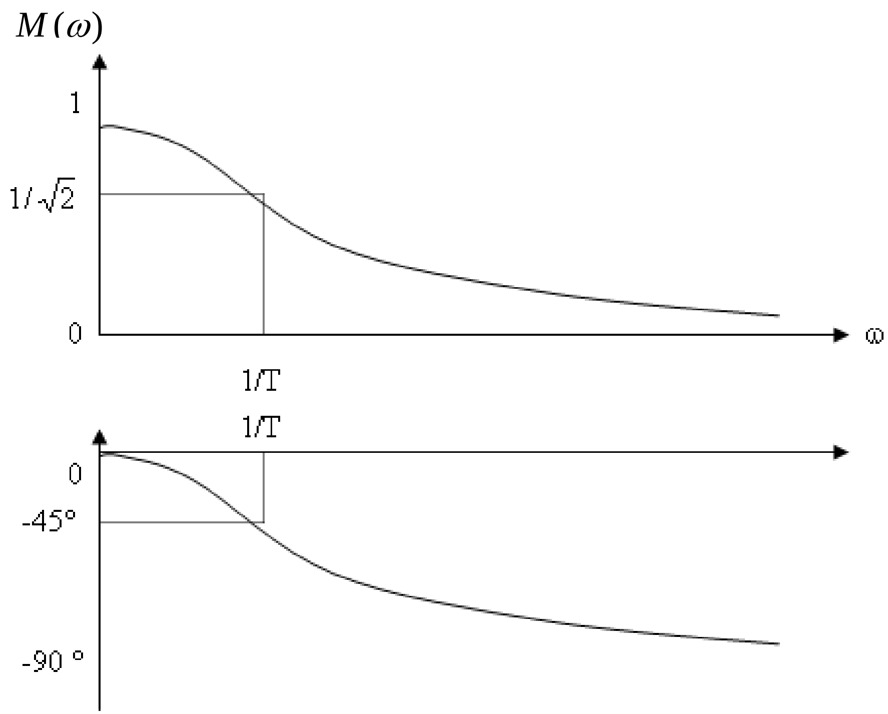
$$G(j\omega) = \frac{1}{Tj\omega + 1}$$

$$M(j\omega) = |G(j\omega)| = \frac{1}{\sqrt{T^2\omega^2 + 1}}$$

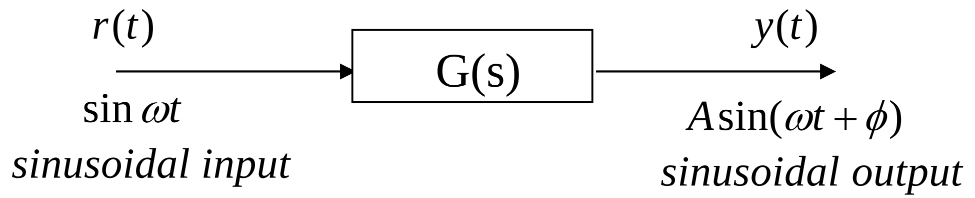
$$\phi(j\omega) = \angle \underline{G(j\omega)} = -\angle(Tj\omega + 1)$$

$$= -\tan^{-1} T\omega$$

$$y(t) = \frac{1}{\sqrt{T^2\omega^2 + 1}} \sin\left(\omega t - \tan^{-1} T\omega\right)$$



Frequency Response of Second Order Systems



$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}, \qquad 0<\zeta<1$$

$$R(s)=\frac{\omega}{s^2+\omega^2}, \qquad r(t)=\sin \omega t$$

$$Y(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\frac{\omega}{s^2+\omega^2}=\frac{as+b}{s^2+2\zeta\omega_n s+\omega_n^2}+\frac{cs+d}{s^2+\omega^2}$$

$$y(t)=ae^{-\zeta\omega_n t}\cos\omega_d t+\frac{b}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t+c\cos\omega_d t+d\sin\omega_d t$$

$$=\underline{Ae^{-\zeta\omega_n t}\sin(\omega_d t+\theta)}+\underline{B\sin(\omega t+\phi)}$$

transient response

Steady-state response

Steady State Frequency Response

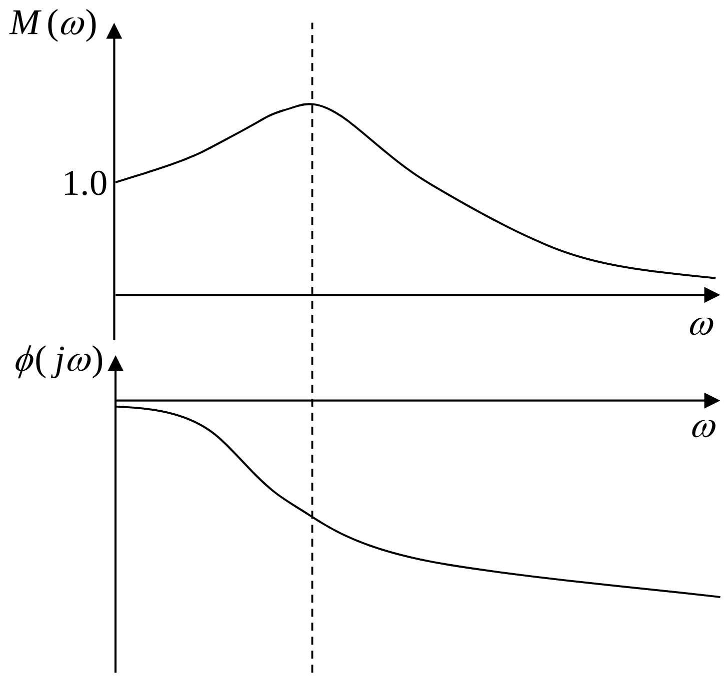
$$y(t) = A(j\omega)\sin(\omega t + \phi)$$

Magnitude ratio

$$M(\omega) = \left|\frac{y(t)}{r(t)}\right| = |G(j\omega)|$$

Phase $\phi(j\omega) = \angle G(j\omega)$

$\Rightarrow$ Frequency response



Steady State Frequency Response

$$\frac{Y(s)}{R(s)} = G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

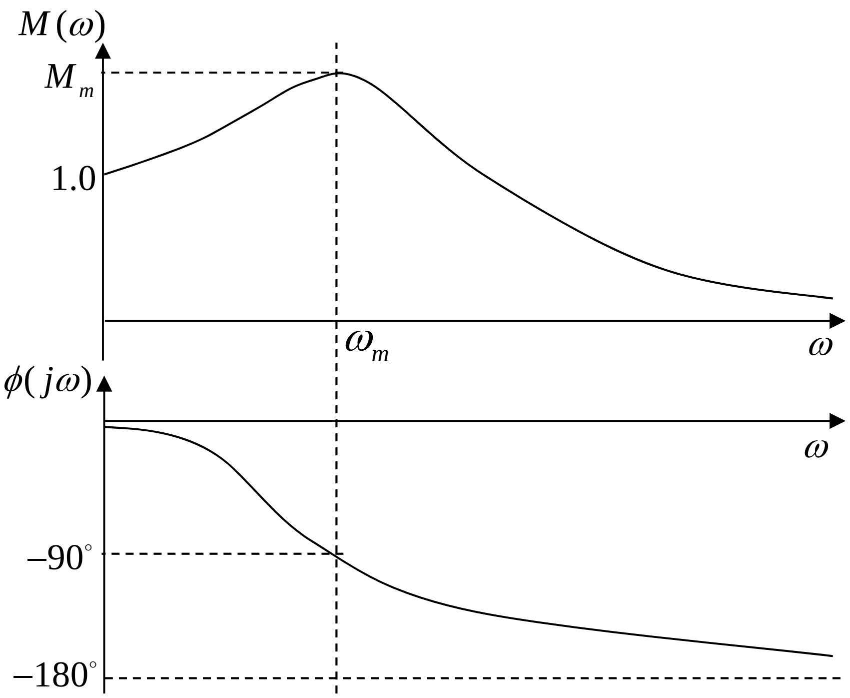
$$\frac{Y(j\omega)}{R(j\omega)} = G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n \omega j + \omega_n^2} = \frac{\omega_n^2}{\omega_n^2 - \omega^2 + 2\zeta\omega_n \omega j}$$

$$M(\omega) = \left|\frac{Y(j\omega)}{R(j\omega)}\right| = \left|\frac{\omega_n^2}{\left(\omega_n^2 - \omega^2\right)^2 + 4\zeta^2 \omega_n{}^2 \omega^2} \cdot \left(\omega_n^2 - \omega^2 - 2\zeta\omega_n \omega j\right)\right|$$

$$= \frac{\omega_n^2}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + 4\zeta^2 \omega_n{}^2 \omega^2}} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2 \cdot \frac{\omega^2}{\omega_n{}^2}}}$$

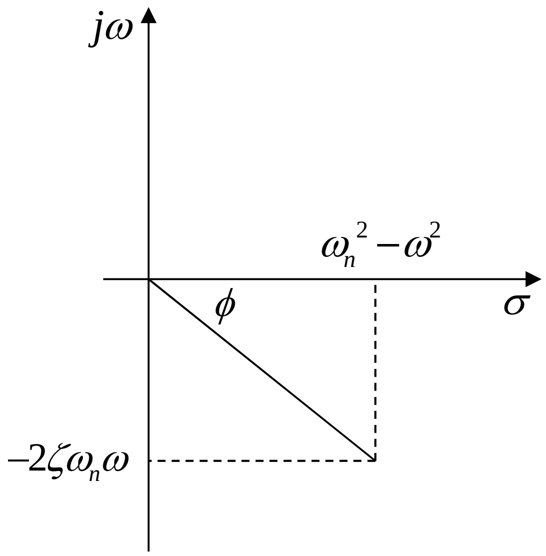
$$\frac{Y(j\omega)}{R(j\omega)} = M(j\omega)\angle G(j\omega)$$

Steady State Frequency Response



$$\phi(j\omega) = -\tan^{-1}\frac{2\zeta\omega_n\omega}{\omega_n^{\ 2} - \omega^2}$$

$$\lim_{\omega\to\infty}\phi(j\omega) = -\tan^{-1}0 = -180^\circ$$



Steady State Frequency Response

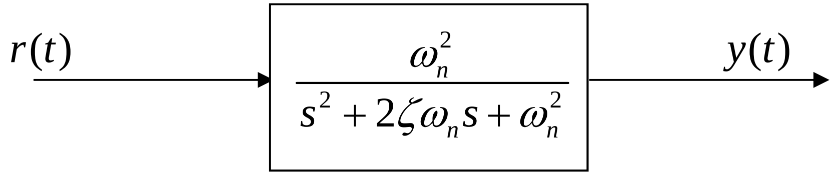
$$M^2(\omega) = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2 \cdot \frac{\omega^2}{\omega_n^{\ 2}}}$$

$$\frac{dM^2(\omega)}{d\omega} = \frac{-\left[2\left(1 - \frac{\omega^2}{\omega_n^2}\right)\left(-\frac{2\omega}{\omega_n^2}\right) + 8\zeta^2 \cdot \frac{\omega}{\omega_n^{\ 2}}\right]}{\left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2 \cdot \frac{\omega^2}{\omega_n^{\ 2}}\right]^2} = 0$$

$$\Rightarrow\ 2\left(1 - \frac{\omega^2}{\omega_n^2}\right)\left(-\frac{2\omega}{\omega_n^2}\right) + 8\zeta^2 \cdot \frac{\omega}{\omega_n^{\ 2}} = 0\,, \qquad -\left(1 - \frac{\omega^2}{\omega_n^2}\right) + 2\zeta^2 = 0$$

$$\therefore\ \omega = \omega_n\sqrt{1 - 2\zeta^2} = \omega_m\,, \qquad M_m = \boxed{\frac{1}{2\zeta\sqrt{1 - \zeta^2}}}$$

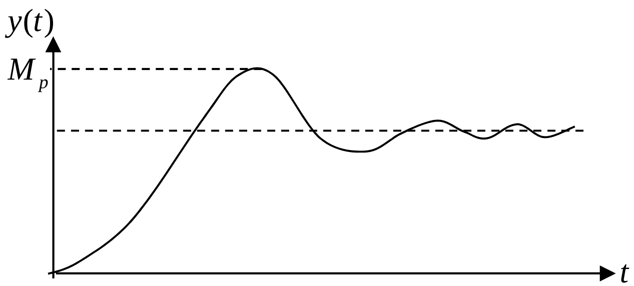
Unit Step Response VS Frequency Response



$$r(t) = u_{-1}(t)$$

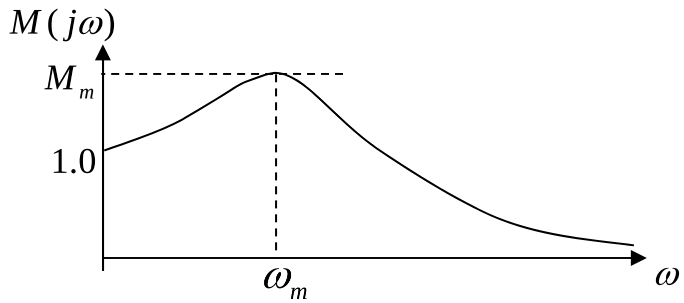
$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \cos^{-1}\zeta\right)$$

Unit step response



$$M_p = 1 + \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

Frequency response



$$\omega_m = \omega_n \sqrt{1-2\zeta^2} \quad , \qquad M_m = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$