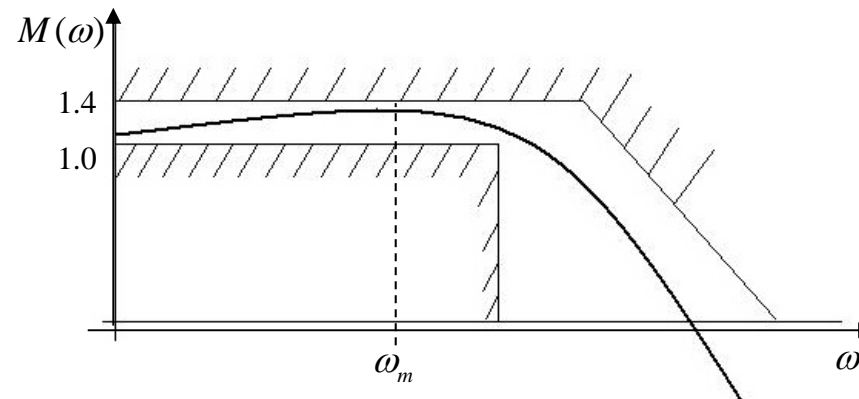


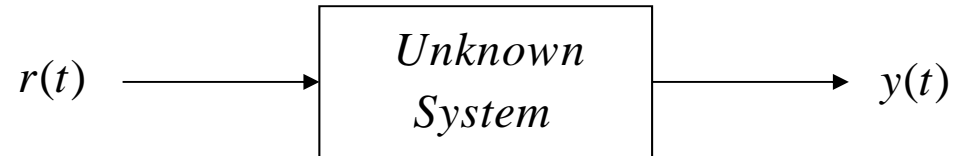
Frequency Domain Analysis II

Control System Design in Frequency Response

1. large $\omega_m \rightarrow$ fast time response
2. M_m, M_p : function of damping ratio ζ
large $M_m \rightarrow$ large M_p
3. good damping characteristics $1 < M_m < 1.4$
4. minimum effect of any undesirable noise



Experimental Determination of Transfer Function

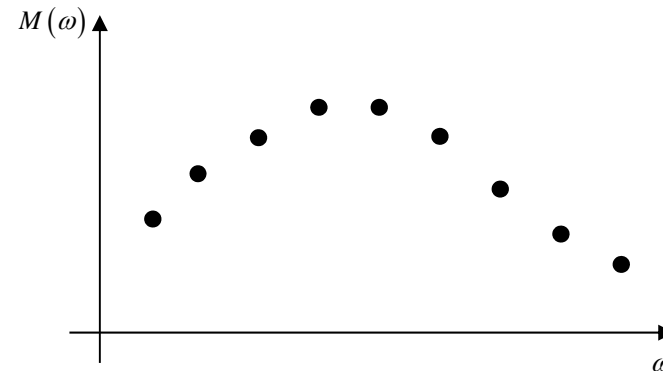


$$r(t) = \sin \omega_i t \quad ; \quad y(t) = M(\omega_i) \sin(\omega_i t - \phi)$$

some frequency $\omega_i \rightarrow$ measure $M(\omega_i)$
steady state

$$G(s) = G(j\omega)$$

$$M(\omega) = |G(j\omega)|$$



Transfer Functions and Frequency Magnitude Plot

- Bode Plot (Logarithmic Plots)

i) Log Magnitude $L_m G(j\omega) = 20 \log |G(j\omega)| \text{ dB}$

ii) dB : decibel, Logarithm of the magnitude

iii) 1 decade : $1\text{Hz} \sim 10\text{Hz}$ frequency width
 $2.5\text{Hz} \sim 25\text{Hz}$

- Drawing the Bode Plots

The reason of using logarithm – mathematical operation $\times, \div \rightarrow +, -$
– typical three types : simple straight line
asymptotic
approximations

Transfer Functions and Frequency Magnitude Plot

ex)
$$G(s) = \frac{K(1+sT_1)(1+sT_2)}{s^2(1+sT_3)(1+as+bs^2)}$$

$$L_m G(j\omega) = L_m K + L_m (1+j\omega T_1) + L_m (1+j\omega T_2) - 2L_m (j\omega) - L_m (1+j\omega T_3) - L_m \{1+aj\omega+b(j\omega)^2\}$$

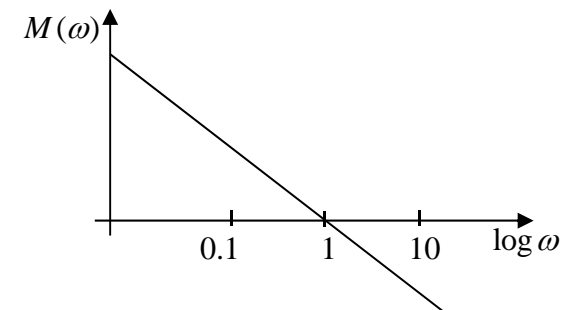
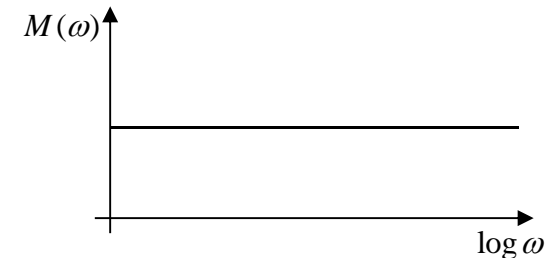
i) $L_m \cdot K = 20 \log K \text{ dB}$

straight line

ii) $L_m \left(\frac{1}{j\omega} \right) = 20 \log \left| \frac{1}{j\omega} \right| = -20 \log \omega$

straight line

With negative slope -20dB/decade



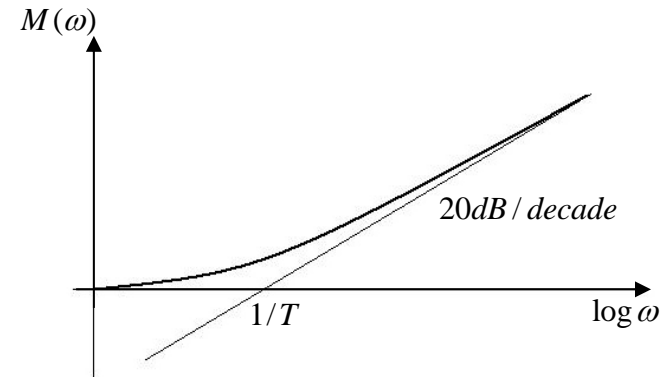
Transfer Functions and Frequency Magnitude Plot

iii) $(1 + j\omega T)$

$$L_m(1 + j\omega T) = 20 \log |1 + j\omega T| = 20 \log \sqrt{1 + \omega^2 T^2}$$

$$\omega T \ll 1 \rightarrow L_m(1 + j\omega T) \cong 20 \log 1 = 0 \text{ dB}$$

$$\begin{aligned} \omega T \gg 1 &\rightarrow L_m(1 + j\omega T) \cong 20 \log \omega T \\ &= 20 \log \omega + 20 \log T \end{aligned}$$

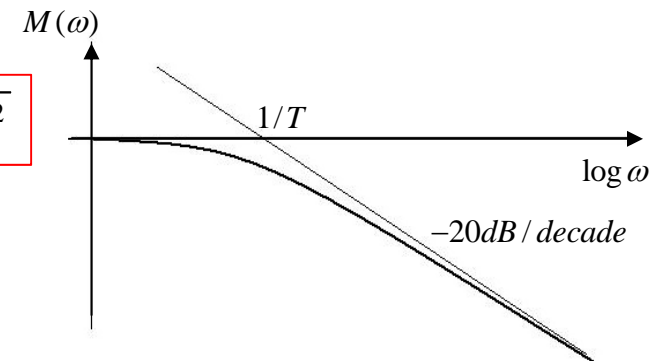


iv) $\frac{1}{1 + j\omega T}$

$$L_m(1 + j\omega T)^{-1} = -20 \log |1 + j\omega T| = -20 \log \sqrt{1 + \omega^2 T^2}$$

$$\omega T \ll 1 \rightarrow L_m(1 + j\omega T)^{-1} \cong -20 \log 1 = 0 \text{ dB}$$

$$\begin{aligned} \omega T \gg 1 &\rightarrow L_m(1 + j\omega T)^{-1} \cong -20 \log \omega T \\ &= -20 \log \omega - 20 \log T \end{aligned}$$



Transfer Functions and Frequency Magnitude Plot

$$v) \frac{1}{\left(1 + \frac{2\zeta}{\omega_n} s + \frac{1}{\omega_n^2} s^2\right)} \Rightarrow \frac{1}{\left[1 + \frac{2\zeta}{\omega_n} j\omega + \frac{1}{\omega_n^2} (j\omega)^2\right]}$$

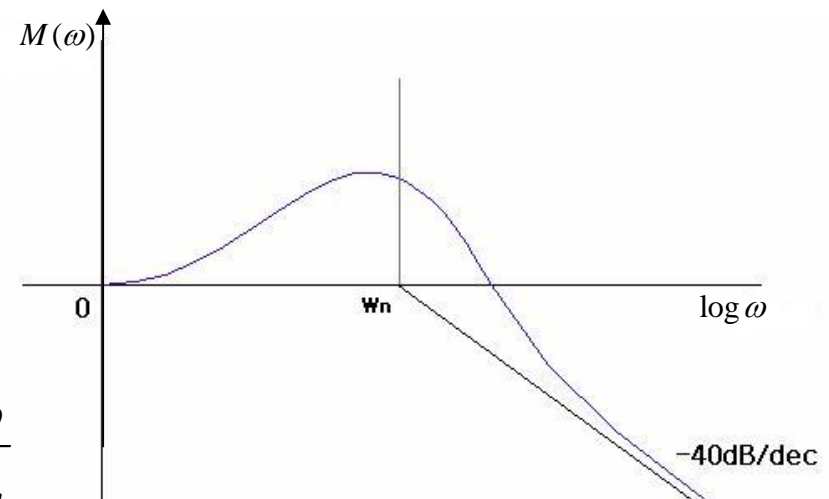
$$L_m \left[\frac{1}{\left[1 + \frac{2\zeta}{\omega_n} j\omega + \frac{1}{\omega_n^2} (j\omega)^2\right]} \right] = 20 \log \left[\frac{1}{\left[1 + \frac{2\zeta}{\omega_n} j\omega + \frac{1}{\omega_n^2} (j\omega)^2\right]} \right] = -20 \log \left[\left(1 - \frac{\omega^2}{\omega_n^2}\right) + \left(\frac{2\zeta\omega}{\omega_n}\right)^2 \right]^{1/2}$$

$$\frac{\omega}{\omega_n} \ll 1$$

$$\rightarrow L_m [G(j\omega)] \cong -20 \log 1 = 0 \text{ dB}$$

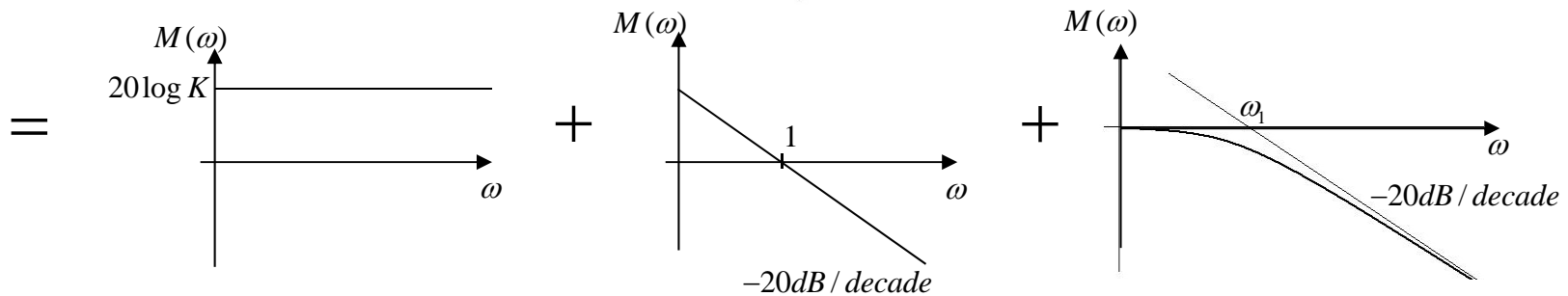
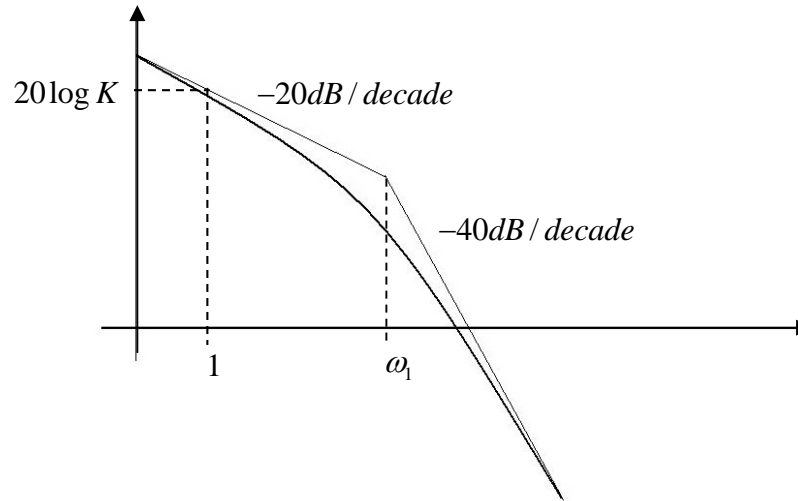
$$\frac{\omega}{\omega_n} \gg 1$$

$$\rightarrow L_m [G(j\omega)] \cong -20 \log \frac{\omega^2}{\omega_n^2} = -40 \log \frac{\omega}{\omega_n}$$



Transfer Functions and Frequency Magnitude Plot

ex)



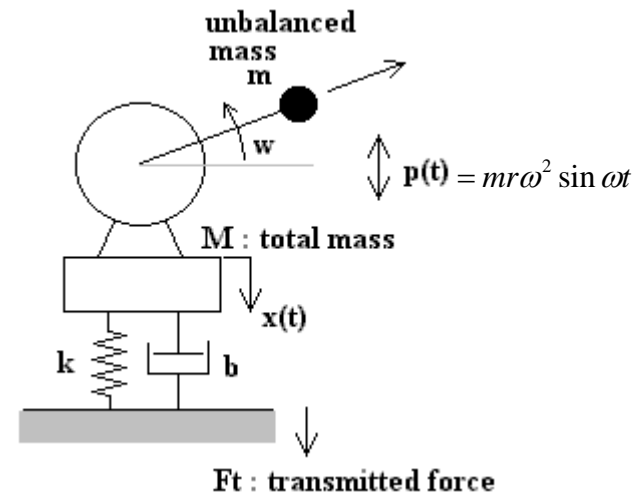
$$= \frac{K}{(j\omega) \left(1 + \frac{j\omega}{\omega_1}\right)} \Rightarrow \frac{K}{s \left(1 + \frac{1}{\omega_1} s\right)}$$

Vibration Isolation in Rotating Systems

Vibration due to rotating unbalance

$$(1) \quad M\ddot{x} + b\dot{x} + kx = p(t) \\ = mr\omega^2 \sin \omega t$$

$$\frac{X(s)}{P(s)} = \frac{1}{Ms^2 + bs + k} = G(s)$$



Vibration Isolation in Rotating Systems

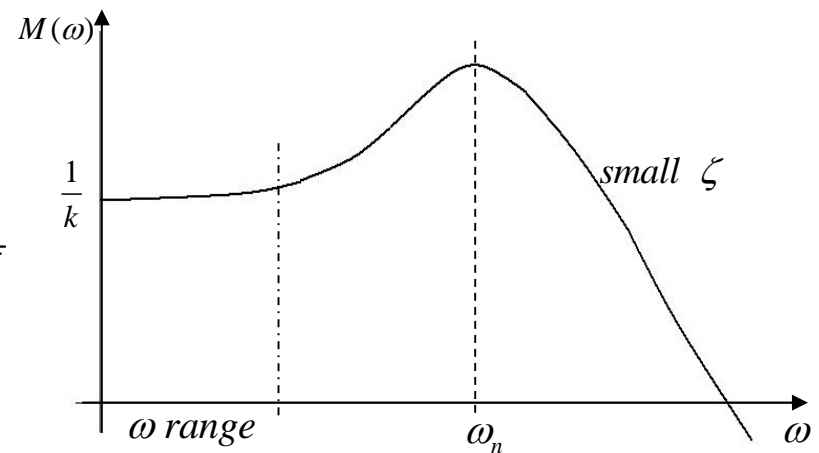
(2) Frequency response

$$\frac{x(j\omega)}{p(j\omega)} = \frac{1}{-M\omega^2 + bj\omega + k} = G(j\omega)$$

$$x(t) = X \sin(\omega t + \phi) = |G(j\omega)| \sin(\omega t + \phi) \cdot m r \omega^2$$

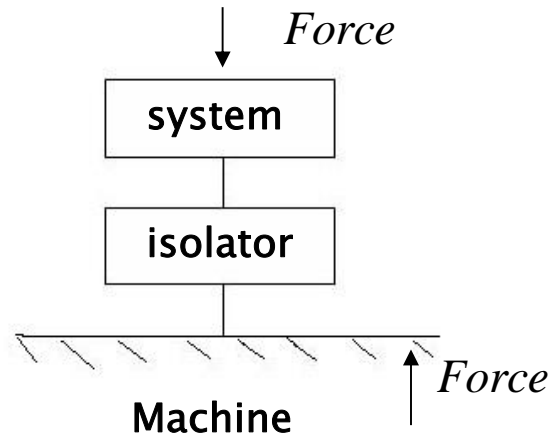
$$\phi = -\tan^{-1} \frac{b\omega}{k - M\omega^2} = -\tan^{-1} \frac{2\zeta\omega / \omega_n}{1 - \omega^2 / \omega_n^2}$$

$$\begin{aligned} |G(j\omega)| &= \frac{1}{\sqrt{(k - M\omega^2)^2 + b^2\omega^2}} \\ &= \frac{1/k}{\sqrt{(1 - \omega^2 / \omega_n^2)^2 + (2\zeta\omega / \omega_n)^2}} \end{aligned}$$

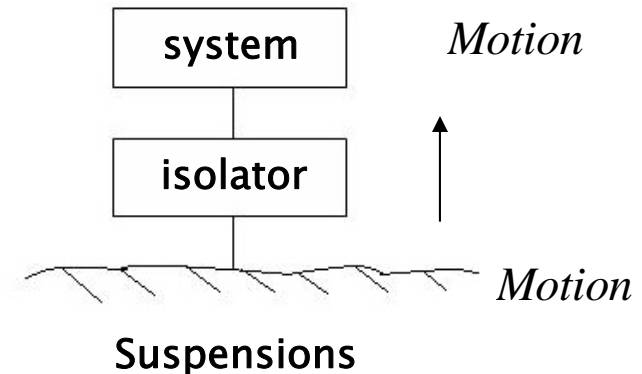


Vibration Isolation in Rotating Systems

Vibration isolators



Reduce the magnitude of force transmitted from a machine to its foundation



Reduce the magnitude of motion transmitted from a vibratory foundation to a system

- Isolator
- i) – load supporting elements, spring.
 - energy-dissipating elements, damper.
 - ii) – Synthetic rubber : both load supporting and energy dissipating

Transmissibility

Transmissibility :

A measure of the reduction of transmitted force or motion afforded by an isolator.

$$TR(\text{machine}) = \frac{\text{the force amplitude transmitted to the foundation}}{\text{the amplitude of the exciting force}}$$

$$TR(\text{suspensions}) = \frac{\text{the vibration amplitude of the system}}{\text{the vibration amplitude of the foundation}}$$

Transmissibility for Force Excitation

$$p(t) = mr\omega^2 \sin \omega t = F_0 \sin \omega t$$

$$f(t) = b\dot{x} + kx = F_t \sin(\omega t + \phi)$$

$$X(s) = \frac{1}{Ms^2 + bs + k} P(s)$$

$$F_t(s) = (bs + k)X(s) = \frac{bs + k}{Ms^2 + bs + k} \cdot P(s)$$

Frequency response

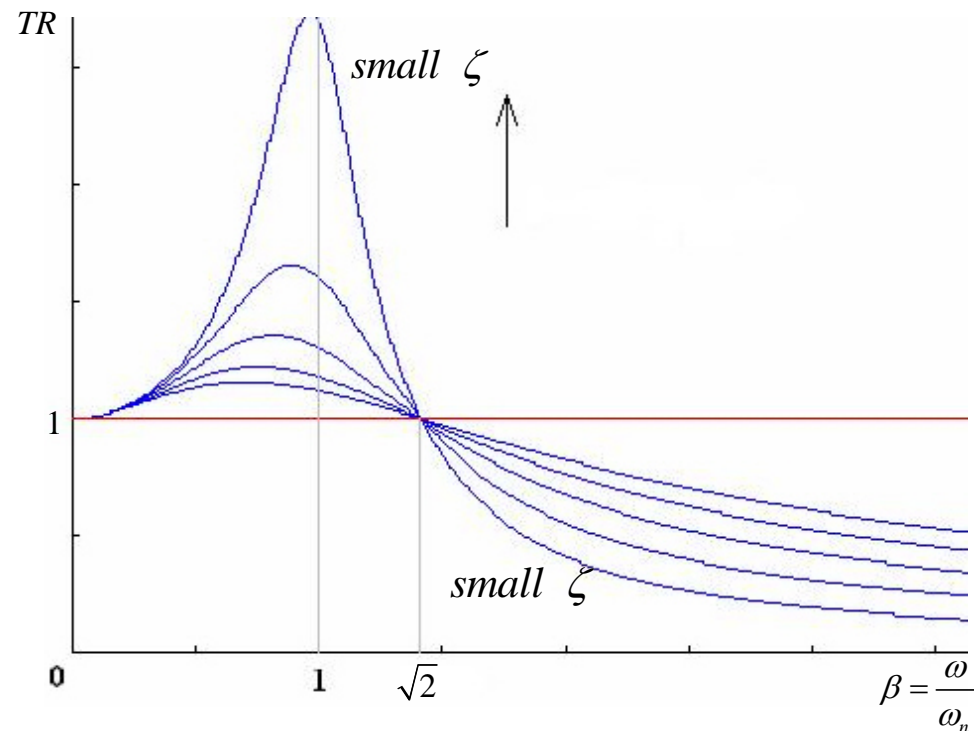
$$\frac{F_t(j\omega)}{p(j\omega)} = \frac{bj\omega + k}{-M\omega^2 + bj\omega + k} = \frac{(b/M)j\omega + (k/M)}{-\omega^2 + (b/M)j\omega + k/M}$$

Transmissibility for Force Excitation

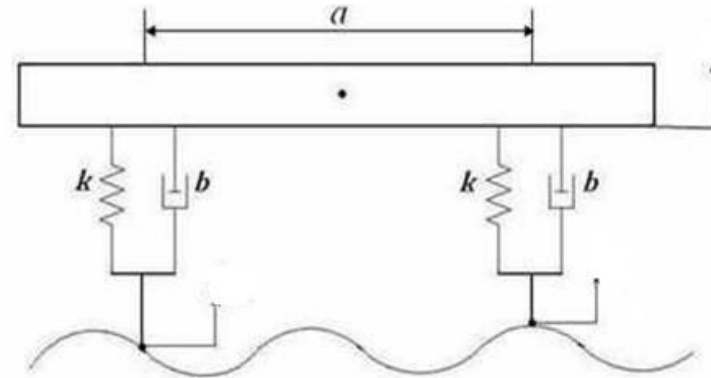
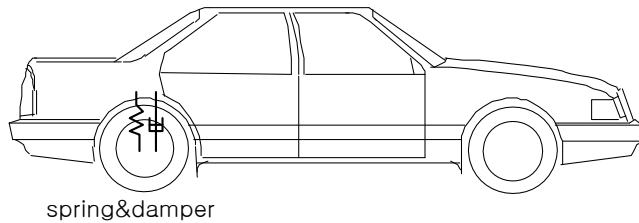
$$k/M = \omega_n^2, \quad b/M = 2\zeta\omega_n \quad \Rightarrow \quad \frac{F_t(j\omega)}{p(j\omega)} = \frac{1 + j(2\zeta\omega/\omega_n)}{1 - \omega^2/\omega_n^2 + j(2\zeta\omega/\omega_n)}$$

$$\begin{aligned} TR &= \frac{F_t}{F_0} = \left| \frac{F_t(j\omega)}{p(j\omega)} \right| \\ &= \frac{\sqrt{1 + (2\zeta\omega/\omega_n)^2}}{\sqrt{(1 - \omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}} \\ &= \frac{\sqrt{1 + (2\zeta\beta)^2}}{\sqrt{(1 - \beta^2)^2 + (2\zeta\beta)^2}} \end{aligned}$$

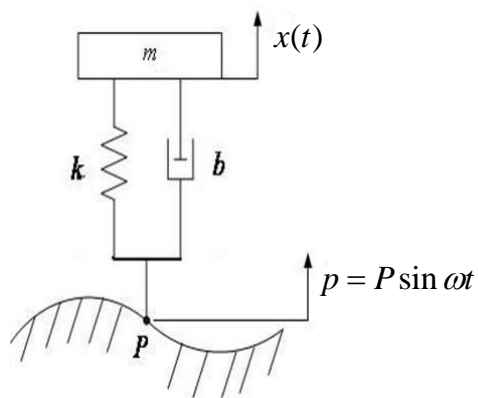
Note, $\beta = \sqrt{2}$, $TR = 1$
for any ζ



Automobile Suspension Systems



Simplified model



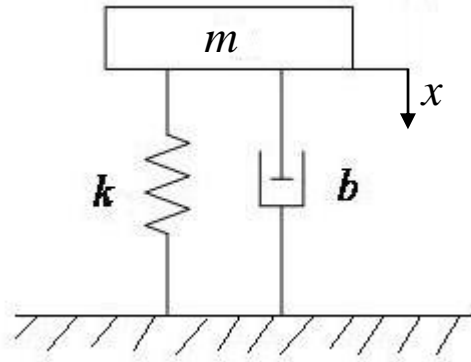
$$m\ddot{x} + b(\dot{x} - \dot{p}) + k(x - p) = 0$$

$$\frac{X(s)}{P(s)} = \frac{bs + k}{ms^2 + bs + k}, \quad \frac{x(j\omega)}{p(j\omega)} = \frac{bj\omega + k}{-m\omega^2 + bj\omega + k}$$

$$TR = \left| \frac{x(j\omega)}{p(j\omega)} \right| = \frac{\sqrt{b^2\omega^2 + k^2}}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}$$

$$= \frac{\sqrt{1 + (2\zeta\beta)^2}}{\sqrt{(1 - \beta^2)^2 + (2\zeta\beta)^2}}$$

Dynamic Vibration Absorbers



$$P = mr\omega^2$$

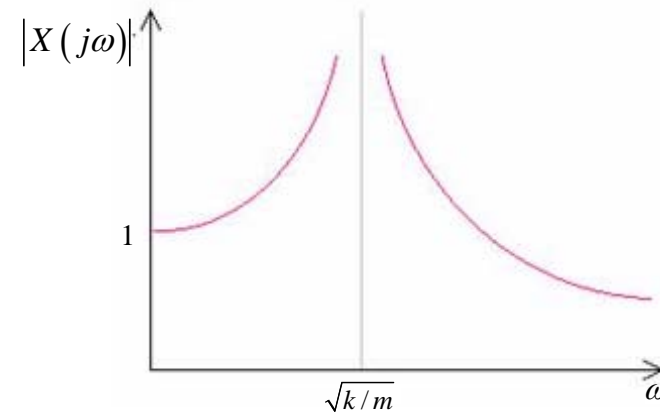
$$TR = \frac{(mr\omega^2)\sqrt{b^2\omega^2 + k^2}}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}$$

If b is small and $\omega = \omega_n$

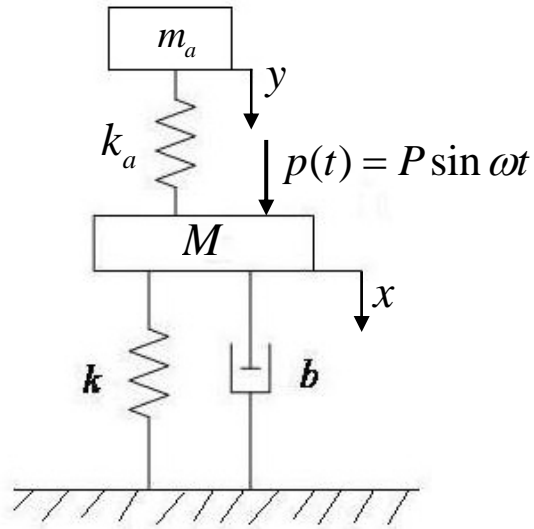
Then resonance – excessive vibration

– extremely large force transmitted

ω is very close to ω_n , critical speed,
=> A dynamic vibration absorber



Dynamic Vibration Absorbers



$$M \ddot{x} = -kx - b\dot{x} - k_a(x - y) + p(t)$$

$$m_a \ddot{y} = -k_a(y - x)$$

Laplace Transform

$$(Ms^2 + bs + k + k_a)X(s) - k_a Y(s) = P(s)$$

$$(m_a s^2 + k_a)Y(s) - k_a X(s) = 0$$

$$\Rightarrow \frac{X(s)}{P(s)} = \frac{m_a s^2 + k_a}{(Ms^2 + bs + k + k_a)(m_a s^2 + k_a) - k_a^2}$$

Frequency response

$$\frac{X(j\omega)}{P(j\omega)} = \frac{-m_a \omega^2 + k_a}{(-M \omega^2 + k + k_a)(-m_a \omega^2 + k_a) - k_a^2} \quad \text{for small } b$$

The transmitted force

$$f(t) = kx + b\dot{x} \cong kx$$

Dynamic Vibration Absorbers

$$\begin{aligned} |X(j\omega)| &= \left| \frac{k_a - m_a \omega^2}{(-M\omega^2 + k + k_a)(k_a + m_a \omega^2) - k_a^2} \right| |P(j\omega)| \\ &= \left| \frac{mr\omega^2 (k_a - m_a \omega^2)}{(-M\omega^2 + k + k_a)(k_a + m_a \omega^2) - k_a^2} \right| \end{aligned}$$

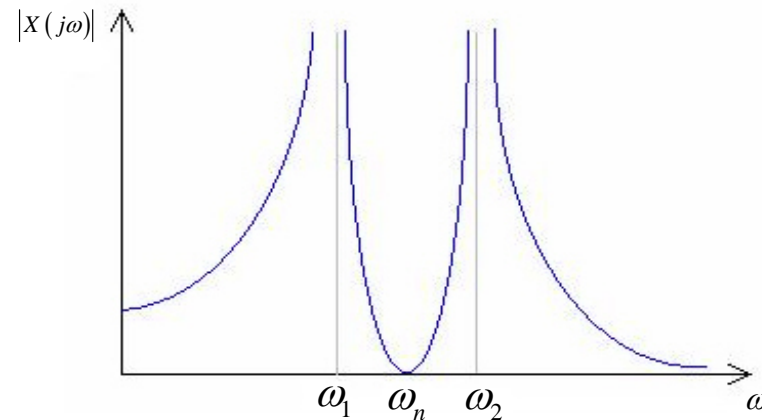
If we choose k_a , m_a so that,

$$k_a - m_a \omega^2 = 0$$

then, $|X(j\omega)| = 0$, $|f(t)| = 0$ at this frequency.

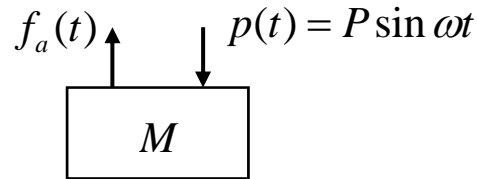
If $\omega_{exc} = \omega_n$

we use dynamic vibration absorbers.



Dynamic Vibration Absorbers

Physically,



$$f_a(t) = -k_a(x - y) = k_a y = p(t), \quad (\text{at } \omega_{exc} = \omega_n, \quad x \cong 0)$$

Spring force $k_a y$ cancels $p(t)$.

$$\frac{Y(j\omega)}{P(j\omega)} = \frac{k_a}{(-M\omega^2 + k + k_a)(-m_a\omega^2 + k_a) - k_a^2}$$

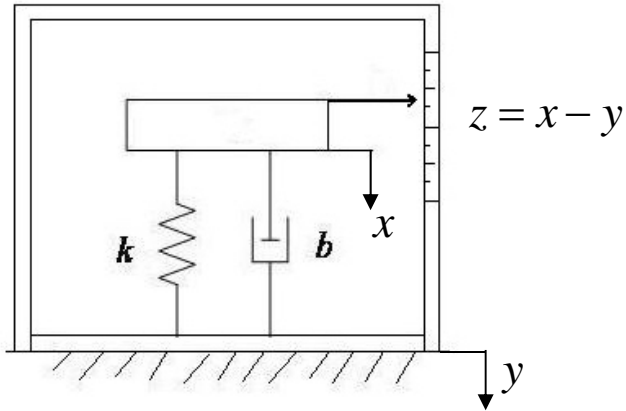
If we choose m_a, k_a so that, $k_a - m_a\omega^2 = 0$

then $\frac{Y(j\omega)}{P(j\omega)} = -\frac{1}{k_a}$ magnitude ratio $\left| \frac{1}{k_a} \right|$

$$p(t) = \sin \omega t$$

$$y(t) = \frac{1}{k_a} P \sin(\omega t - 180^\circ) = -\frac{P}{k_a} \sin \omega t \quad (\text{phase } -180^\circ)$$

Seismograph



A device used to measure ground displacement during earthquakes

$$\text{Equation of motion : } m\ddot{x} = k(y - x) + b(\dot{y} - \dot{x})$$

$$\text{measurement : } z = x - y$$

$$\Rightarrow m(\ddot{y} + \ddot{z}) + b\dot{z} + kz = 0$$

$$m\ddot{z} + b\dot{z} + kz = -m\ddot{y}$$

$$\Rightarrow \text{Transfer Function : } \frac{Z(s)}{Y(s)} = \frac{-ms^2}{ms^2 + bs + k} = \frac{-s^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\left(\frac{k}{m} = \omega_n^2, \quad \frac{b}{m} = 2\zeta\omega_n \right)$$

Seismograph

$$\frac{Z(j\omega)}{Y(j\omega)} = \frac{\omega^2}{-\omega^2 + 2\zeta\omega_n j\omega + \omega_n^2} = \frac{\beta^2}{1 - \beta^2 + 2\zeta\beta j} \quad (\beta = \omega/\omega_n)$$

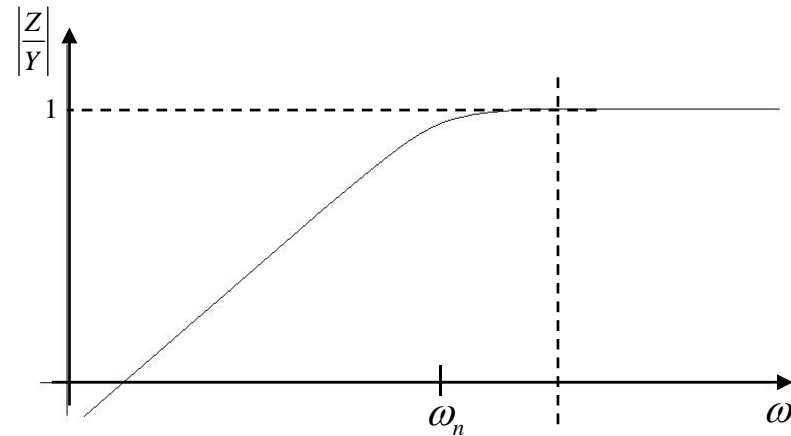
We hope, $z(t) \cong y(t)$

If, $\beta \gg 1$, $\frac{Z(j\omega)}{Y(j\omega)} = -1$

$z(t) \cong y(t)$ for $\omega \gg \omega_n$

Choose ω_n as small as possible.
(large mass, soft spring)

$$\left(\omega_n = \sqrt{\frac{k}{m}} \right)$$



Accelerometer



The system configuration is basically the same as the seismograph, but choice of undamped natural frequency is different.

Equation of motion : the same as the seismograph, $m\ddot{z} + b\dot{z} + kz = -m\ddot{y}$

$$\Rightarrow \text{Transfer Function} : \frac{Z(s)}{s^2 Y(s)} = \frac{-m}{ms^2 + bs + k} = \frac{-1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Frequency response

$$\omega_n \gg \omega$$

$$\frac{Z(s)}{s^2 Y(s)} \cong -\frac{1}{\omega_n^2}$$

