Optimal Design of Energy Systems

Chapter 8 Lagrange Multipliers

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Chapter 8. Lagrange Multipliers

8.1 Calculus methods of optimization

- Function – differentiable
- Constraints - equality

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8.2 Lagrange Multiplier equations

optimize $\quad y = y(x_1, x_2, \cdots, x_n)$

subject to $\quad \phi_1 = \phi_1(x_1, x_2, \cdots, x_n) = 0$

$$\vdots$$

$$\phi_m = \phi_m(x_1, x_2, \cdots, x_n) = 0$$

→ optimum occurs at x values

$$\nabla y - \lambda_1 \nabla \phi_1 - \cdots - \lambda_m \nabla \phi_m = 0$$

(λ_1 ←) constants : Lagrange multipliers

$$\nabla y = \frac{\partial y}{\partial x_1}\hat{i}_1 + \frac{\partial y}{\partial x_2}\hat{i}_2 + \cdots + \frac{\partial y}{\partial x_n}\hat{i}_n$$

($\hat{i}_n$ ←) unit vectors

(∇y ←) Gradient vector : normal to y=const. surface

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8.2 Lagrange Multiplier equations

- n scalar equations

$$\left[\begin{array}{l} \hat{i}_1 : \dfrac{\partial y}{\partial x_1} - \lambda_1 \dfrac{\partial \phi_1}{\partial x_1} - \cdots - \lambda_m \dfrac{\partial \phi_m}{\partial x_1} = 0 \\ \qquad \vdots \\ \hat{i}_n : \dfrac{\partial y}{\partial x_n} - \lambda_1 \dfrac{\partial \phi_1}{\partial x_n} - \cdots - \lambda_m \dfrac{\partial \phi_m}{\partial x_n} = 0 \end{array}\right.$$

- m constraints equations

- unknowns $\underbrace{\lambda_1 \cdots \lambda_m}_{m},\ \overbrace{x_1^* \cdots x_n^*}^{n}$ ← at optimum point

$$m < n$$

if $m = n$ fixed values of x's (no optimization)

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8.5 Unconstrained optimization

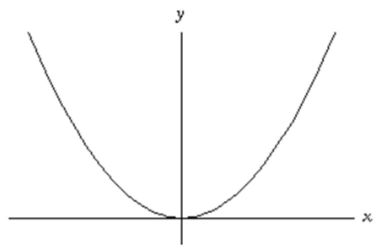
$$y = y(x_1, x_2, \cdots, x_n)$$

$$\rightarrow \quad \nabla y = 0$$

$$\frac{\partial y}{\partial x_1} = \frac{\partial y}{\partial x_2} = \cdots = \frac{\partial y}{\partial x_n} = 0$$

ex) $y = x^2$ minimize (optimize)

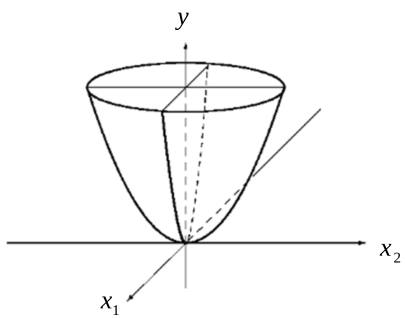
$$\frac{\partial y}{\partial x} = 2x = 0$$



$$y = x_1^2 + x_2^2$$

$$\frac{\partial y}{\partial x_1} = 2x_1 = 0$$

$$\frac{\partial y}{\partial x_2} = 2x_2 = 0$$



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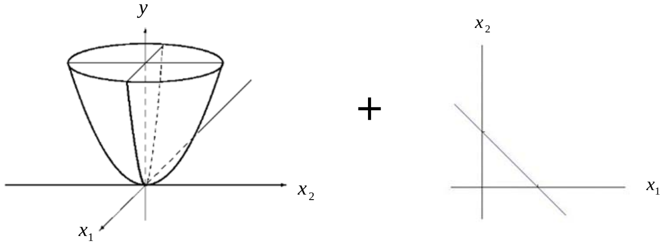
8.6 Constrained optimization

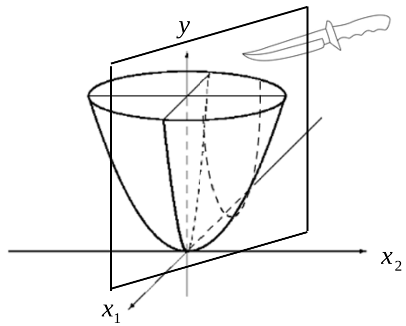


ex) minimize $y = x_1^2 + x_2^2$

constraint : $x_1 + x_2 = 1$

$\rightarrow \phi = x_1 + x_2 - 1 = 0$

✓ Lagrange multiplier



$$\nabla y = \frac{\partial y}{\partial x_1}\hat{i}_1 + \frac{\partial y}{\partial x_2}\hat{i}_2 = 2x_1\hat{i}_1 + 2x_2\hat{i}_2$$

$$\nabla \phi = \frac{\partial \phi}{\partial x_1}\hat{i}_1 + \frac{\partial \phi}{\partial x_2}\hat{i}_2 = \hat{i}_1 + \hat{i}_2$$

$$\nabla y - \lambda \nabla \phi = (2x_1 - \lambda)\hat{i}_1 + (2x_2 - \lambda)\hat{i}_2 = 0$$

$$2x_1 - \lambda = 0 \qquad x_1 = \lambda / 2, \quad x_2 = \lambda / 2$$

$$\rightarrow$$

$$2x_2 - \lambda = 0 \qquad x_1 = 0.5, \quad x_2 = 0.5 \rightarrow y_{\min} = 0.5$$

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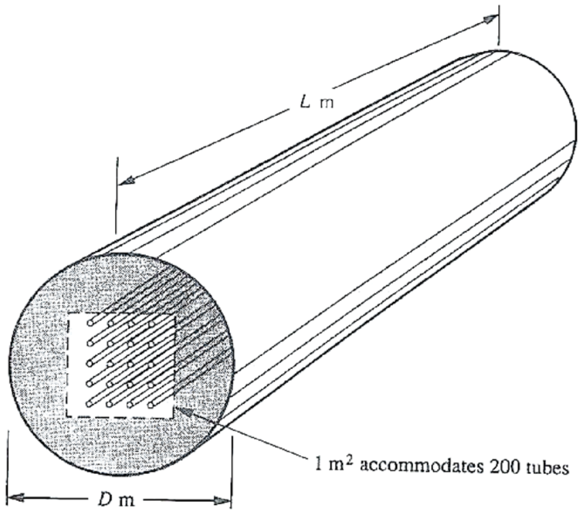
8.6 Constrained optimization

<Example 8.3> A total length of 100 m of tubes must be installed in a shell-and-tube heat exchanger, in order to provide the necessary heat- transfer area. Total cost of the installation in dollars includes

1. The cost of the tubes, which is constant at $ 900
2. The cost of the shell =$1100D^{2.5}L$
3. The cost of the floor space occupied by the heat exchanger=320 DL

where L is the length of the heat exchanger and D is the diameter of the shell, both in meters. The spacing of the tubes in such that **200 tubes will fit in a cross-sectional area of 1 m²** in the shell. Determine the diameter and length of the heat exchanger for **minimum first cost**.



8.6 Constrained optimization

<Solution>

minimize $y = 900 + 1100 D^{2.5} L + 320 DL$

subject to $\frac{\pi D^2}{4} \; (m^2) \times L \; (m) \times 200 \; (tubes / m^2) = 100 \; (m)$

$\rightarrow 50 \pi D^2 L = 100$

$$\nabla y - \lambda \nabla \phi$$

$$\begin{cases} D : \hat{i}_1 & (2750 D^{1.5} L + 320 L) - \lambda (100 \pi DL) = 0 \\ L : \hat{i}_2 & (1100 D^{2.5} + 320 D) - \lambda (50 \pi D^2) = 0 \end{cases}$$

$$\lambda = \frac{2750 D^{1.5} + 320}{100 \pi D}$$

$$D^* = 0.7 \; m, \;\; L^* = 1.3 \; m, \;\; \lambda = 8.78$$

$$y^* = 900 + (1100)(0.7)^{2.5}(1.3) + (320)(0.7)(1.3) = \$1777.45$$

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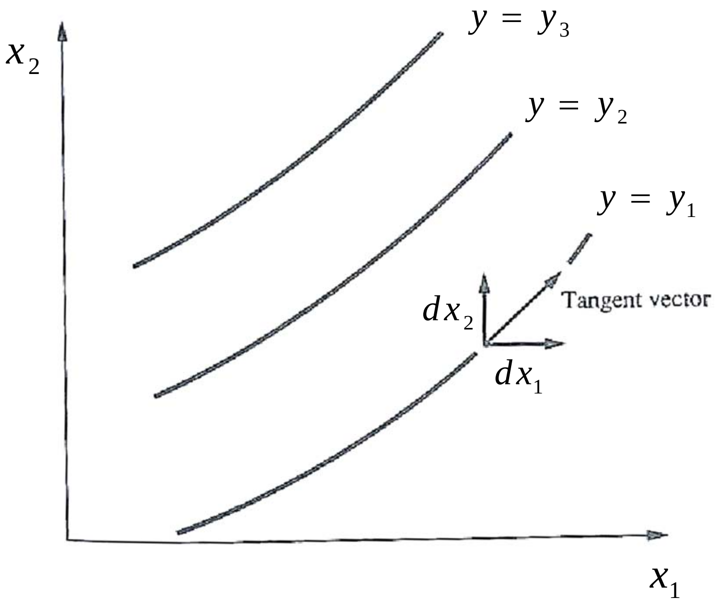
8.7 Gradient vector

- gradient vector is normal to contour line



$$y = y(x_1, x_2)$$

$$dy = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2$$

Along the line of $y = const.$

$$dy = 0 \quad \rightarrow \quad dx_1 = -dx_2 \frac{\partial y / \partial x_2}{\partial y / \partial x_1}$$

substitution into arbitrary unit vector

$$\frac{dx_1 \hat{i} + dx_2 \hat{j}}{\sqrt{(dx_1)^2 + (dx_2)^2}}$$

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8.7 Gradient vector

Tangent vector

$$\vec{T} = \frac{dx_1 \hat{i} + dx_2 \hat{j}}{\sqrt{(dx_1)^2 + (dx_2)^2}} = \frac{-\dfrac{\partial y / \partial x_2}{\partial y / \partial x_1} \hat{i} + \hat{j}}{\sqrt{\left(\dfrac{\partial y / \partial x_2}{\partial y / \partial x_1}\right)^2 + 1}} = \frac{-\dfrac{\partial y}{\partial x_2} \hat{i} + \dfrac{\partial y}{\partial x_1} \hat{j}}{\sqrt{\left(\dfrac{\partial y}{\partial x_1}\right)^2 + \left(\dfrac{\partial y}{\partial x_2}\right)^2}}$$

Gradient vector

$$\vec{G} = \frac{\dfrac{\partial y}{\partial x_2} \hat{i} + \dfrac{\partial y}{\partial x_1} \hat{j}}{\sqrt{\left(\dfrac{\partial y}{\partial x_1}\right)^2 + \left(\dfrac{\partial y}{\partial x_2}\right)^2}}$$

$$\vec{T} \bullet \vec{G} = 0$$

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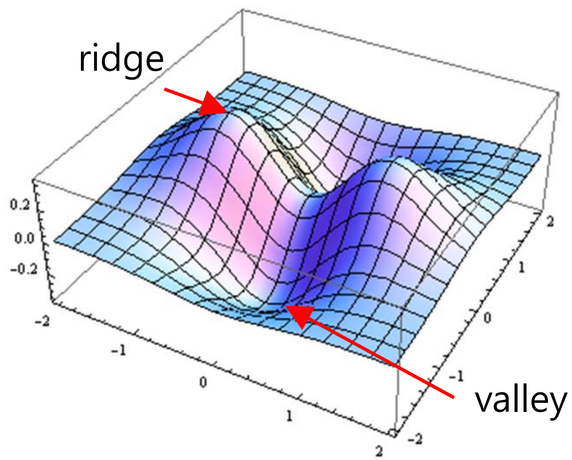
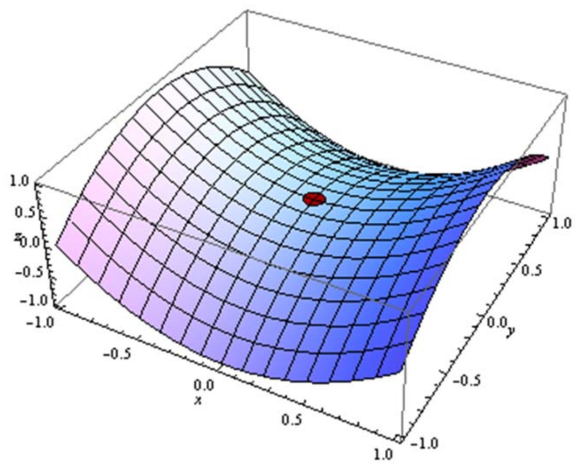
8.9 Test for maximum or minimum

- decide whether the point is a maximum, minimum, saddle point, ridge, or valley



ridge & valley point



saddle point

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8.9 Test for maximum or minimum

- $y = y(x)$ expected minimum occurs at $x = a$

$$y(x) = y(a) + \frac{dy}{dx}(x-a) + \frac{1}{2}\frac{d^2y}{dx^2}(x-a)^2 + \cdots$$ Taylor series expansion near $x = a$

If $\frac{dy}{dx} > 0$

$y(x) > y(a)$ when $x > a$ $y(a)$ is still considered minimum

$y(x) < y(a)$ when $x < a$ $y(a)$ is not minimum

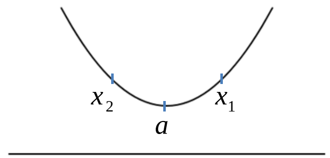
$\rightarrow \frac{dy}{dx}$ should be zero

If $\frac{d^2y}{dx^2} > 0$

$y(x_1) > y(a)$ when $x_1 > a$

$y(x_2) > y(a)$ when $x_2 < a$

minimum

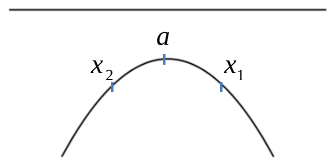


If $\frac{d^2y}{dx^2} < 0$

$y(x_1) < y(a)$ when $x_1 > a$

$y(x_2) < y(a)$ when $x_2 < a$

maximum



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8.9 Test for maximum or minimum

- $y = y(x_1, x_2)$ expected minimum or maximum occurs at (a_1, a_2)

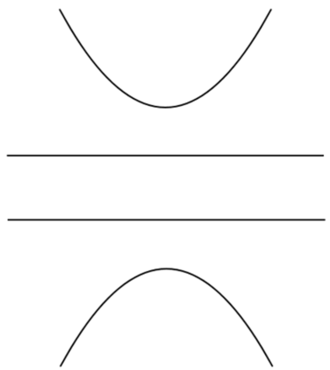
$$y(x_1, x_2) = y(a_1, a_2) + \underbrace{\boxed{\frac{\partial y}{\partial x_1}}}_{=0}(x_1 - a_1) + \underbrace{\boxed{\frac{\partial y}{\partial x_2}}}_{=0}(x_2 - a_2)$$

$$+ \frac{1}{2} y''_{11}(x_1 - a_1)^2 + y''_{12}(x_1 - a_1)(x_2 - a_2) + \frac{1}{2} y''_{22}(x_2 - a_2)^2 + \cdots$$

$$D = \begin{vmatrix} y''_{11} & y''_{12} \\ y''_{21} & y''_{22} \end{vmatrix} = y''_{11} y''_{22} - {y''_{12}}^2$$

$D > 0$ and $y''_{11} > 0$ $(y''_{22} > 0)$ min

$D > 0$ and $y''_{11} < 0$ $(y''_{22} < 0)$ max



$D < 0$ not min or max

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8.9 Test for maximum or minimum

<Example 8.4> optimal value x_1 and x_2 & maximum or minimum ?

$$y = \frac{x_1^2}{4} + \frac{2}{x_1 x_2} + 4x_2$$

<Solution>

$$\frac{\partial y}{\partial x_1} = \frac{x_1}{2} - \frac{2}{x_1^2 x_2} \quad \text{and} \quad \frac{\partial y}{\partial x_2} = -\frac{2}{x_1 x_2^2} + 4 \quad \rightarrow \quad x_1^* = 2, \quad x_2^* = \frac{1}{2}$$

$$\frac{\partial^2 y}{\partial x_1^2} = \frac{3}{2} \quad \frac{\partial^2 y}{\partial x_2^2} = 16 \quad \frac{\partial^2 y}{\partial x_1 \partial x_2} = 2$$

$$D = \begin{vmatrix} 3/2 & 2 \\ 2 & 16 \end{vmatrix} > 0 \qquad y''_{11} > 0 \quad \rightarrow \quad \text{minimum}$$

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8.10 Sensitivity coefficient

- represents the effect on the optimal value of slightly relaxing the constraints
- variation of optimized objective function y^*

In example 8.3, if the total length of the tube increases from 100m to random value 'H', what would be the increase in minimum cost ?

$$50\pi D^2 L = 100 \rightarrow 50\pi D^2 L = H$$

$$Cost^* = 900 + 1100 D^{2.5} L + 320 DL = 900 + 8.78 H$$

$$SC = \frac{\partial (Cost^*)}{\partial H} = 8.78 = \lambda$$

Extra meter of tube for the heat exchanger would cost additional $8.78