Optimal Design of Energy Systems

Chapter 11 Geometric Programming

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Chapter 11. Geometric Programming

11.1 Introduction

- Sum of polynomials ┬ objective function

  └ constraints

- Find optimum value of the function first

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11.3 Degree of difficulty

$$DOD = T - (N + 1)$$

T: # of terms in objective function + constraints

N: # of variables

ex) $y = 6 - 3x + 10x^{1/2}$

T=2 ($-3x,\ 10x^{1/2}$)
N=1 (x)
DOD=0

$y = 5x_1x_2^{0.5} + 2x_1^2 + x_2^{1/5}$

$x_1x_2 = 50$

T=4 (3 from objective function + 1 from constraint)
N=2
DOD=1

in this chapter DOD=0

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11.4 Solution for one independent variable, unconstrained

$$y = c_1 x^{a_1} + c_2 x^{a_2}$$

$$\underbrace{c_1 x^{a_1}}_{= u_1} \qquad \underbrace{c_2 x^{a_2}}_{= u_2}$$

optimized $\quad y^* = \left(\frac{c_1 x^{a_1}}{w_1}\right)^{w_1} \left(\frac{c_2 x^{a_2}}{w_2}\right)^{w_2}$

$$w_1 + w_2 = 1$$

$$a_1 w_1 + a_2 w_2 = 0$$

(at optimum) $\quad = \left(\frac{c_1}{w_1}\right)^{w_1} \left(\frac{c_2}{w_2}\right)^{w_2}$

$$a_1 w_1 + a_2 (1 - w_1) = 0$$

$$w_1 = \frac{a_2}{a_2 - a_1}, \quad w_2 = \frac{-a_1}{a_2 - a_1}$$

$$y^* = \left(\frac{u_1^*}{w_1}\right)^{w_1} \left(\frac{u_2^*}{w_2}\right)^{w_2}$$

$$y^* = u_1^* + u_2^* = \frac{(u_1^*)^{w_1}}{w_1^{w_1}} \frac{(u_2^*)^{w_2}}{w_2^{w_2}}$$

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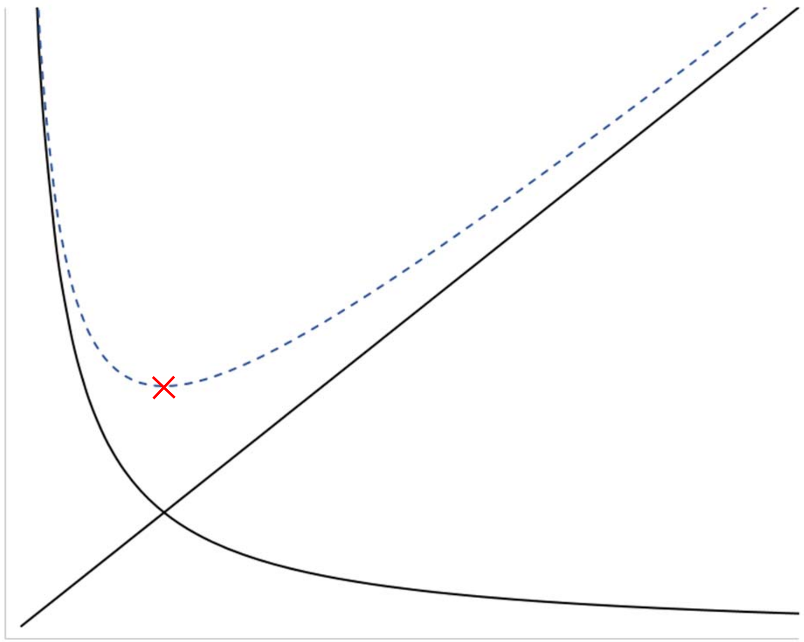
11.4 Solution for one independent variable, unconstrained

Ex) $y = x + x^{-1} \geq 2\sqrt{x \cdot x^{-1}}$



$$w_1 + w_2 = 1$$
$$w_1 - w_2 = 0$$
$$w_1 = w_2 = 1/2$$
$$y^* = (2)^{1/2} \times (2)^{1/2} = 2$$

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11.5 How Geometric Programming works?

$$y = c_1 x^{a_1} + c_2 x^{a_2} = u_1 + u_2$$

$$\begin{cases} g = \left(\dfrac{u_1}{w_1}\right)^{w_1} \left(\dfrac{u_2}{w_2}\right)^{w_2} = \left(\dfrac{c_1 x^{a_1}}{w_1}\right)^{w_1} \left(\dfrac{c_2 x^{a_2}}{w_2}\right)^{w_2} \\ w_1 + w_2 = 1 \end{cases}$$

(maximum of g = maximum of $\ln g$)

optimize $\ln g = w_1(\ln u_1 - \ln w_1) + w_2(\ln u_2 - \ln w_2)$

subject to $\phi = w_1 + w_2 - 1 = 0$

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11.5 How Geometric Programming works?

Lagrange multiplier method $\nabla(\ln g) - \lambda \nabla \phi = 0$

w_1 : $\ln u_1 - \ln w_1 + w_1 \dfrac{-1}{w_1} - \lambda = 0$

w_2 : $\ln u_2 - \ln w_2 + w_2 \dfrac{-1}{w_2} - \lambda = 0$

$$\ln u_1 - \ln u_2 = \ln w_1 - \ln w_2 \rightarrow \frac{u_1}{u_2} = \frac{w_1}{w_2}$$

$$1 = w_1 + w_2$$

$$1 = \frac{u_1}{u_2} w_2 + w_2$$

$$w_2 = \frac{u_2}{u_1 + u_2}, \quad w_1 = \frac{u_1}{u_1 + u_2}$$

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11.5 How Geometric Programming works?

$$g = \left(\frac{u_1}{u_1/(u_1+u_2)}\right)^{\frac{u_1}{u_1+u_2}} \left(\frac{u_2}{u_2/(u_1+u_2)}\right)^{\frac{u_2}{u_1+u_2}} = u_1 + u_2$$

At optimum $\dfrac{dy(x^*)}{dx} = c_1 a_1 x^{*a_1 - 1} + c_2 a_2 x^{*a_2 - 1} = 0$

$$(\times\ x^*) \rightarrow c_1 a_1 x^{*a_1} + c_2 a_2 x^{*a_2} = a_1 u_1^* + a_2 u_2^* = 0$$

$$w_1 = \frac{u_1^*}{u_1^* + \left(-\frac{a_1}{a_2} u_1^*\right)} = \frac{a_2}{a_2 - a_1}$$

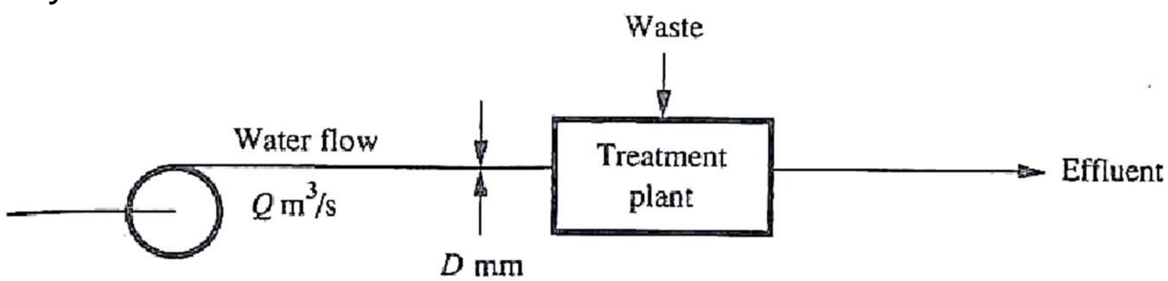
$$w_2 = \frac{a_1}{a_1 - a_2}$$

$$a_1 w_1 + a_2 w_2 = 0$$

$$g^* = \left(\frac{c_1 x^{a_1}}{w_1}\right)^{\frac{a_2}{a_2 - a_1}} \left(\frac{c_2 x^{a_2}}{w_2}\right)^{\frac{a_1}{a_1 - a_2}} = \left(\frac{c_1}{w_1}\right)^{w_1} \left(\frac{c_2}{w_2}\right)^{w_2} = y^*$$

11.7 unconstrained, multivariable optimization

<Example 11.4> The pump and piping of example 11.1 are actually part of a waste-treatment complex, as shown in Fig. 11-1. The system accomplishes the treatment by a combination of dilution and chemical action so that the effluent meets code requirements. The size of the reactor can decrease as the dilution increases. The cost of the reactor is 150/Q, where Q is the flow rate in cubic meters per second. The equation for the pumping cost with Q broken out of the combined constant is $(220 \times 10^{15} Q^2)/D^5$ and the cost of pipe is 160D where D is diameter of pipe in milimiters. Use geometric programming to optimize the total system.



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11.7 unconstrained, multivariable optimization

<solution>

$$DOD = T - (N+1)$$

3 2

$$y = 160D + \frac{220 \times 10^{15} Q^2}{D^5} + \frac{150}{Q}$$

$$y^* = \left(\frac{u_1}{w_1}\right)^{w_1} \left(\frac{u_2}{w_2}\right)^{w_2} \left(\frac{u_3}{w_3}\right)^{w_3}$$

$$w_1 + w_2 + w_3 = 1$$

D: $w_1 - 5w_2 = 0$

Q: $2w_2 - w_3 = 0$

$$w_1 + \frac{w_1}{5} + \frac{2w_1}{5} = 1 \qquad w_1 = \frac{5}{8},\ w_2 = \frac{1}{8},\ w_3 = \frac{2}{8}$$

11.7 unconstrained, multivariable optimization

<solution>

$$y^* = \left(\frac{160}{w_1}\right)^{w_1} \left(\frac{220 \times 10^{15}}{w_2}\right)^{w_2} \left(\frac{150}{w_3}\right)^{w_3} = \$\ 30{,}224$$

$$u_1^* = w_1 y^* = 160 D^* \rightarrow D^* = 118\ mm$$

$$u_2^* = w_2 y^*$$

$$u_3^* = w_3 y^* = 150 / Q^* \rightarrow Q^* = 0.0198\ m^3 / s$$

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11.7 unconstrained, multivariable optimization

<solution>

$\nabla y = 0$ Lagrange Multiplier Method

$$\frac{\partial y}{\partial D} = 160 + \frac{(-5)\times 220\times 10^{15} Q^2}{D^6} = 0 \qquad Q = \sqrt{\frac{160}{5\times 220\times 10^{15}} D^3}$$

$$\frac{\partial y}{\partial Q} = \frac{220\times 10^{15}\times 2Q}{D^5} + \frac{-150}{Q^2} = 0$$

$$= \frac{220\times 10^{15}\times 2\sqrt{\frac{160}{5\times 220\times 10^{15}} D^3}}{D^5} + \frac{-150}{\frac{160}{5\times 220\times 10^{15}} D^6} = 0$$

$$\begin{cases} D = 118 \ mm \\ Q = 0.0198 \ m^3/s \end{cases}$$

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11.8 Constrained optimization with DOD=0

objective function

$$y = u_1 + u_2 + u_3$$

variables

$$x_1, \quad x_2, \quad x_3, \quad x_4$$

$$DOD = T - (N+1) = 0$$

5 4

subject to

$$u_4 + u_5 = 1$$

should be 1

geometric programming

$$c_1 x_1^{a_{11}} x_2^{a_{12}} x_3^{a_{13}} x_4^{a_{14}}$$

$$y = g = \left(\frac{u_1}{w_1}\right)^{w_1} \left(\frac{u_2}{w_2}\right)^{w_2} \left(\frac{u_3}{w_3}\right)^{w_3}$$

$$w_1 + w_2 + w_3 = 1$$

11.8 Constrained optimization with DOD=0

$$w_1 = \frac{u_1}{u_1 + u_2 + u_3} \qquad w_2 = \frac{u_2}{u_1 + u_2 + u_3} \qquad w_3 = \frac{u_3}{u_1 + u_2 + u_3}$$

$$u_4 + u_5 = 1 = \left(\frac{u_4}{w_4}\right)^{w_4} \left(\frac{u_5}{w_5}\right)^{w_5}$$

$$w_4 + w_5 = 1$$

M : arbitrary constant

$$w_4 = \frac{u_4}{u_4 + u_5} = u_4 \qquad w_5 = \frac{u_5}{u_4 + u_5} = u_5$$

$$1 = \left(\frac{u_4}{w_4}\right)^{Mw_4} \left(\frac{u_5}{w_5}\right)^{Mw_5}$$

$$y = g = \left(\frac{u_1}{w_1}\right)^{w_1} \left(\frac{u_2}{w_2}\right)^{w_2} \left(\frac{u_3}{w_3}\right)^{w_3} \left(\frac{u_4}{w_4}\right)^{Mw_4} \left(\frac{u_5}{w_5}\right)^{Mw_5}$$

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11.8 Constrained optimization with DOD=0

$$\nabla(u_1 + u_2 + u_3) - \lambda \nabla(u_4 + u_5) = 0$$

$$u_4 + u_5 = 1$$

$$\begin{cases} a_{11}u_1^* + a_{21}u_2^* + a_{31}u_3^* - \lambda a_{41}u_4^* - \lambda a_{51}u_5^* = 0 \\ a_{12} \ \cdots \\ a_{13} \ \cdots \\ a_{14} \ \cdots \end{cases}$$

divide these equations by y^*

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11.8 Constrained optimization with DOD=0

$$\begin{cases} a_{11}\dfrac{u_1^*}{y^*} + a_{21}\dfrac{u_2^*}{y^*} + a_{31}\dfrac{u_3^*}{y^*} - a_{41}\lambda\dfrac{u_4^*}{y^*} - a_{51}\lambda\dfrac{u_5^*}{y^*} = 0 \\ \qquad\quad w_1 \qquad\quad w_2 \qquad\quad w_3 \qquad\quad Mw_4 \qquad\quad Mw_5 \\ a_{12} \cdots \\ a_{13} \cdots \\ a_{14} \cdots \end{cases}$$

6 unknowns : w_i ,M

$$w_1 + w_2 + w_3 = 1$$

$$Mw_4 + Mw_5 = M$$

$$y^* = \left(\frac{u_1}{w_1}\right)^{w_1}\left(\frac{u_2}{w_2}\right)^{w_2}\left(\frac{u_3}{w_3}\right)^{w_3}\left(\frac{u_4}{w_4}\right)^{Mw_4}\left(\frac{u_5}{w_5}\right)^{Mw_5}$$

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11.8 Constrained optimization with DOD=0

Example 11.5. A water pipeline extends 30 km across a desert from a desalination plant at the seacoast to a city. The pipeline, as shown schematically in Fig. 11-2, conveys 0.16 m^3/s of water. The first costs of the pipeline are

$$\text{Cost of each pump} = 2500 + 0.00032\Delta p^{1.2} \text{ dollars}$$

$$\text{Cost of 30 km of pipe} = 2{,}560{,}000 D^{1.5} \text{ dollars}$$

where Δp = pressure drop in each pipe section, Pa
D = diameter of pipe, m

Assume a friction factor of 0.02.

Use geometric programming on a constrained objective function to select the number of pumps and the pipe diameter that results in the minimum total first cost for the system.

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11.8 Constrained optimization with DOD=0

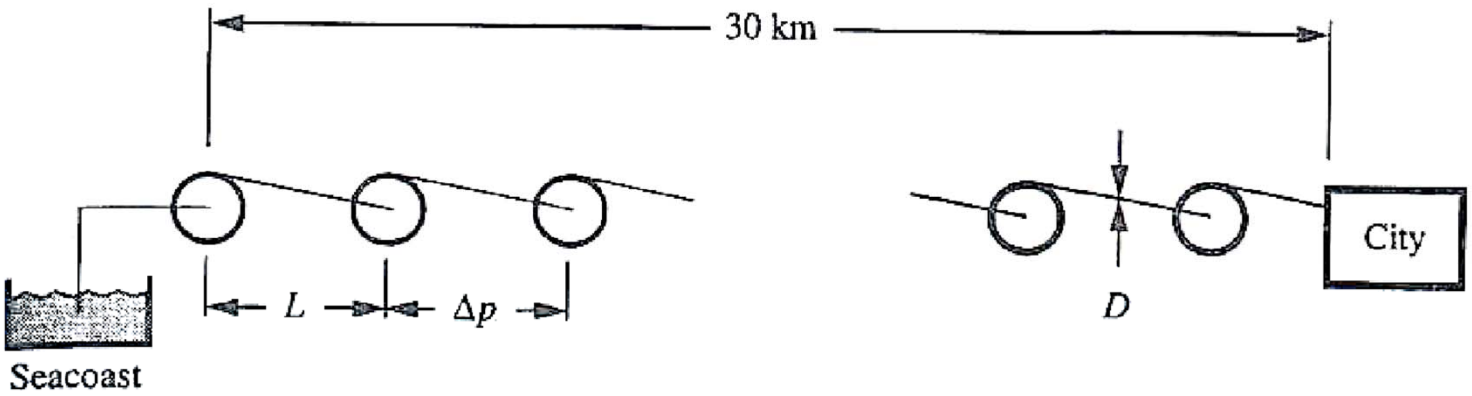

FIGURE 11-2
Water pipeline in Example 11.5.

11.8 Constrained optimization with DOD=0

<solution> total cost y

$$y = n(2500 + 0.00032\Delta p^{1.2}) + 2,560,000D^{1.5}$$

$$n = \frac{30,000\ \ m}{L}$$

$$\Delta p = f\,\frac{L}{D}\frac{V^2}{2}\rho = (0.02)\frac{L}{D}\left(\frac{0.16}{\pi D^2/4}\right)^2\frac{1}{2}(1000\ \ kg/m^3) = 0.4150\frac{L}{D^5}$$

optimize $$y = \frac{75,000,000}{L} + \frac{9.6\Delta p^{1.2}}{L} + 2,560,000D^{1.5}$$

subject to $$\frac{2.410\Delta p D^5}{L} = 1$$

11.8 Constrained optimization with DOD=0

<solution>

$$y^* = \left(\frac{75,000,000}{w_1}\right)^{w_1}\left(\frac{9.6}{w_2}\right)^{w_2}\left(\frac{2,560,000}{w_3}\right)^{w_3}\left(\frac{2.41}{w_4}\right)^{Mw_4}$$

L : $-w_1 - w_2 - Mw_4 = 0$

Δp : $1.2w_2 + Mw_4 = 0$

D : $1.5w_3 + 5Mw_4 = 0$

$$w_1 + w_2 + w_3 = 1$$

$$Mw_4 = M$$

$$w_1 = 0.0385,\ w_2 = 0.1923,\ w_3 = 0.7692,\ M = -0.2308,\ Mw_4 = -0.2308$$

11.8 Constrained optimization with DOD=0

<solution>

$$y^* = \left(\frac{75,000,000}{0.0385}\right)^{0.0385}\left(\frac{9.6}{0.1923}\right)^{0.1923}\left(\frac{2,560,000}{0.7692}\right)^{0.7692}\left(\frac{2.41}{1}\right)^{-0.2308}$$

$$y^* = \$410,150$$

$$u_1^* = (410,150)(0.0385) = \frac{75,000,000}{L}$$

$$L^* = 4750 \ \ m$$

$$u_3^* = (410,150)(0.769) = 2,560,000D^{1.5}$$

$$D^* = 0.246 \ \ m$$

$$\Delta p^* = \frac{L^*}{2.410D^{*5}} = 2,188,000 \ \ Pa = 2188 \ \ kPa$$