Optimal Design of Energy Systems

Chapter 12 Linear Programming

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Chapter 12. Linear Programming

12.1 The origins of Linear Programming

- objective function
- constraints
  - equality or inequality

objective function and constraints: linear combinations

1930s, economic models

1947 USAF simplex method

(United States Air Force)

12.2 Some examples

(1) blending application – oil company

(2) machine allocation – manufacturing plant

(3) inventory and production planning

(4) transportation

Chapter 12. Linear Programming

12.3 Mathematical statement

objective function

$$y = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

constraints

$$\left.\begin{aligned} \phi_1 &= a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \geq r_1 \\ &\vdots \\ \phi_m &= a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \geq r_m \end{aligned}\right\} \text{\underline{inequality} constraints}$$

cf) <u>Lagrange method</u> is applicable for <u>equality</u> constraints

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12.4 Developing the mathematical statement

<Example 12.1> A simple power plant consist of an extraction turbine that drives a generator, as show in Fig. 12-1. The turbine receives 3.2 kg/s of steam, and the plant can sell either electricity or extraction steam for processing purposes. The revenue rates are

Electricity, $0.03 per kilowatthour

Low-pressure steam, $1.10 per megagram

High-pressure steam, $1.65 per megagram

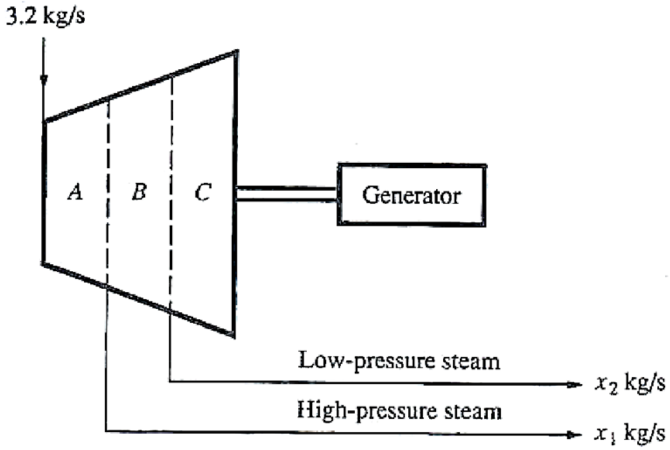

FIGURE 12-1
Power plant in Example 12.1.

12.4 Developing the mathematical statement

<Example 12.1 continued>The generation rate of electric power depends upon the flow rate of steam passing through each of the sections A, B and C; these flow rates are w_A, w_B and w_C, respectively. The relationships are

$$
\begin{aligned}
P_A,\ kW &= 48 w_A \\
P_B,\ kW &= 56 w_B \\
P_C,\ kW &= 80 w_C
\end{aligned}
$$

Where the w's are in kilograms per second. The plant can sell as much electricity as it generates, but there are other restrictions.

12.4 Developing the mathematical statement

<Example 12.1 continued> To Prevent overheating the low-pressure section of the turbine, no less than 0.6 kg/s must always flow through section C. Furthermore, to prevent unequal loading on the shaft, the permissible combination of extraction rates is such that if $x_1 = 0$, then $x_2 \leq 1.8$ kg/s, and for each kilogram of x_1 extracted 0.25 kg less can be extracted of x_2.

The customer of the process steam is primarily interested in total energy and will purchase no more than

$$4x_1 + 3x_2 \leq 9.6$$

Develop the objective function for the total revenue from the plant and also the constraint equations

12.4 Developing the mathematical statement

<Solution>

Revenue $= \frac{1.65}{1000}(3600x_1) + \frac{1.10}{1000}(3600x_2) + 0.03(48w_A + 56w_B + 80w_C)$

$\left(w_A = 3.2 \ \ kg / s, \ \ w_B = 3.2 - x_1, \ \ w_C = 3.2 - x_1 - x_2\right)$

$= 17.66 + 1.86x_1 + 1.56x_2$

Maximize $y = 1.86x_1 + 1.56x_2$

constraints $x_1 + x_2 \leq 2.6$

$x_1 + 4x_2 \leq 7.2$

$4x_1 + 3x_2 \leq 9.6$

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12.5 Geometric Visualization of the Linear-Programming Problem

<Solution>

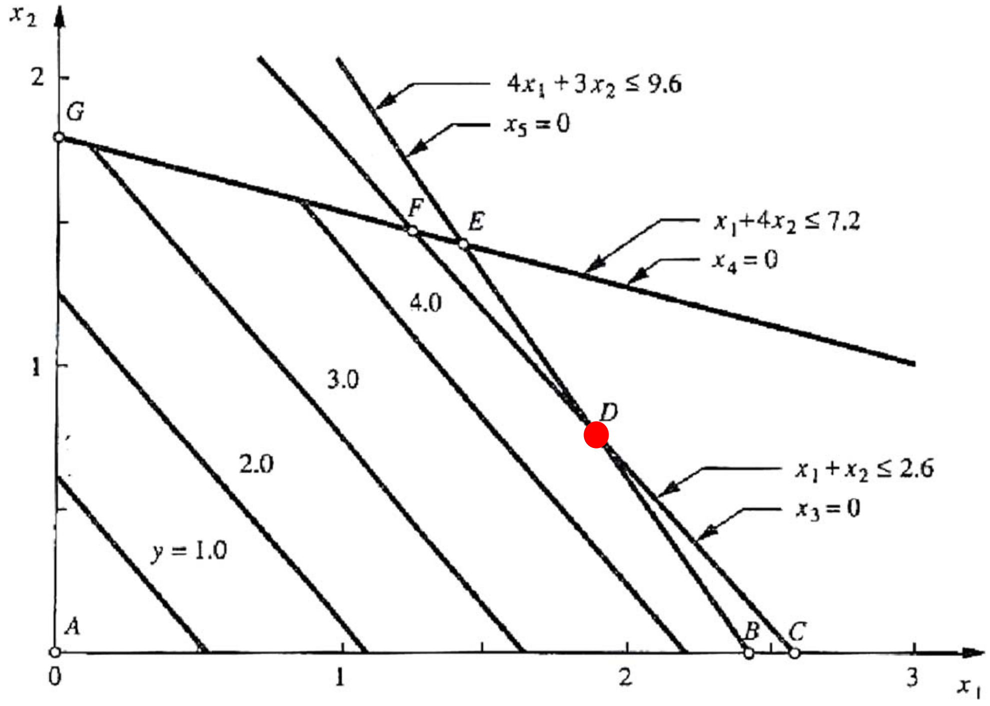

FIGURE 12-2
Constraints and lines of constant profit in Example 12.1.

- Permitted region : ABDFG
- Optimal point : D
- Optimum solution lies at a corner

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12.6 Introduction of Slack Variables

From Ex 12.1 inequalities can be converted into equalities by introduction of another variable in each equation.

$$\begin{aligned} x_1 + x_2 + x_3 &= 2.6 & \qquad x_3 &\geq 0 \\ x_1 + 4x_2 + x_4 &= 7.2 & \qquad x_4 &\geq 0 \\ 4x_1 + 3x_2 + x_5 &= 9.6 & \qquad x_5 &\geq 0 \end{aligned}$$

slack variables : $x_3, \ x_4, \ x_5$

12.7 Preparation for simplex algorithm

objective function : $y - 1.86x_1 - 1.56x_2 = 0$

	x_1	x_2	x_3	x_4	x_5	
	1	1	1			2.6
	1	4		1		7.2
	4	3			1	9.6
	-1.86	-1.56				0

current value of objective function

12.9 Starting at the origin

Move from one corner to the next corner starting point $x_1=0$, $x_2=0$

	$x_1=0$	$x_2=0$	x_3	x_4	x_5	
	1	1	1			2.6
	1	4		1		7.2
	4	3			1	9.6
	-1.86	-1.56				0

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12.10 The simplex algorithm

1. Decide the variable

   Maximization – largest negative difference coefficient
   Minimization – largest positive difference coefficient

2. Determine the controlling constraint

3. Transfer of the controlling constraint

4. For all other boxes

   New value = $v - wz$

   old value

   coefficient of the variable being programmed

   value in the preview controlling equation

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12.11 Solution of Example 12.1

Table 1	$x_1=0$	$x_2=0$	x_3	x_4	x_5	
2.6/1=2.6	1	1	1			2.6
7.2/1=7.2	1	4		1		7.2
9.6/4=2.4	4	3			1	9.6
	-1.86	-1.56				0

controlling constraint
(smallest)

Step 1 : largest negative

x_1 should be programmed
(increased from zero)

Step 2 : How much x_1 can be increased?

$$x_1 = 0 \quad \rightarrow \quad x_1 \neq 0$$

$$x_2 = 0 \quad \rightarrow \quad x_2 = 0$$

$$x_5 \neq 0 \quad \rightarrow \quad x_5 = 0 \quad \Leftarrow \quad x_1 \text{ increases until } x_5 \text{ becomes zero.}$$

(In fig 12-2, A→B)

12.11 Solution of Example 12.1

Step 3 :

	x_1	$x_2=0$	x_3	x_4	$x_5=0$	
	1-(1)(1)	1-(1)(0.75)	1-(1)(0)	0-(1)(1)	0-(1)(0.25)	2.6-(1)(2.4)
	1-(1)(1)	4-(1)(0.75)	0-(1)(0)	1-(1)(0)	0-(1)(0.25)	7.2-(1)(2.4)
÷ 4	1	0.75	0	0	0.25	2.4
	-1.86-(-1.86)(1)	-1.56-(-1.86)(0.75)	0-(-1.86)(0)	0-(-1.86)(0)	0-(-1.86)(0.25)	0-(-1.86)(2.4)

Step 4 :

12.11 Solution of Example 12.1

Table 2	x_1	$x_2=0$	x_3	x_4	$x_5=0$	
0.2/0.25=0.8	0	0.25	1	0	-0.25	0.20
4.8/3.25=1.48	0	3.25	0	1	-0.25	4.8
2.4/0.75=3.2	1	0.75	0	0	0.25	2.4
	0	-0.165	0	0	0.465	4.464

controlling constraint
(smallest)

Step 1 : largest negative (x$_2$ is programmed next)

$$\rightarrow x_3 = 0.2,\ \ x_4 = 4.8,\ \ x_1 = 2.4,\ \ y = 4.464$$

12.11 Solution of Example 12.1

Step 2 : x_2 increases to its limit until x_3 becomes zero

Step 3 :

	x_1	x_2	$x_3=0$	x_4	$x_5=0$	
÷0.25	0	1	4	0	-1	0.8
	0-(3.25)(0)	3.25-(3.25)(1)	0-(3.25)(4)	1-(3.25)(0)	-0.25-(3.25)(-1)	4.8-(3.25)(0.8)
	1-(0.75)(0)	0.75-(0.75)(1)	0-(0.75)(4)	0-(0.75)(0)	0.25-(0.75)(-1)	2.4-(0.75)(0.8)
	0-(-0.165)(0)	-0.165-(-0.165)(1)	0-(-0.165)(4)	0-(-0.165)(0)	0.465-(-0.165)(-1)	4.464-(-0.165)(0.8)

Step 4 :

12.11 Solution of Example 12.1

Table 3	x_1	x_2	$x_3=0$	x_4	$x_5=0$	
	0	1	4	0	-1	0.80
	0	0	-13	1	3	2.2
	1	0	-3	0	1	1.8
	0	0	0.66	0	0.3	4.596

$$\therefore\ x_2 = 0.8,\quad x_1 = 1.8,\quad x_4 = 2.2,\quad y = 4.596$$

→ no negative coefficients

→ no further improvement is possible (second constraint has no influence)

Chapter 12. Linear Programming

12.12 Another Geometric Interpretation of Table Transformation

└ by changing the coordinates so that the current point is always at origin

$$y = 1.86x_1 + 1.56x_2$$

$$\begin{cases} x_1 + x_2 + x_3 = 2.6 \\ x_1 + 4x_2 + x_4 = 7.2 \\ 4x_1 + 3x_2 + x_5 = 9.6 \end{cases}$$

Table 1	$x_1, \ x_2$	- physical variables	$x_1 = 0, \ x_2 = 0$	origin
Table 2	$x_2, \ x_5$		$x_2 = 0, \ x_5 = 0$	
Table 3	$x_3, \ x_5$		$x_3 = 0, \ x_5 = 0$	

12.12 Another Geometric Interpretation of Table Transformation

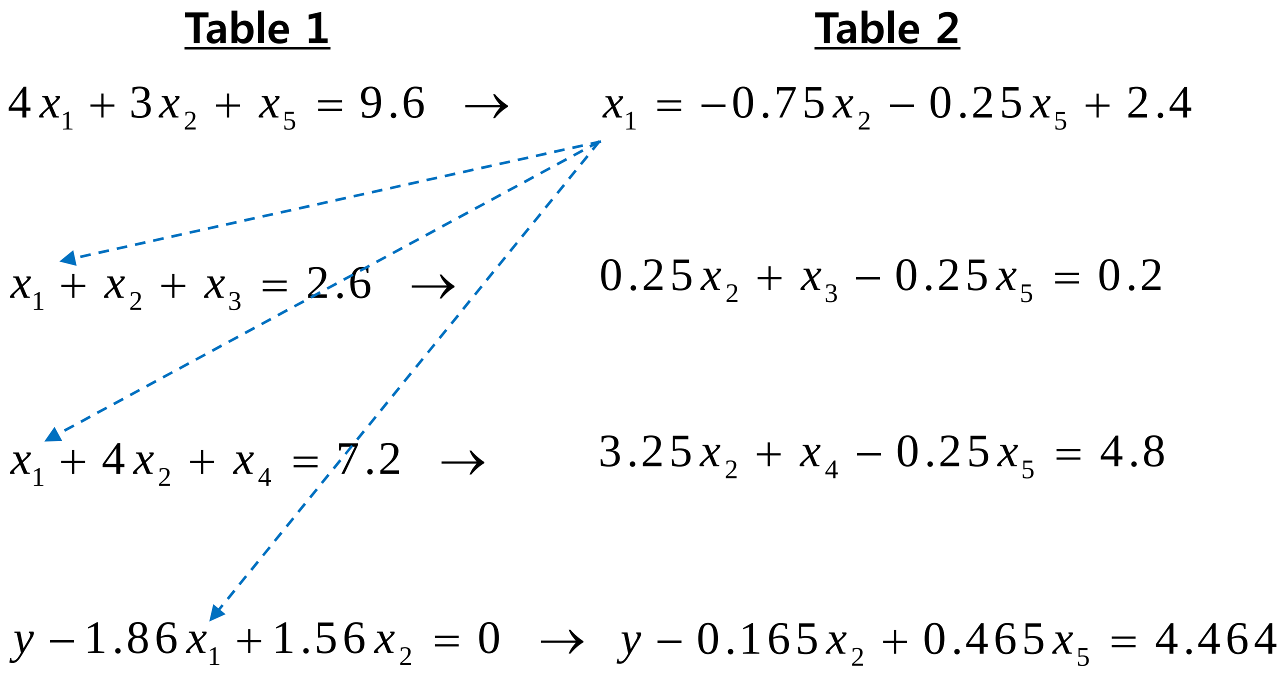

	Table 1		**Table 2**
3rd constraint	$4x_1 + 3x_2 + x_5 = 9.6$	$\rightarrow$	$x_1 = -0.75x_2 - 0.25x_5 + 2.4$
1st constraint	$x_1 + x_2 + x_3 = 2.6$	$\rightarrow$	$0.25x_2 + x_3 - 0.25x_5 = 0.2$
2nd constraint	$x_1 + 4x_2 + x_4 = 7.2$	$\rightarrow$	$3.25x_2 + x_4 - 0.25x_5 = 4.8$
Objective function	$y - 1.86x_1 + 1.56x_2 = 0$	$\rightarrow$	$y - 0.165x_2 + 0.465x_5 = 4.464$

12.12 Another Geometric Interpretation of Table Transformation

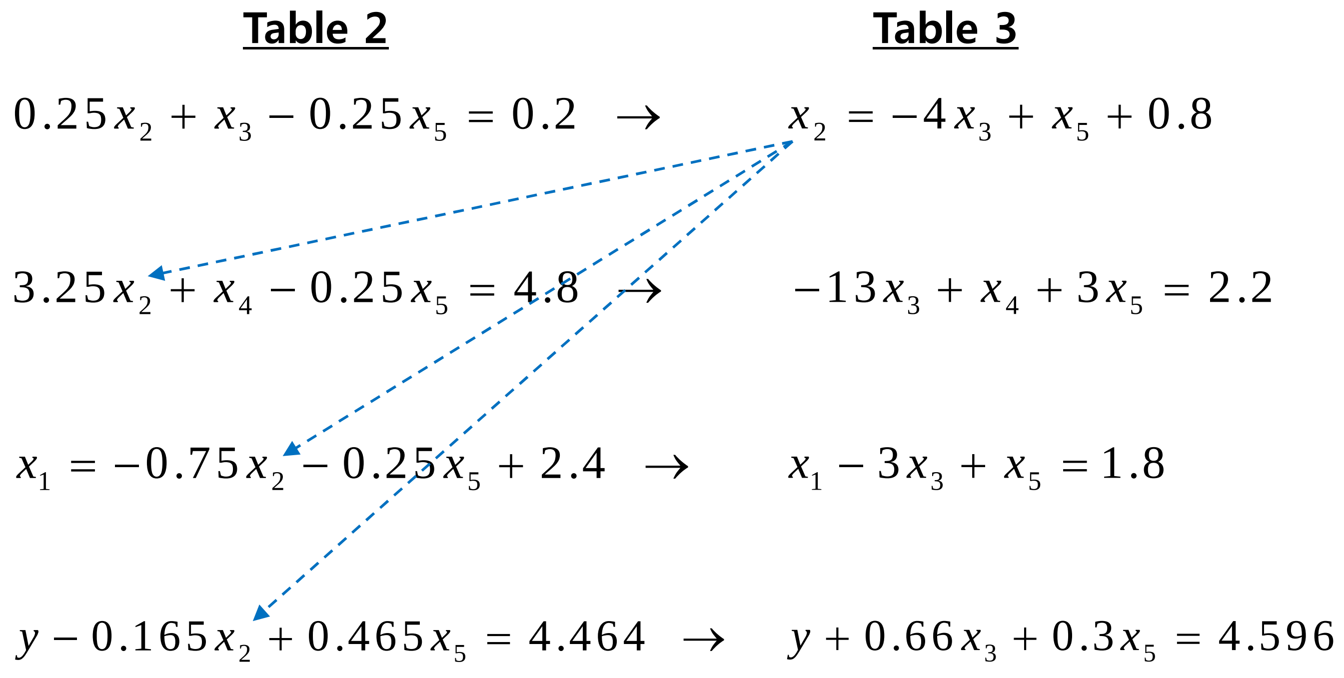

	Table 2		**Table 3**
1st constraint	$0.25x_2 + x_3 - 0.25x_5 = 0.2$	→	$x_2 = -4x_3 + x_5 + 0.8$
2nd constraint	$3.25x_2 + x_4 - 0.25x_5 = 4.8$	→	$-13x_3 + x_4 + 3x_5 = 2.2$
3rd constraint	$x_1 = -0.75x_2 - 0.25x_5 + 2.4$	→	$x_1 - 3x_3 + x_5 = 1.8$
Objective function	$y - 0.165x_2 + 0.465x_5 = 4.464$	→	$y + 0.66x_3 + 0.3x_5 = 4.596$

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12.12 Another Geometric Interpretation of Table Transformation

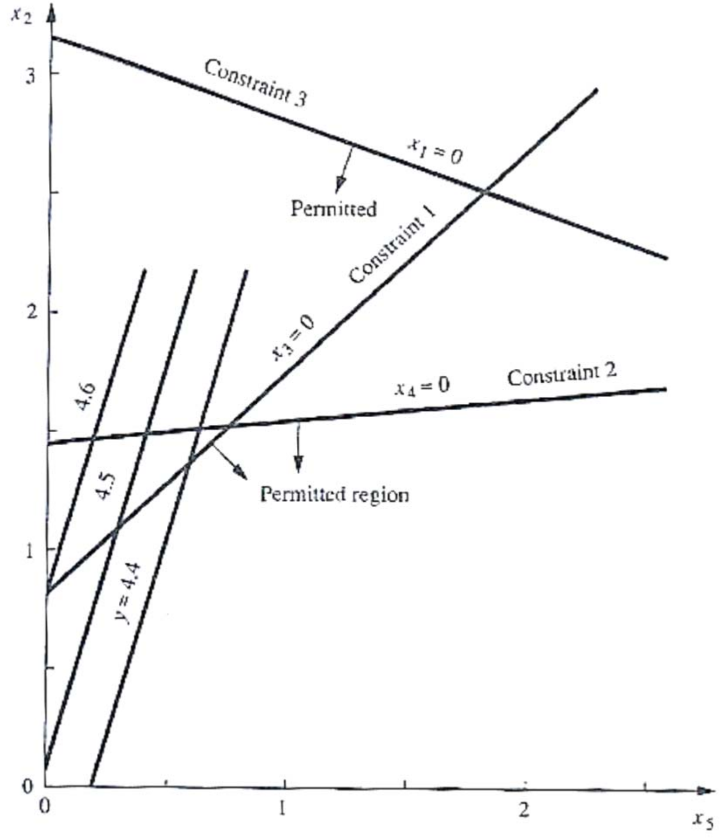

FIGURE 12-3
Tableau 2 expressed on x_5x_2 coordinates.

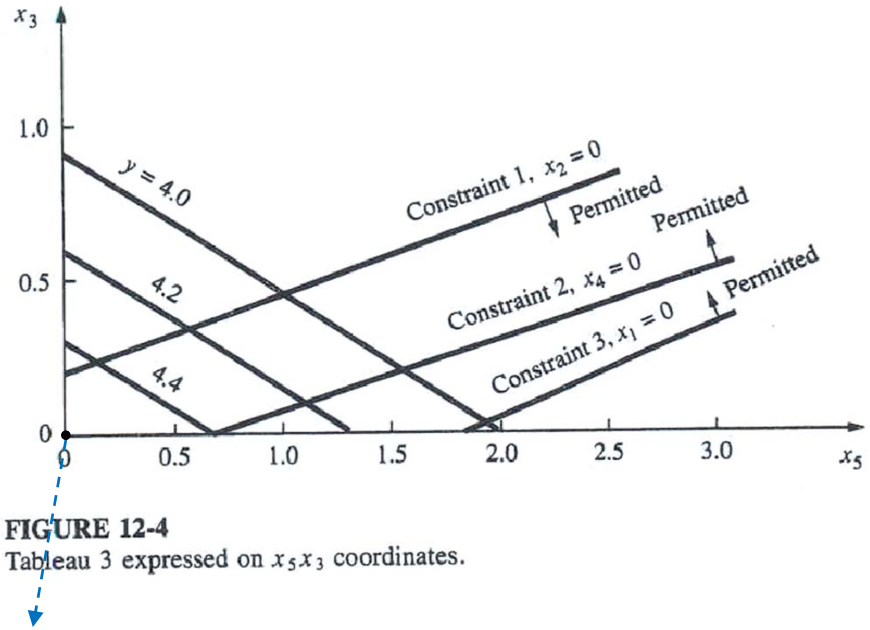

FIGURE 12-4
Tableau 3 expressed on x_5x_3 coordinates.

$y = 4.596 \ at \ x_3 = 0, \ x_5 = 0$

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12.14 # of variables and # of constraints

n (m = # of slack variables)

At optimum, n variables are zero (corner)

m>n m-n constraints play no role

m<n n-m variables are zero

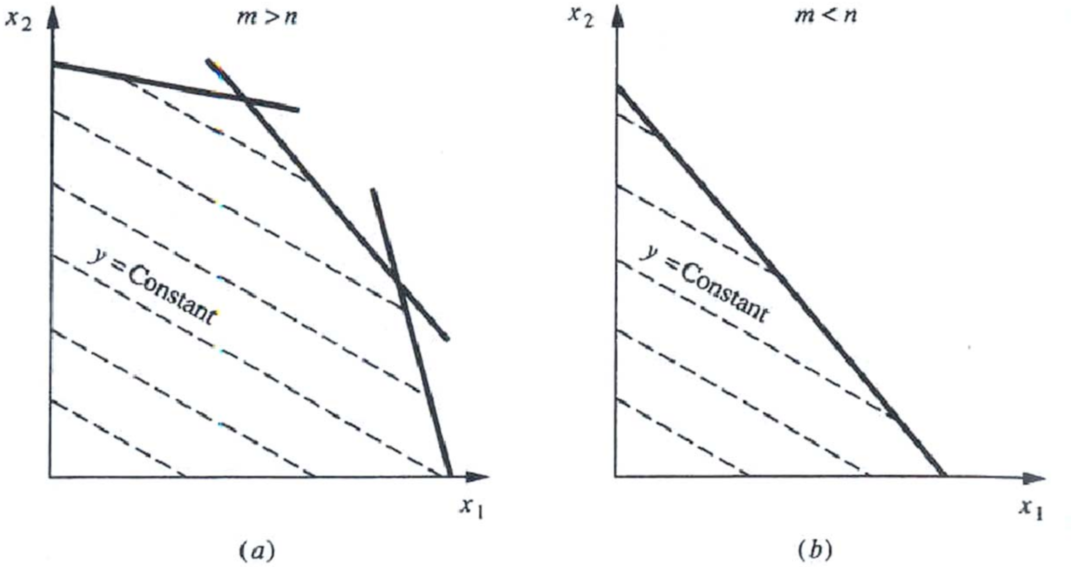

FIGURE 12-5
Relation of number of physical and slack variables.

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12.15 Minimization with greater than constraints

- ✓ Maximization with less than constraints
  - Moving from one corner to another adjacent corner (start from the origin)
- ✓ Minimization with greater than constraints
  - Locating the first feasible point - difficult

    → introduction of artificial variable (12.16)

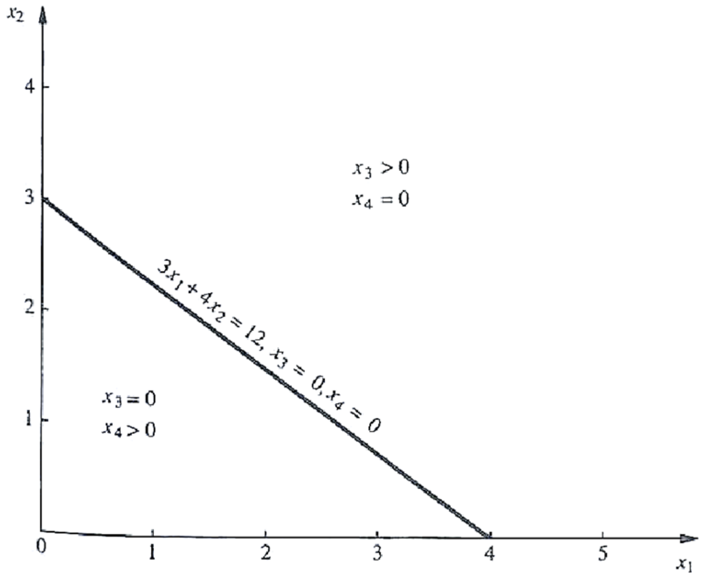
12.16 Artificial variables

$$3x_1 + 4x_2 \geq 12$$

$$3x_1 + 4x_2 - x_3 = 12 \qquad\qquad x_3 \geq 0$$

If $x_1 = x_2 = 0$ (origin), it is not realistic

$$3x_1 + 4x_2 - \underset{\substack{\uparrow \\ \text{slack variable}}}{x_3} + \underset{\substack{\uparrow \\ \text{artificial variable}}}{x_4} = 12$$



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12.17 Simplex algorithm to minimization problem

<Example 12.2> Determine the minimum value of y and the magnitudes of x1 and x2 at this minimum, where

$$y = 6x_1 + 3x_2$$

Subject to the constraints

$$5x_1 + x_2 \geq 10$$

$$9x_1 + 13x_2 \geq 74$$

$$x_1 + 3x_2 \geq 9$$

Chapter 12. Linear Programming

12.17 Simplex algorithm to minimization problem

<solution>

$$\begin{cases} 5x_1 + x_2 - x_3 + x_6 = 10 \\ 9x_1 + 13x_2 - x_4 + x_7 = 74 \\ x_1 + 3x_2 - x_5 + x_8 = 9 \end{cases}$$

$$y = 6x_1 + 3x_2 + Px_6 + Px_7 + Px_8$$

P → a numerical value which is extremely large

$x_3, \ x_4, \ x_5 \ \rightarrow$ slack variables

$x_6, \ x_7, \ x_8 \ \rightarrow$ artificial variables

12.17 Simplex algorithm to minimization problem

<solution>

Starting point-origin with all slack variable=0

artificial variable>0

$$x_6 = 10 - 5x_1 - x_2 + x_3$$

$$x_7 = 74 - 9x_1 - 13x_2 + x_4$$

$$x_8 = 9 - x_1 - 3x_2 + x_5$$

$$y = (6 - 15P)x_1 + (3 - 17P)x_2 + Px_3 + Px_4 + Px_5 + 93P$$

Chapter 12. Linear Programming

12.17 Simplex algorithm to minimization problem

<solution>

Table 1		x_1	x_2	x_3	x_4	x_5	
10	x_6	5	1	-1	0	0	10
74/13	x_7	9	13	0	-1	0	74
3	x_8	1	3	0	0	-1	9
		15P-6	17P-3	-P	-P	-P	93P

controlling constraint
(smallest)

largest positive
difference coefficient

Chapter 12. Linear Programming

12.17 Simplex algorithm to minimization problem

<solution>

Table 1 **Table 2**

$$x_6 = 10 - 5x_1 - x_2 + x_3 \quad \rightarrow \quad x_6 = 7 - (14/3)x_1 + x_3 - (1/3)x_5 + (1/3)x_8$$

$$x_7 = 74 - 9x_1 - 13x_2 + x_4 \quad \rightarrow \quad x_7 = 35 - (14/3)x_1 + x_4 - (13/3)x_5 + (13/3)x_8$$

$$x_8 = 9 - x_1 - 3x_2 + x_5 \quad \rightarrow \quad x_2 = 3 - (1/3)x_1 + (1/3)x_5 - (1/3)x_8$$

$$y = (6 - 15P)x_1 + (3 - 17P)x_2 + Px_3 + Px_4 + Px_5 + 93P$$

$$\rightarrow \quad y = \frac{15 - 28P}{3}x_1 + Px_3 + Px_4 + \frac{3 - 14P}{3}x_5 + \frac{-3 + 17P}{3}x_8 + 42P + 9$$

Chapter 12. Linear Programming

12.17 Simplex algorithm to minimization problem

<solution>

Table 2		x_1	x_3	x_4	x_5	x_8	
3/2	x_6	14/3	-1	0	1/3	-1/3	7
105/14	x_7	14/3	0	-1	13/3	-13/3	35
9	x_2	1/3	0	0	-1/3	1/3	3
		(28P-15)/3	-P	-P	(14P-3)/3	(-17P+3)/3	42P+9

controlling constraint
(smallest)

largest positive
difference coefficient

Chapter 12. Linear Programming

12.17 Simplex algorithm to minimization problem

<solution>

Table 2 **Table 3**

$$x_6 = 7 - (14/3)x_1 + x_3 - (1/3)x_5 + (1/3)x_8 \rightarrow x_1 = 3/2 + (3/14)x_3 - (1/14)x_5 - (3/14)x_6 + (1/14)x_8$$

$$x_7 = 35 - (14/3)x_1 + x_4 - (13/3)x_5 + (13/3)x_8 \rightarrow x_7 = 28 - x_3 + x_4 - 4x_5 + x_6 + 4x_8$$

$$x_2 = 3 - (1/3)x_1 + (1/3)x_5 - (1/3)x_8 \rightarrow x_2 = 5/2 - (1/14)x_3 + (5/14)x_5 + (1/14)x_6 - (5/14)x_8$$

$$y = \frac{15 - 28P}{3}x_1 + Px_3 + Px_4 + \frac{3 - 14P}{3}x_5 + \frac{-3 + 17P}{3}x_5 + 42P + 9$$

$$\rightarrow y = \frac{15 - 14P}{14}x_3 + Px_4 + \frac{9 - 56P}{14}x_5 + \frac{-15 + 28P}{14}x_6 + \frac{-9 + 70P}{14}x_8 + \frac{56P + 33}{2}$$

12.17 Simplex algorithm to minimization problem

<solution>

Table 3		x_3	x_4	x_5	x_6	x_8	
21	x_1	-3/14	0	1/14	3/14	-1/14	3/2
7	x_7	1	-1	4	-1	-4	28
-7	x_2	1/14	0	-5/14	-1/14	5/14	5/2
		(14P-15) /14	-P	(56P-9)/14	(-28P+15) /14	(-70P+9) /14	(56P+33)/ 2

controlling constraint
(smallest)

largest positive
difference coefficient

Chapter 12. Linear Programming

12.17 Simplex algorithm to minimization problem

<solution>

Table 3 **Table 4**

$$x_1 = 3/2 + (3/14)x_3 - (1/14)x_5 - (3/14)x_6 + (1/14)x_8 \;\rightarrow\; x_1 = 1 + (13/56)x_3 - (1/56)x_4 - (13/56)x_6 + (1/56)x_7$$

$$x_7 = 28 - x_3 + x_4 - 4x_5 + x_6 + 4x_8 \;\rightarrow\; x_5 = 7 - (1/4)x_3 + (1/4)x_4 + (1/4)x_6 - (1/4)x_7 + x_8$$

$$x_2 = 5/2 - (1/14)x_3 + (5/14)x_5 + (1/14)x_6 - (5/14)x_8 \;\rightarrow\; x_2 = 5 - (9/56)x_3 + (5/56)x_4 + (9/56)x_6 - (5/56)x_7$$

$$y = \frac{15-14P}{14}x_3 + Px_4 + \frac{9-56P}{14}x_5 + \frac{-15+28P}{14}x_6 + \frac{-9+70P}{14}x_8 + \frac{56P+33}{2}$$

$$\rightarrow\; y = \frac{51}{56}x_3 + \frac{9}{56}x_4 + \left(P - \frac{15}{56}\right)x_6 + \left(P - \frac{9}{56}\right)x_7 + Px_8 + 21$$

12.17 Simplex algorithm to minimization problem

<solution>

Table 4	x_3	x_4	x_6	x_7	x_8	
x_1	-13/56	1/56	13/56	-1/56	0	1
x_5	1/4	-1/4	-1/4	¼	-1	7
x_2	9/56	-5/56	-9/56	5/56	0	5
	-51/56	-9/56	-P+51/56	-P+9/56	-P	21

$$\therefore\ x_1 = 1,\ \ x_5 = 7,\ \ x_2 = 5,\ \ y = 21$$

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12.17 Simplex algorithm to minimization problem

<solution>

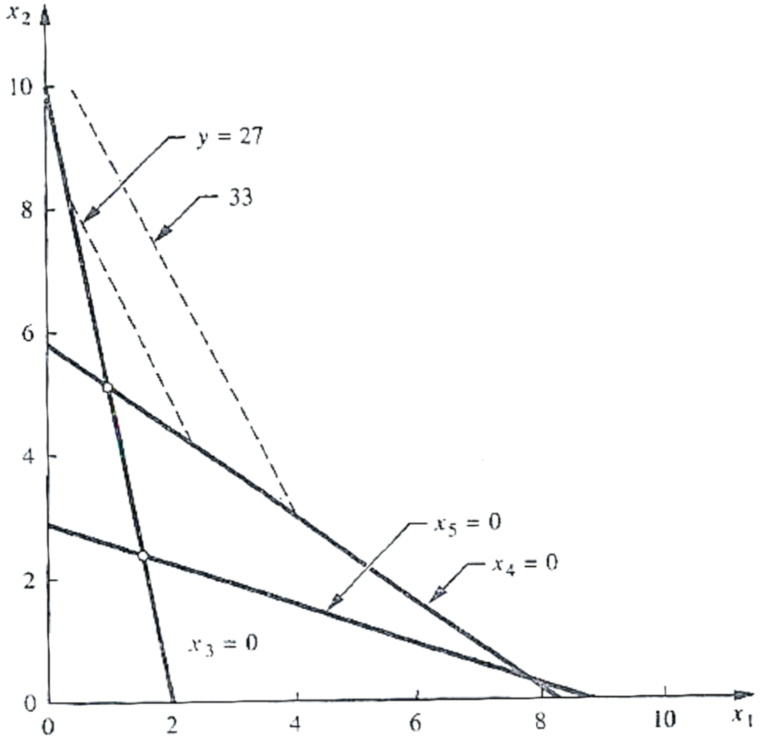

FIGURE 12-7
Minimization in Example 12.2.

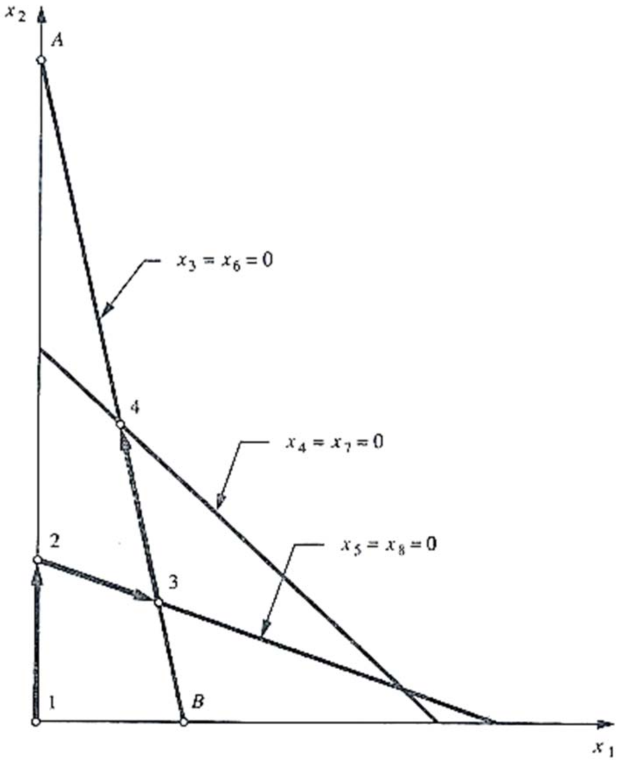

FIGURE 12-8
Points represented by successive tableaux in Example 12.2.