3 General forced response

- So far, all of the driving forces have been sine or cosine excitations
- In this chapter we examine the response to any form of excitation such as
  - Impulse
  - Sums of sines and cosines
  - Any integrable function

Linear Superposition allows us to break up complicated forces into sums of simpler forces, compute the response and add to get the total solution

If x_1, x_2 are solutions of a linear homogeneous equation, then

$$x = a_1 x_1 + a_2 x_2 \text{ is also a solution.}$$

If x_1 is the particular sol of $\ddot{x} + \omega_n^2 x = f_1$

and x_2 the particular sol of $\ddot{x} + \omega_n^2 x = f_2$

$$\Rightarrow ax_1 + bx_2 \text{ solves } \ddot{x} + \omega_n^2 x = af_1 + bf_2$$

3.1 Impulse Response Function

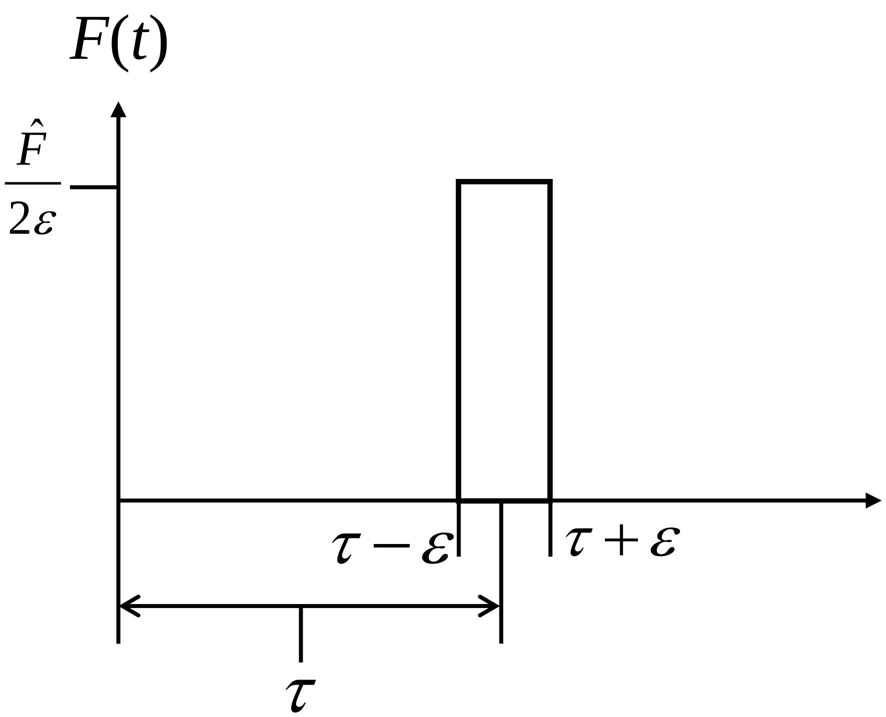


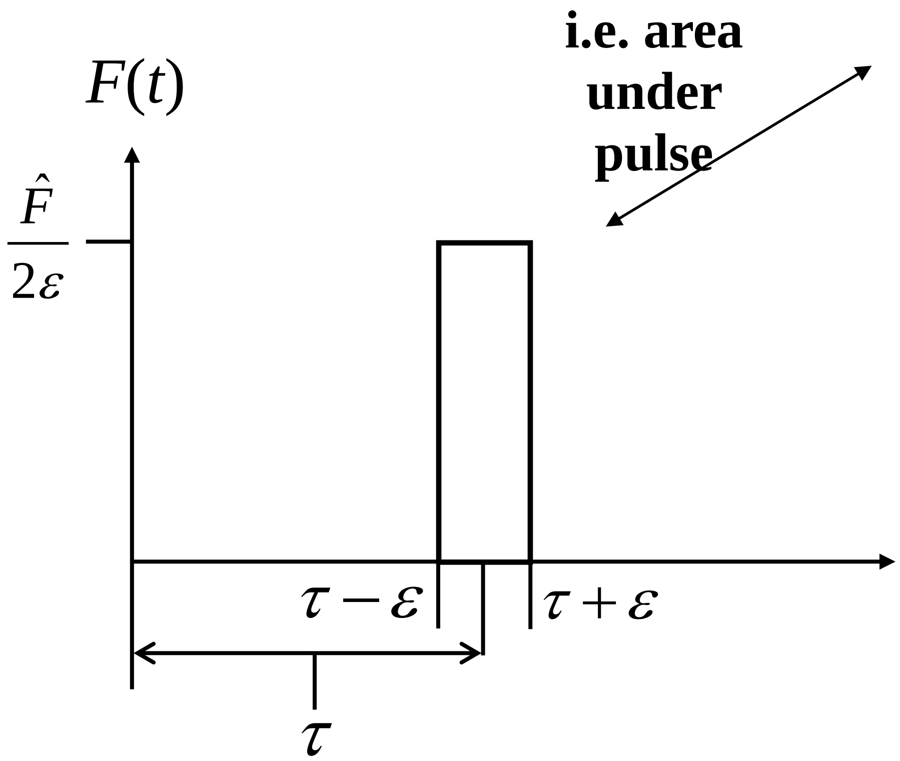
Figure 3.1

Impulse excitation

$$F(t)=\begin{cases} 0 & t<\tau-\varepsilon \\ \dfrac{\hat{F}}{2\varepsilon} & \tau-\varepsilon<t<\tau+\varepsilon \\ 0 & t>\tau+\varepsilon \end{cases}$$

ε is a small positive number

From sophomore dynamics **The impulse imparted to an object is equal to the change in the objects momentum**

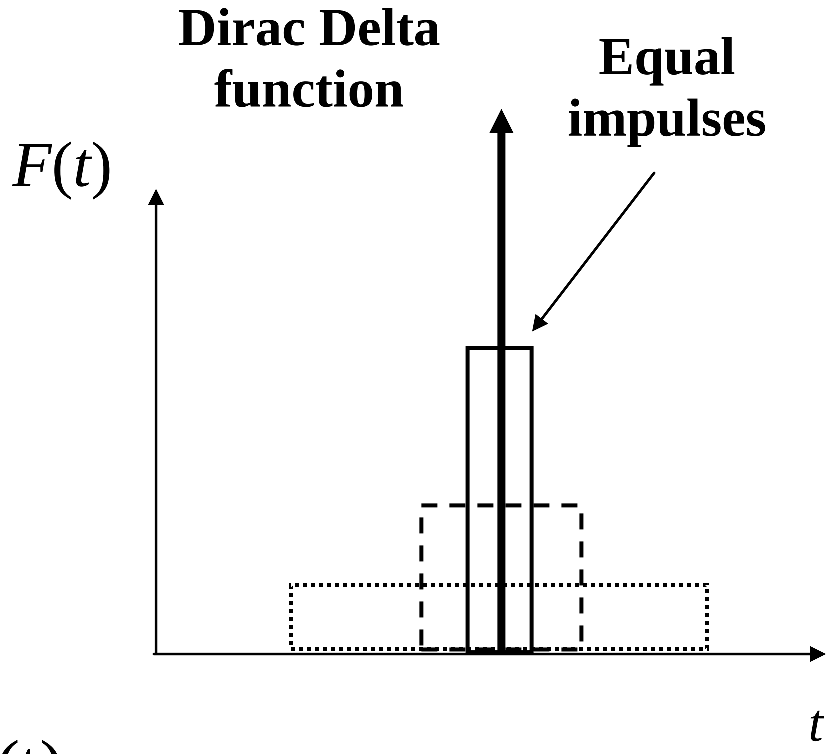


$$\text{impulse force } = \int F(t)dt = F\Delta t$$

$$I(\varepsilon) = \int_{\tau-\varepsilon}^{\tau+\varepsilon} F(t)dt = \int_{-\infty}^{\infty} F(t)dt \ \mathrm{N\cdot s}$$

$$= \frac{\hat{F}}{2\varepsilon}2\varepsilon = \hat{F}$$

We use the properties of impulse to define the impulse function:



$$F(t-\tau)=0, \quad t \neq \tau$$

$$\int_{-\infty}^{\infty} F(t-\tau)dt = \hat{F}$$

If $\hat{F} = 1$, this is the Dirac Delta δ(t)

The effect of an impulse on a spring-mass-damper is related to its change in momentum.

Just after impulse

Just before impulse

impulse=momentum change

$$\overbrace{F\Delta t = \Delta mv}^{\text{impulse=momentum change}} = m[v(t_0^+) - v(t_0^-)]$$

$$\hat{F} = mv_0 \Rightarrow v_0 = \frac{\hat{F}}{m} = \frac{F\Delta t}{m}$$

Thus the *response to impulse* with zero IC is equal to the *free response* with IC: x_0=0 and $v_0 = F\Delta t/m$

Recall that the free response to just non zero initial conditions is:

The solution of:

$$m\ddot{x}+c\dot{x}+kx=0 \quad x(0)=x_0 \quad \dot{x}(0)=v_0$$

in underdamped case:

$$x(t)=\frac{\sqrt{\left(v_0+\zeta\omega_n x_0\right)^2+\left(x_0\omega_d\right)^2}}{\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t+\tan^{-1}\frac{x_0\omega_d}{v_0+\zeta\omega_n x_0})$$

For $x_0=0$ this becomes:

$$x(t)=\frac{v_0 e^{-\zeta\omega_n t}}{\omega_d}\sin\omega_d t$$

Next compute the response to *x*(0)=0 and *v*(0) =*FΔt*/*m*

The solution of:

$$m\ddot{x} + c\dot{x} + kx = 0 \quad x(0) = x_0 \quad \dot{x}(0) = F\Delta t / m = \frac{\hat{F}}{m}$$

in underdamped case from the previous slide is:

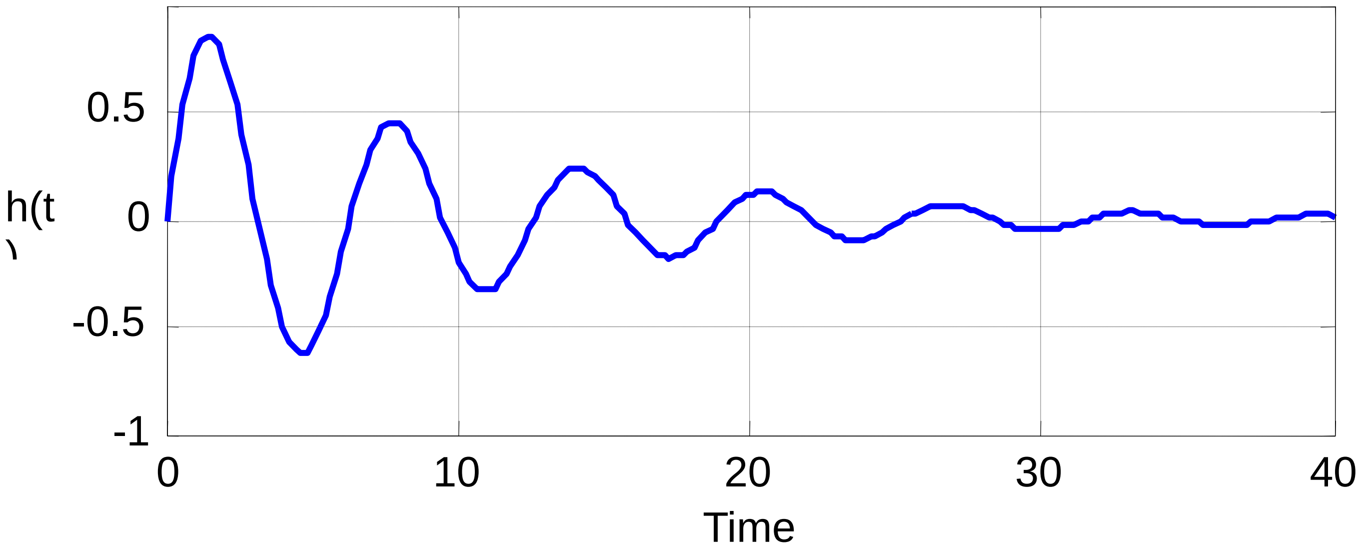
$$x(t) = \frac{\hat{F}e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t$$

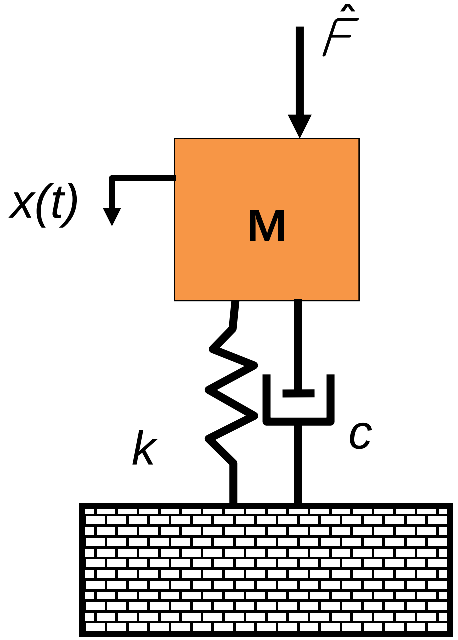
Response to an impulse at *t* = 0, and zero initial conditions

So for an underdamped system the impulse response is (x_0 = 0)

$$x(t) = \frac{\hat{F}e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t \quad \text{(response to } \hat{F}\text{)} \qquad (3.6)$$

$$x(t) = \hat{F}h(t), \text{ where } \underbrace{h(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t}_{\text{unit impulse response function}} \qquad (3.8)$$





The response to an impulse is thus defined in terms of the impulse response function, *h*(*t*).

So, the response to $\delta(t)$ is given by $h(t)$.

$$h(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_d} \sin \omega_d t \quad (3.8)$$

What is the response to a unit impulse applied at a time different from zero?

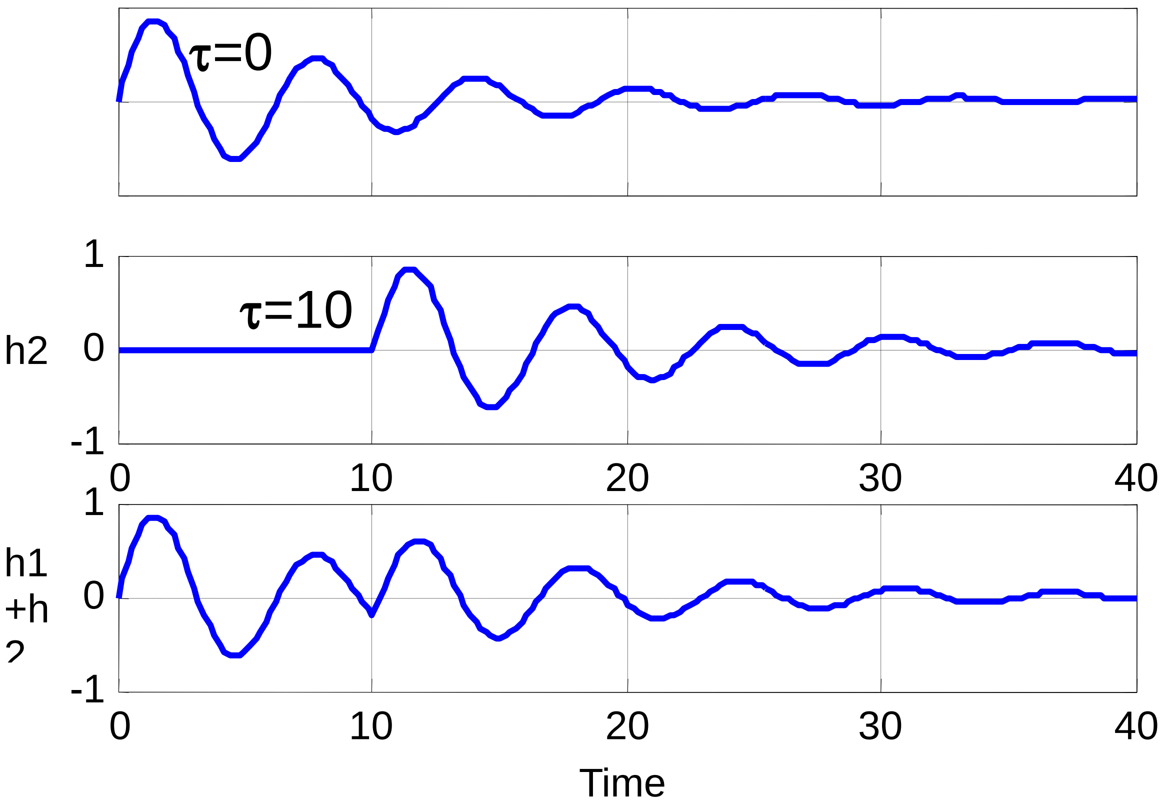
The response to $\delta(t\text{-}\tau)$ is $h(t\text{-}\tau)$.

This is given on the following slide

$$h(t-\tau)=\begin{cases} 0 & t<\tau \\ \dfrac{e^{-\zeta\omega_n(t-\tau)}}{m\omega_d}\sin\omega_d(t-\tau) & t>\tau \end{cases}$$

for the case that the impulse occurs at τ note that the effects of non-zero initial conditions and other forcing terms must be super imposed on this solution (see Equation (3.9))

For example: If two pulses occur at two different times then their impulse responses will superimpose



Consider the undamped impulse response

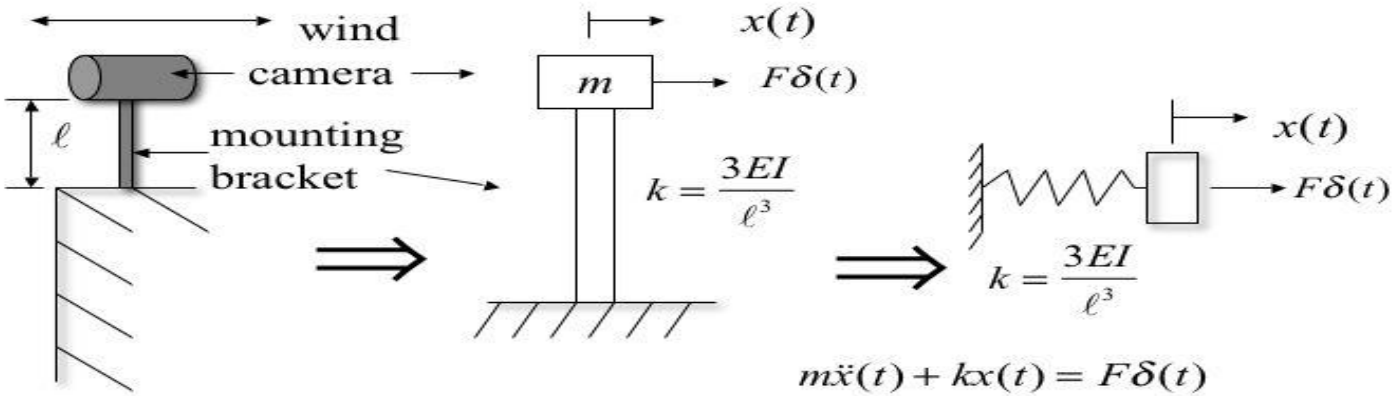
Setting $\zeta = 0$ in the equation (3.8)

Response to unit impulse applied at $t = \tau$,

i.e. $\delta(t\text{-}\tau)$ is:

$$h(t-\tau) = \frac{1}{m\omega_n} \sin \omega_n (t-\tau)$$

Example 3.1.2 Design a camera mount with a vibration constraint



Consider example 2.1.3 of the security camera again only this time with an impulsive load

Using the stiffness and mass parameters of Example 2.1.3, does the system stay with in vibration limits if hit by a 1 kg bird traveling at 72 kmh?

The natural frequency of the camera system is

$$\omega_n = \sqrt{\frac{k}{m_c}} = \sqrt{\frac{3Ebh^3}{12m_c\ell^3}}$$

$$= \sqrt{\frac{(7.1 \text{ x } 10^{10} \text{ N/m})(0.02 \text{ m})(0.02 \text{ m})^3}{4(3 \text{ kg})(0.55)^3}} = 75.43 \text{ rad/s}$$

From equations (3.7) and (3.8) with ζ = 0, the impulsive response is:

$$x(t) = \frac{F\Delta t}{m_c\omega_n}\sin\omega_n t = \frac{m_b v}{m_c\omega_n}\sin\omega_n t$$

The magnitude of the response due to the impulse is thus $X = \left|\frac{m_b v}{m_c\omega_n}\right|$

Next compute the momentum of the bird to complete the magnitude calculation:

$$m_b v = 1\text{ kg} \bullet 72\ \frac{\text{km}}{\text{hour}} \bullet \frac{1000\text{ m}}{\text{km}} \bullet \frac{\text{hour}}{3600\text{ s}} = 20\text{ kg m/s}$$

Next use this value in the expression for the maximum value:

$$X = \left|\frac{m_b v}{m_c \omega_n}\right| = \left|\frac{20\text{ kg m/s}}{3\text{ kg} \bullet 75.45\text{ rad/s}}\right| = \underline{0.088\text{ m}}$$

This max value exceeds the camera tolerance

Example 3.1.3: two impacts, zero initial conditions (double hit).

$$m = 1 \text{ kg}, \; c = 0.5 \text{ kg/s}, \; k = 4 \text{ N/m}$$

$$\hat{F} = 2 \text{ N}\cdot\text{s} \;\; \text{and} \; F(t) = 2\delta(t) + \delta(t-\tau)$$

$$\omega_n = 2, \zeta = 0.125$$

$$x_1(t) = \frac{2e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t = 1.008e^{-0.25t}\sin(1.984t), t > 0$$

$$x_2(t) = 0.504e^{-0.25(t-\tau)}\sin(1.984(t-\tau)), t > \tau$$

$$x(t) = x_1 + x_2$$

$$= \begin{cases} 1.008e^{-0.25(t)}\sin(1.984t) & 0 < t < \tau \\ 1.008e^{-0.25t}\sin(1.984t) + 0.504e^{-0.25(t-\tau)}\sin(1.984(t-\tau)) & t > \tau \end{cases}$$

Example 3.1.3 two impacts and initial conditions

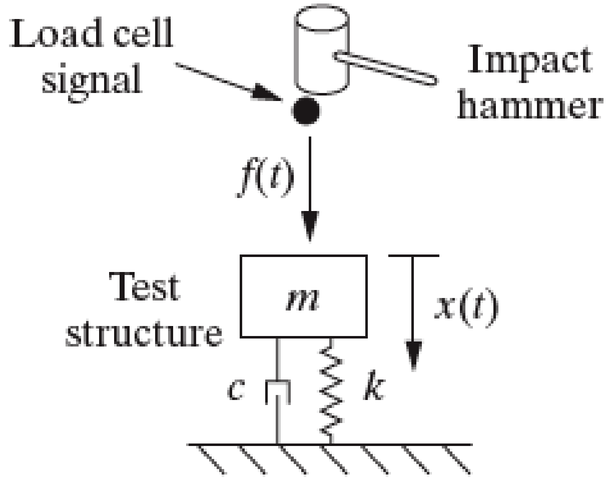
$$\ddot{x} + 2\dot{x} + 4x = \delta(t) - \delta(t-4), \quad x_0 = 1 \text{ mm}, \ \dot{x}_0 = -1 \text{ mm/s}$$

Solve three simple problems and add the results.

Homogeneous solution ($\omega_n = 2$rad/s, ζ=0.5, $\omega_d = \sqrt{3}$ rad/s)

$$x_h(t) = e^{-\zeta\omega_n t}[\frac{v_0 + x_0\zeta\omega_n}{\omega_d}\sin\omega_d t + x_0\cos\omega_d t]$$

$$= e^{-t}[\frac{-1+1}{\sqrt{3}}\sin\sqrt{3}t + \cos\sqrt{3}t] = e^{-t}\cos\sqrt{3}t$$



Note, no need to redo constants of integration

for impulse excitation (others, yes)

Computation of the response to first impulse:

Treat $\delta(t)$ as $x_0 = 0$ and $v_0 = 1,\ 0 < t < 4$

$$x_I(t) = e^{-\zeta\omega_n t}\left[\frac{v_0}{\omega_d}\sin\omega_d t\right] = \frac{1}{\sqrt{3}}e^{-t}\sin\sqrt{3}t$$

$$0 < t < 4$$

Total Response for $0 < t < 4$

$$x_1(t) = x_h(t) + x_I(t)$$

$$= e^{-t}\left(\cos\sqrt{3}t + \frac{1}{\sqrt{3}}\sin\sqrt{3}t\right),$$

$$\text{for} \quad 0 \le t < 4$$

Next compute the response to the second impulse:

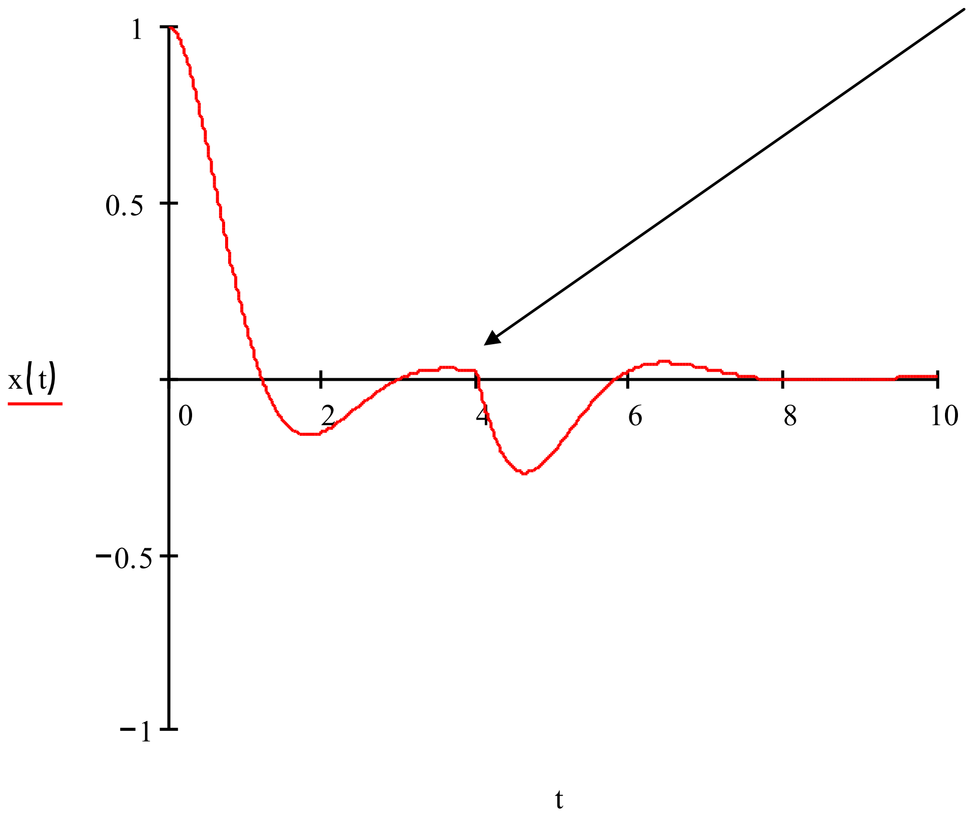
$$x_2 = \frac{-1}{\sqrt{3}} e^{-t+4} \sin\sqrt{3}(t-4), \quad t > 4$$

$$= -\frac{e^{-t+4}}{\sqrt{3}} \sin\sqrt{3}(t-4) \quad \underbrace{H(t-4)}_{\text{Heaviside Step function}}$$

Here the Heaviside step function is used to "turn on" the response to the impulse at *t* = 4 seconds.

To get the total response add the partial solutions:

$$x(t) = e^{-t}(\underbrace{\frac{1}{\sqrt{3}}\sin\sqrt{3}t}_{\text{frist impulse}} + \underbrace{\cos\sqrt{3}t}_{\text{initial condition}}) - \underbrace{\frac{e^{-t+4}}{\sqrt{3}}\sin\sqrt{3}(t-4)H(t-4)}_{\text{second impulse}}$$

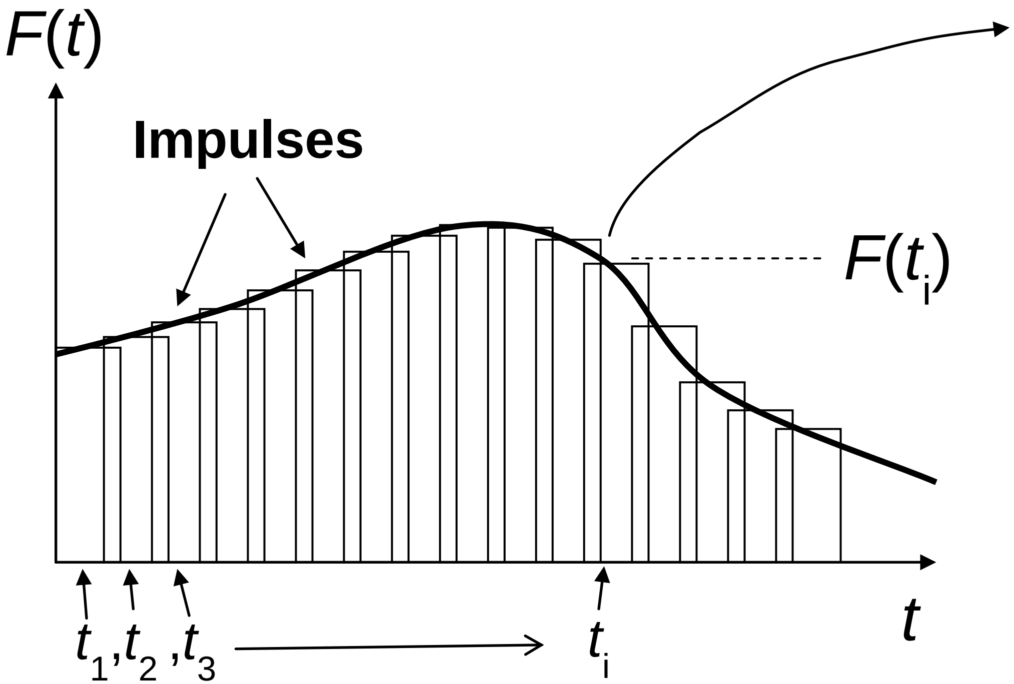


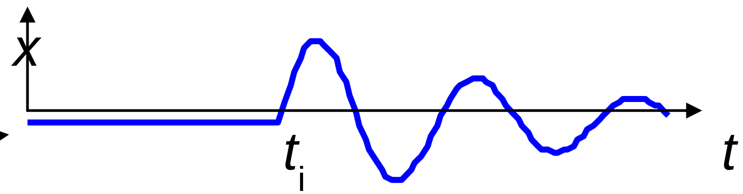
3.2 Response to an Arbitrary Input

The response to general force, $F(t)$, can be viewed as a series of impulses of magnitude $F(t_i)\Delta t$

Response at time *t* due to the i^{th} impulse **zero IC**

$x_i(t) = [F(t_i)\mathrm{D}t\,]\,h(t-t_i)$ for $t>t_i$





If $t = t_I$ (the i^{th} time interval)

$$x(t_I) = \sum_{i=1}^{I} [F(t_i)\Delta t] h(t - t_i)$$

$$\Delta t \to 0, t_i \to \tau \Rightarrow$$

$$x(t) = \underbrace{\int_0^t F(\tau) h(t-\tau) d\tau}_{\text{convolution integral}} \quad (3.12)$$

Properties of convolution integrals: It is symmetric meaning:

Let $\alpha = t - \tau, t$ fixed so that $\tau = t - \alpha$

and $d\tau = -d\alpha$. Also $\tau : 0 \rightarrow t \Rightarrow \alpha : t \rightarrow 0$

$$x(t) = \int_0^t F(\tau)h(t-\tau)d\tau = \int_t^0 F(t-\alpha)h(\alpha)(-d\alpha)$$

$$= \int_0^t F(t-\alpha)h(\alpha)d\alpha$$

The convolution integral, or Duhamel integral, for underdamped systems is:

$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t \left[F(\tau) e^{\zeta\omega_n \tau} \sin \omega_d (t-\tau) \right] d\tau$$

$$= \frac{1}{m\omega_d} \int_0^t F(t-\tau) e^{-\zeta\omega_n \tau} \sin \omega_d \tau d\tau \qquad (3.13)$$

- The response to *any* integrable force can be computed with either of these forms
- Which form to use depends on which is easiest to compute

Example 3.2.1: Step function input

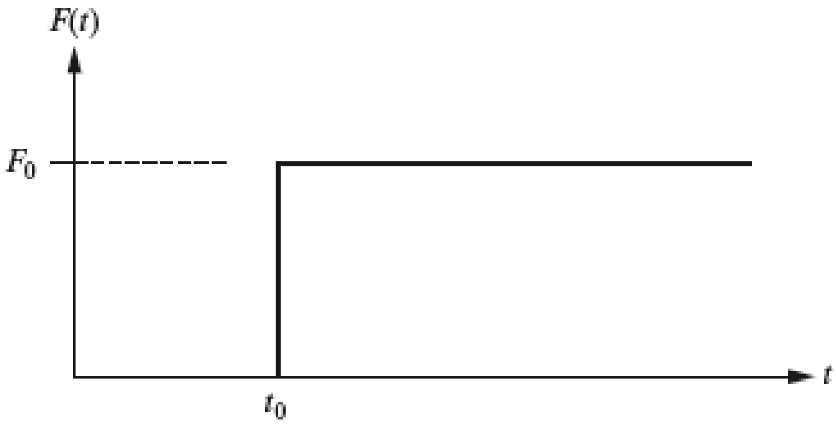


Figure 3.6 Step function

$$m\ddot{x} + c\dot{x} + kx = \begin{cases} 0 & 0 < t < t_0 \\ F_0 & t_0 \le t \end{cases}$$

$$x_0 = 0, \quad v_0 = 0, \quad 0 < \zeta < 1$$

To solve apply (3.13):

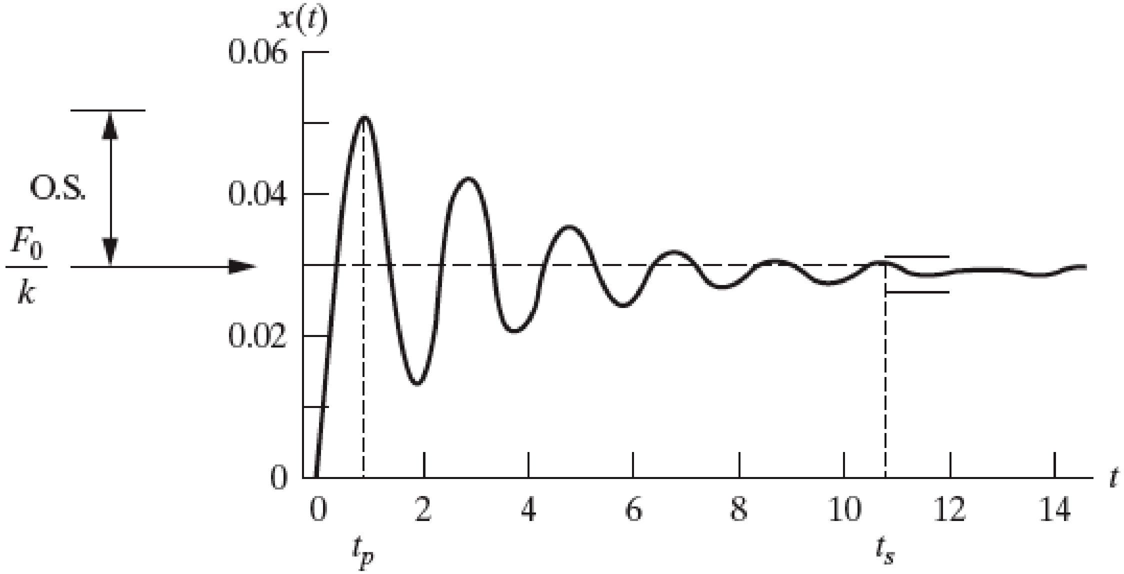
$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \int_0^{t_0} (0) e^{\zeta\omega_n \tau} \sin\omega_d (t-\tau) d\tau + \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \int_{t_0}^{t} F_0 e^{\zeta\omega_n \tau} \sin\omega_d (t-\tau) d\tau$$

$$= \frac{F_0}{m\omega_d} e^{-\zeta\omega_n t} \int_{t_0}^{t} e^{\zeta\omega_n \tau} \sin\omega_d (t-\tau) d\tau$$

Integrating (use a table, code or calculator) yields the solution:

$$x(t) = \frac{F_0}{k} - \frac{F_0}{k\sqrt{1-\zeta^2}} e^{-\zeta\omega_n (t-t_0)} \cos\left(\omega_d (t-t_0) - \theta\right), \quad t \geq t_0 \quad (3.15)$$

$$\theta = \tan^{-1} \frac{\zeta}{\sqrt{1-\zeta^2}} \qquad (3.16)$$



Example: undamped oscillator under IC and constant force

For an undamped system:

$$h(t) = \frac{1}{m\omega_n} \sin \omega_n t$$

The homogeneous solution is

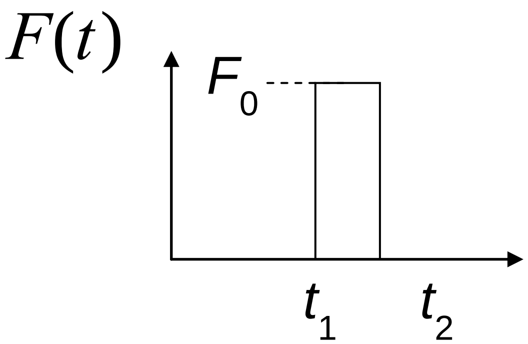
$$x_h = \frac{v_0}{\omega_n} \sin \omega_n t + x_0 \cos \omega_n t, \quad t < t_1$$

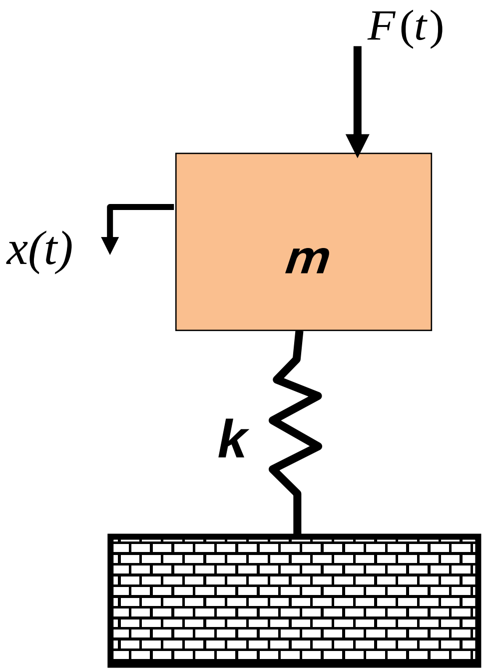
Good until the applied force acts at t_1, then:

$$x_{1\to 2} = \int_0^t F(\tau) h(t-\tau) d\tau, \quad t_1 < t < t_2$$

$$= \int_0^{t_1} F(\tau) h(t-\tau) d\tau + \int_{t_1}^{t} F(\tau) h(t-\tau) d\tau$$

(first integral → 0)





Next compute the solution between t_1 and t_2

For $t_1 < t < t_2$

$$x_{1\to 2} = \int_{t_1}^{t} F_0 \frac{1}{m\omega_n} \sin\omega_n (t-\tau) d\tau$$

$$= \frac{F_0}{m\omega_n} \left\{ \frac{(-1)(-1)}{\omega_n} \cos\omega_n (t-\tau) \Big|_{t_1}^{t} \right\}$$

$$= \frac{F_0}{m\omega_n^2} [1 - \cos\omega_n (t - t_1)]$$

Now compute the solution for time greater than t_2

For $t > t_2$

$$x_{2\rightarrow} = \int_0^{t_1} \cancelto{0}{F(\tau)} h(t-\tau)d\tau + \int_{t_1}^{t_2} F(\tau)h(t-\tau)d\tau + \int_{t_2}^{t} \cancelto{0}{F(\tau)} h(t-\tau)d\tau$$

$$= \frac{F_0}{m\omega_n}\left\{ \frac{1}{\omega_n}\cos\omega_n(t-\tau)\Big|_{t_1}^{t_2} \right\}$$

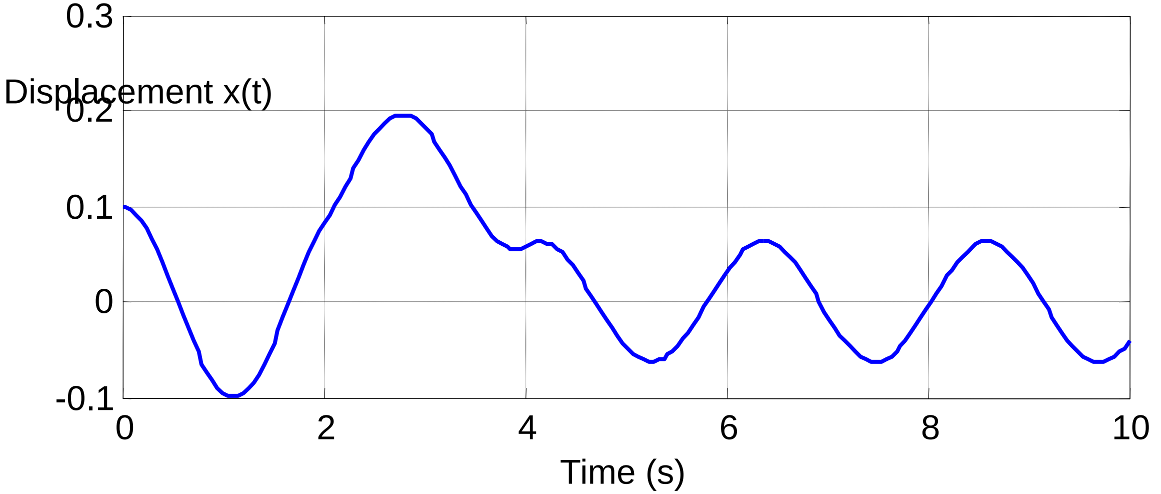
$$= \frac{F_0}{m\omega_n^2}[\cos\omega_n(t-t_2) - \cos\omega_n(t-t_1)]$$

Total solution is superposition:

$$x(t)=\begin{cases} \dfrac{v_0}{\omega_n}\sin\omega_n t + x_0\cos\omega_n t & t<t_1 \\ \dfrac{v_0}{\omega_n}\sin\omega_n t + x_0\cos\omega_n t + \dfrac{F_0}{m\omega_n^2}[1-\cos\omega_n(t-t_1)] & t_1<t<t_2 \\ \dfrac{v_0}{\omega_n}\sin\omega_n t + x_0\cos\omega_n t + \dfrac{F_0}{m\omega_n^2}\left[\cos\omega_n(t-t_2)-\cos\omega_n(t-t_1)\right] & t>t_2 \end{cases}$$

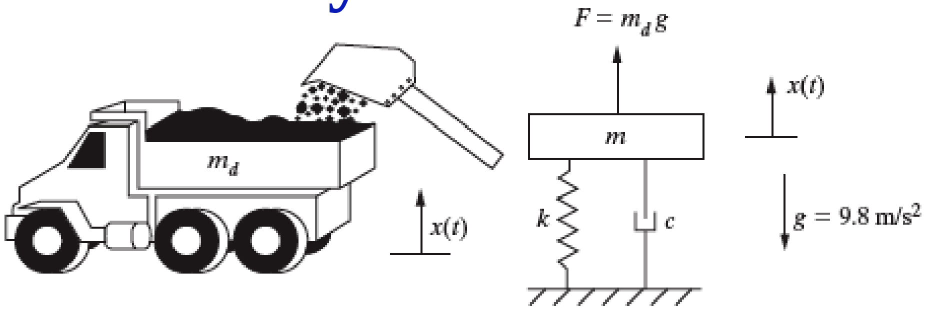
$m=F_0=1, \omega_n=\sqrt{8}, t_1=2, t_2=4, x_0=0.1, v_0=0$

Check points: x increases after application of F. Undamped response around $x=0$



Example 3.2.3: Static versus dynamic load

$$m\ddot{x}+c\dot{x}+kx=\begin{cases} m_d g & t\geq 0 \\ 0 & t<0 \end{cases}$$



Truck being filled with dirt

Vibration model

$$\Rightarrow x(t)=\frac{m_d g}{k}\left[1-\frac{1}{\sqrt{1-\zeta^2}}\right]e^{-\zeta\omega_n t}\cos\left(\omega_d t-\theta\right)$$

$$\zeta=0\Rightarrow x(t)=\frac{m_d g}{k}(1-\cos\omega_d t)$$

This has max value of $x_{\max}=2\dfrac{m_d g}{k}$, twice the static load

Numerical simulation and plotting

- **At the end of this chapter, numerical simulation is used to solve the problems of this section.**
- **Numerical simulation is often easier then computing these integrals**
- **It is wise to check the two approaches against each other by plotting the analytical solution and numerical solution on the same graph**

3.3 Response to an Arbitrary Periodic Input

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = F(t) \quad \text{where} \quad F(t) = F(t+T)$$

- **We have solutions to sine and cosine inputs.**
- **What about periodic but non-harmonic inputs?**
- **We know that periodic functions can be represented by a series of sines and cosines (Fourier)**
- **Response is superposition of as many RHS terms as you think are necessary to represent the forcing function accurately**

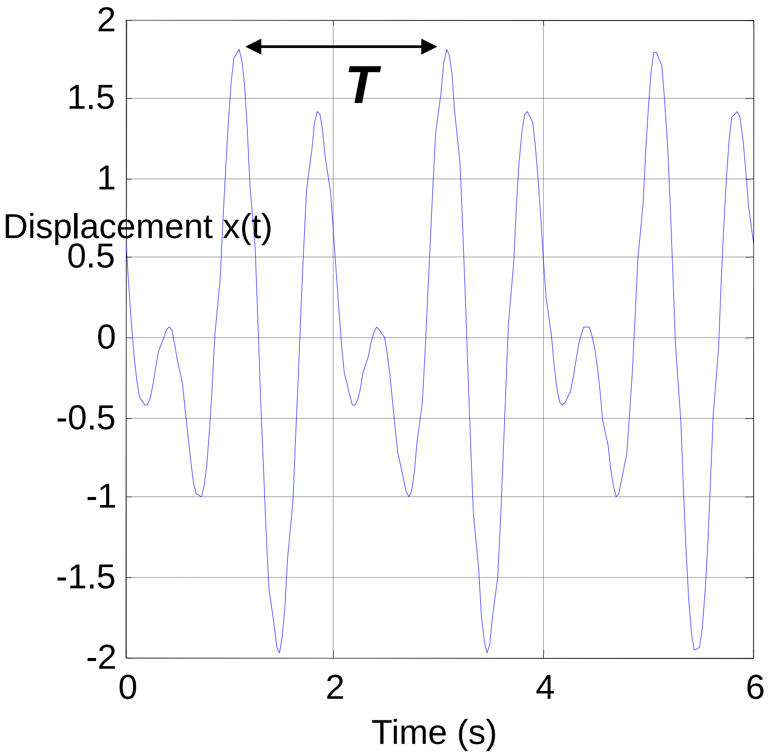


Figure 3.11

Recall the Fourier Series Definition:

$$\text{Assume } F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos \Omega_n t + b_n \sin \Omega_n t\right) \qquad (3.20)$$

$$\text{where} \quad \Omega_n = \frac{2\pi n}{T} = n\omega$$

$$a_0 = \tfrac{2}{T}\int_0^T F(t)\, dt \qquad (3.21): \text{twice the average}$$

$$a_n = \tfrac{2}{T}\int_0^T F(t) \cos \Omega_n t\, dt \qquad (3.22): \text{Oscillations around average}$$

$$b_n = \tfrac{2}{T}\int_0^T F(t) \sin \Omega_n t\, dt \qquad (3.23)$$

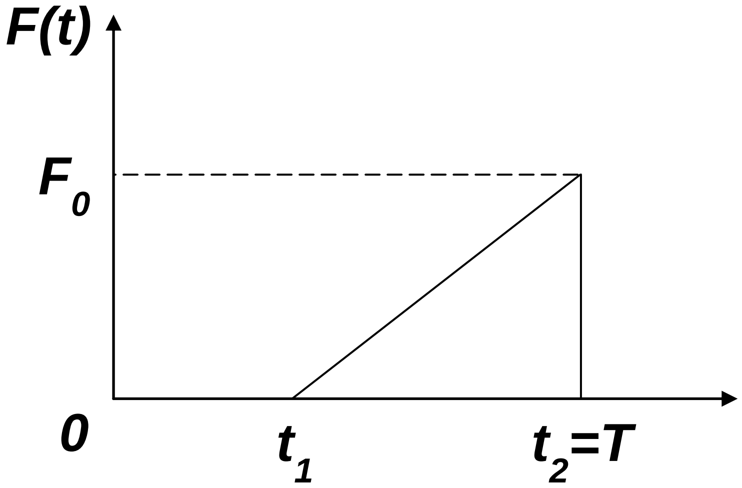
The terms of the Fourier series satisfy orthogonality conditions:

$$\int_0^T \sin n\omega_T t \sin m\omega_T t dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases} \qquad (3.24)$$

$$\int_0^T \cos n\omega_T t \cos m\omega_T t dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases} \qquad (3.25)$$

$$\int_0^T \cos n\omega_T t \sin m\omega_T t dt = 0 \qquad (3.26)$$

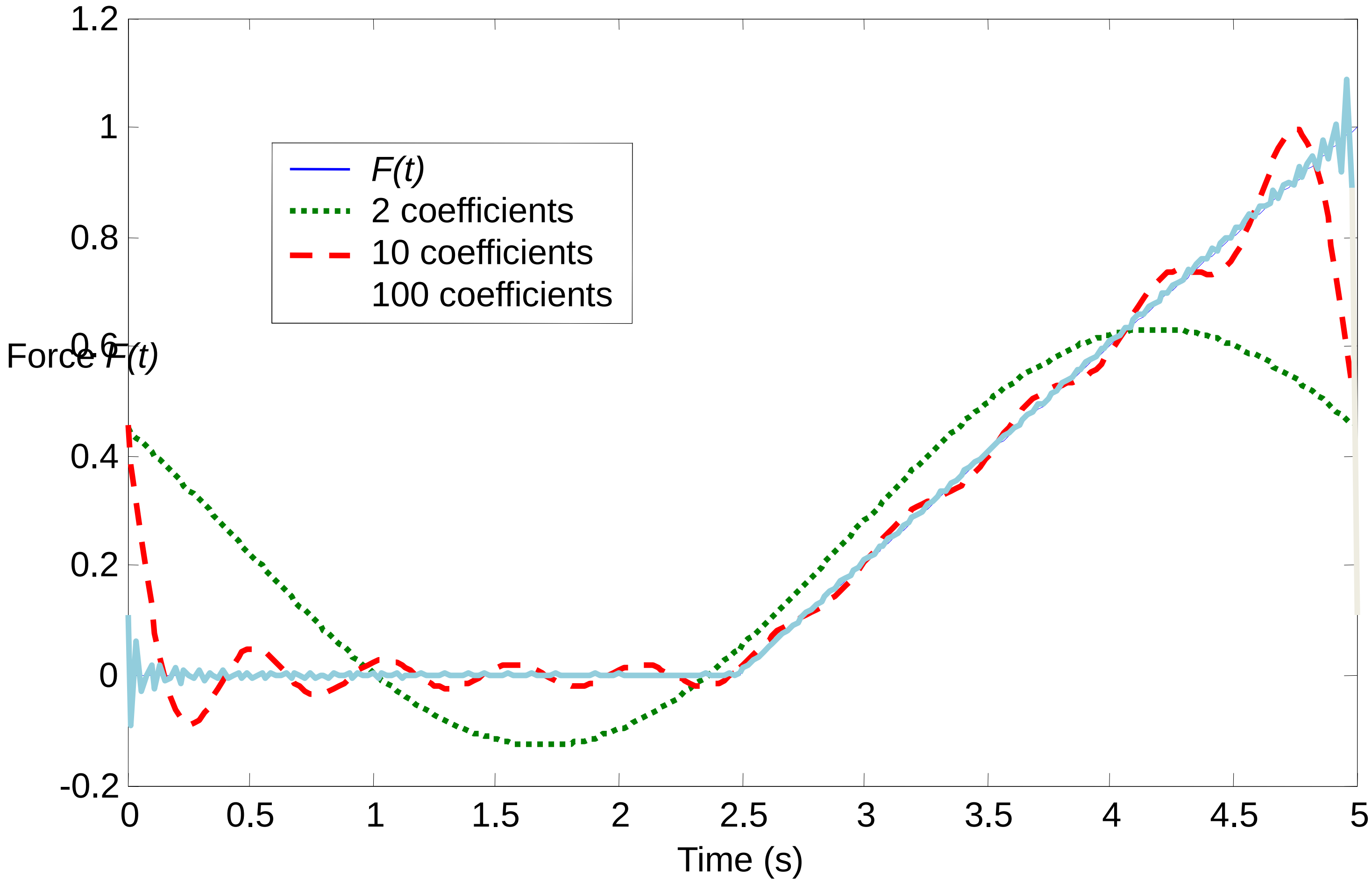
Fourier Series Example



Step 1: find the F.S. and determine how many terms you need

$$F(t) = \begin{cases} 0, & t < t_1 \\ \dfrac{F_0}{t_2 - t_1}\left(t - t_1\right), & t_1 < t \le t_2 \end{cases}$$

Fourier Series Example



Having obtained the FS of input

- **The next step is to find responses to each term of the FS**
- **And then, just add them up!**
- **Danger!!: Resonance occurs whenever a multiple of excitation frequency equals the natural frequency.**
- **You may excite at 100rad/s and observe resonance while natural frequency is 500rad/s!!** **Backwards?**

Solution as a series of sines and cosines to

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = F(t)$$

The solution can be written as a summation

$$x_p(t) = x_0(t) + \sum_{n=1}^{\infty} x_{cn}(t) + x_{sn}(t)$$

where $x_0(t)$ is a solution to

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{a_0}{2} \Rightarrow x_0(t) = \frac{a_0}{2\omega_n^2}$$

and $x_{cn}(t)$ and $x_{sn}(t)$ are a solutions to

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = a_n \cos(n\omega_T t)$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = b_n \sin(n\omega_T t)$$

Solutions calculated from equations of motion (see section Example 3.3.2)

3.4 Transform Methods

An alternative to solving the previous problems, similar to section 2.3

Laplace Transform

- **Laplace transformation**

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt = \mathbf{L}\{f(t)\} \quad (3.41)$$

Laplace transforms are very useful because they change differential equations into simple algebraic equations

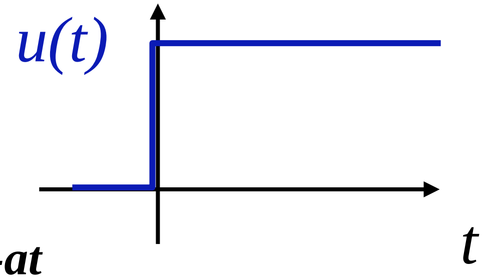
- **Examples of Laplace transforms (see page 244) in book)**

$f(t)$	$F(s)$
Step function, $u(t)$	$1/s$
e^{-at}	$1/(s+a)$
$\sin(\omega t)$	$\omega/(s^2+\omega^2)$

Laplace Transform

- **Example: Laplace transform of a step function *u(t)***

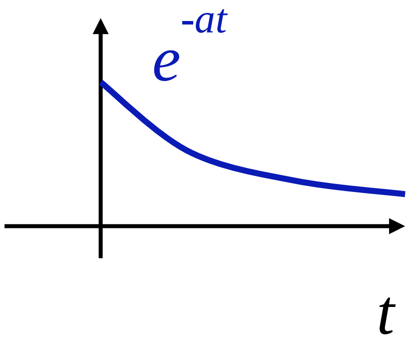
$$\mathbf{L}\{u(t)\} = \int_0^\infty e^{-st}dt = \left[\frac{-e^{-st}}{s}\right]_0^\infty = \frac{1}{s}$$



- **Example: Laplace transform of e^{-at}**

$$\mathbf{L}\{e^{-at}\} = \int_0^\infty e^{-at}e^{-st}dt = \int_0^\infty e^{-(s+a)t}dt$$

$$\mathbf{L}\{e^{-at}\} = \left[\frac{-e^{-(s+a)t}}{(s+a)}\right]_0^\infty = \frac{1}{(s+a)}$$



Laplace Transforms of Derivatives

- **Laplace transform of the derivative of a function**

$$\mathbf{L}\left\{\frac{df(t)}{dt}\right\} = \int_0^\infty \frac{df(t)}{dt} e^{-st} dt$$

Integration by parts gives,

$$\mathbf{L}\left\{\frac{df(t)}{dt}\right\} = \left[f(t)e^{-st} \right]_0^\infty + s\int_0^\infty f(t)e^{-st} dt$$

$$\mathbf{L}\left\{\frac{df(t)}{dt}\right\} = -f(0) + s\mathbf{L}\left\{ f(t) \right\}$$

Laplace Transform Procedures

- **Laplace transform of the integral of a function**

$$\mathbf{L}\left\{\int_{-\infty}^{t} f(t)dt\right\} = \frac{1}{s}\mathbf{L}\{f(t)\} + \int_{-\infty}^{0} f(t)dt$$

Steps in using the Laplace transformation to solve DE's

- **Find differential equations**
- **Find Laplace transform of equations**
- **Rearrange equations in terms of variable of interest**
- **Convert back into time domain to find resulting response (inverse transform using tables)**

Laplace Transform Shift Property

Note these shift properties in *t* and *s* spaces...

$$e^{at} f\left(t\right) \xrightarrow{\mathrm{L}} F\left(s-a\right)$$

$$f\left(t-a\right)\Phi(t-a) \xrightarrow{\mathrm{L}} e^{-as}F\left(s\right)$$

thus

$$\delta(t) \xrightarrow{\mathrm{L}} 1 \Rightarrow \delta(t-a) \xrightarrow{\mathrm{L}} e^{-as}$$

Example 3.4.3: compute the forced response of a spring mass system to a step input using LT

The equation of motion is

$$m\ddot{x}(t) + kx(t) = \Phi(t)$$

Taking the Laplace Transform (zero initial conditions)

$$(ms^2 + k)X(s) = \frac{1}{s} \Rightarrow X(s) = \frac{1}{s(ms^2 + k)} = \frac{1/m}{s(s^2 + \omega_n^2)}$$

Taking the inverse Laplace Transform yields:

$$x(t) = \frac{1/m}{\omega_n^2}\left(1 - \cos\omega_n t\right) = \frac{1}{k}\left(1 - \cos\omega_n t\right)$$

Compare this to the solution given in (3.18)

Fourier Transform

- From Fourier series of non-periodic functions
- Allow period to go to infinity
- Similar to Laplace Transform
- Useful for random inputs

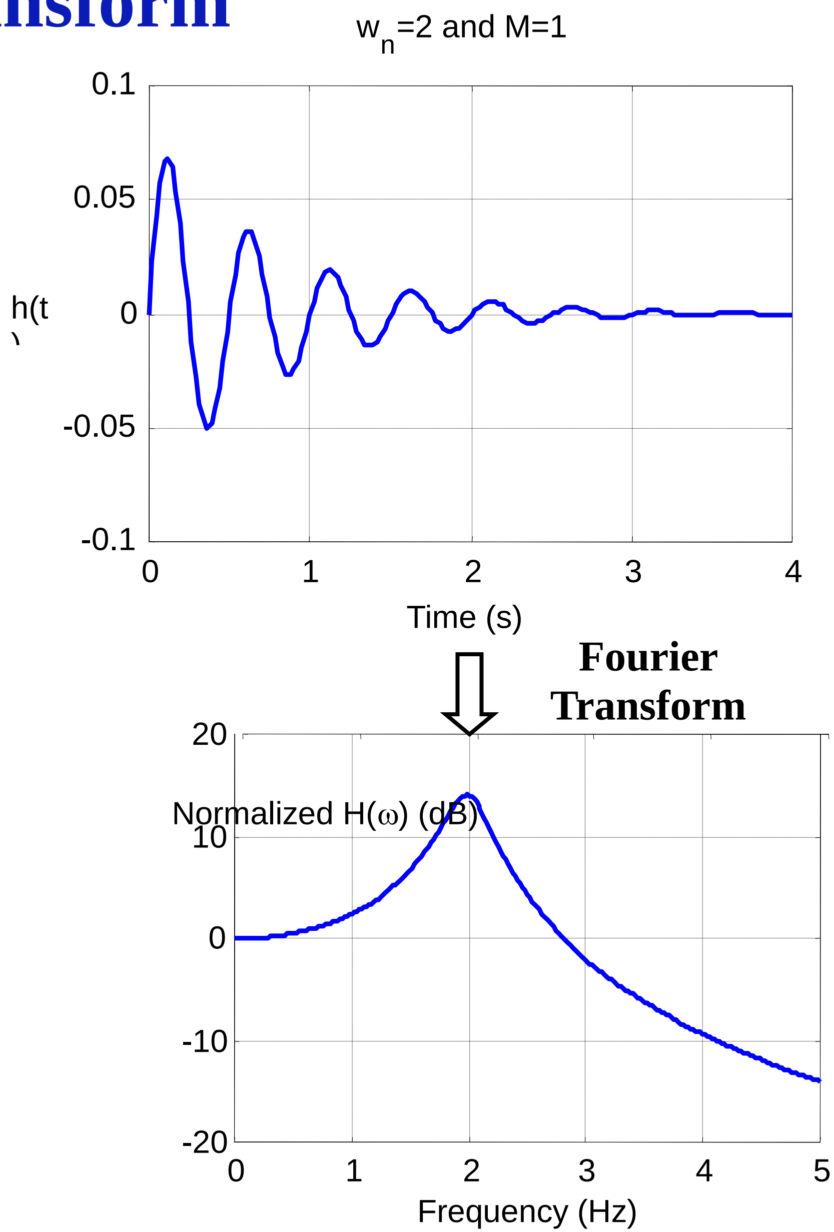
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

- Corresponding inverse transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

- Fourier transform of the unit impulse response is the frequency response function

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$



3.5 Random Vibrations

- **So far our excitations have been harmonic, periodic, or at least known in advance**
- **These are examples of *deterministic* excitations, i.e., known in advance for all time**
  - **That is given t we can predict the value of $F(t)$ exactly**
- **Responses are deterministic as well**
- **Many physical excitations are *nondeterministic*, or random, i.e., can't write explicit time descriptions**
  - **Rockets**
  - **Earthquakes**
  - **Aerodynamic forces**
  - **Rough roads and seas**
- **The responses $x(t)$ are also nondeterministic**

Random Vibrations

- *Stationary* signals are those whose statistical properties do not vary over time
- Functions are described in terms of probabilities
  - Mean values*
  - Standard deviations
- Random outputs related to random input via system transfer function

*ie given t we do not know $x(t)$ exactly, but rather

we only know statistical properties of the response

such as the average value

Autocorrelation function and power spectral density

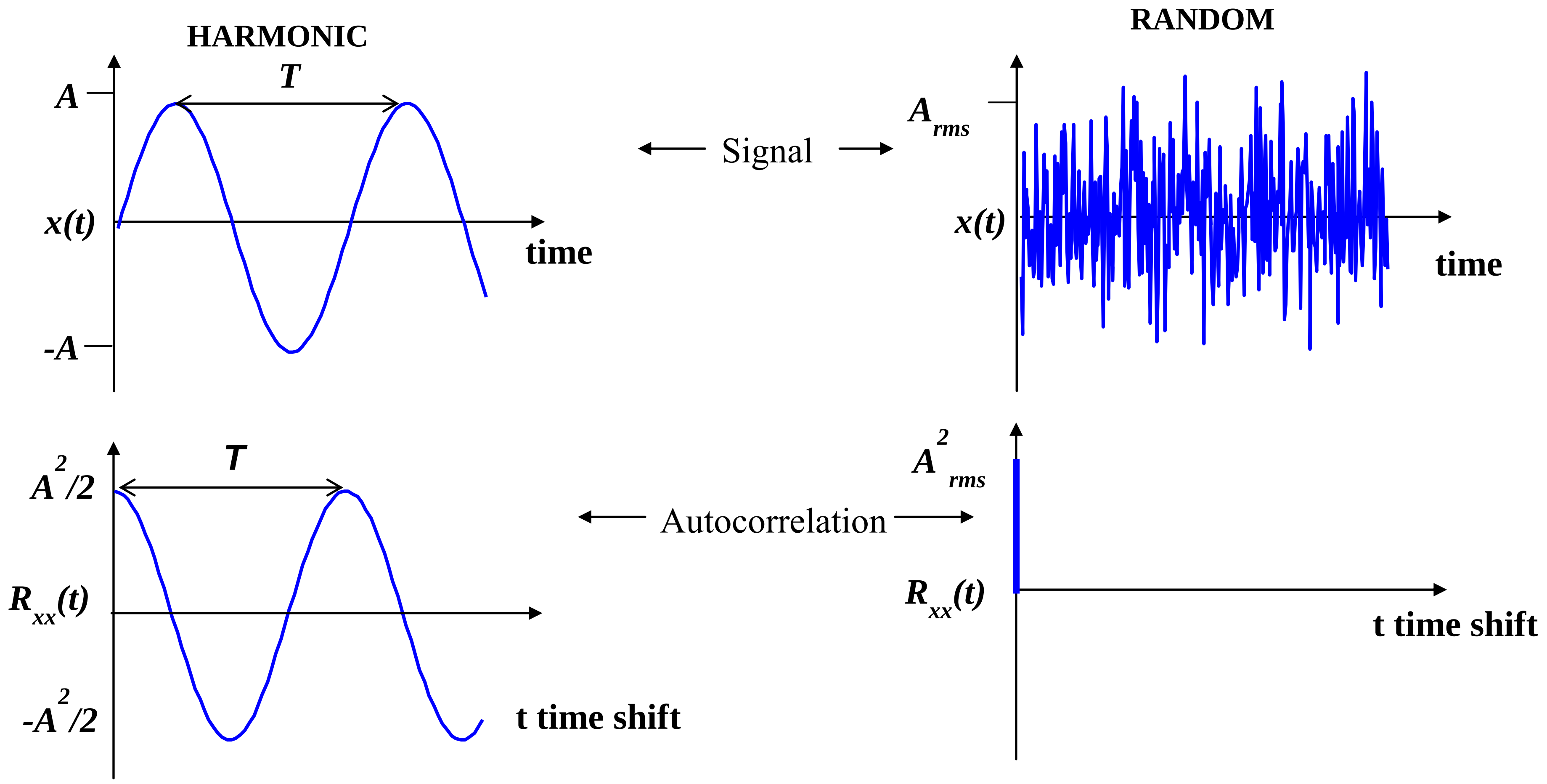
The autocorrelation function describes how a signal is changing in time or how correlated the signal is at two different points in time.

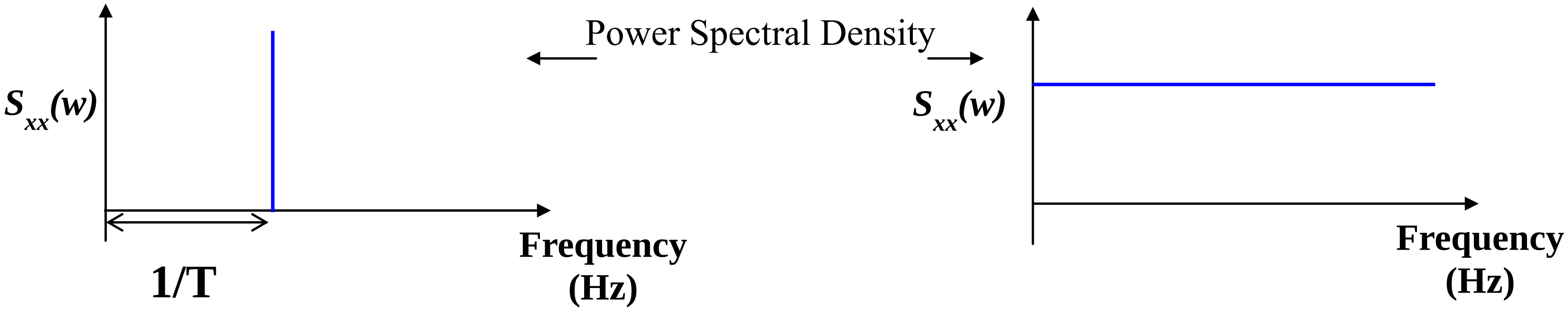
$$R_{xx}(t) = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\,x(t+\tau)d\tau$$

The Power Spectral Density describes the power in a signal as a function of frequency and is the Fourier transform of the autocorrelation function.

$$S_{xx}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{xx}(\tau)e^{-j\omega\tau}d\tau$$

Examples of signals





Power Spectral Density
$S_{xx}(w)$
1/T
Frequency (Hz)
$S_{xx}(w)$
Frequency (Hz)

More Definitions

Average :

$$\bar{x} = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)dt \qquad (3.47)$$

Mean-square:

$$\overline{x^2} = \lim_{T\to\infty} \frac{1}{T} \int_0^T x^2(t)dt \qquad (3.48)$$

rms:

$$x_{\text{rms}} = \sqrt{\overline{x^2}} = \sqrt{\lim_{T\to\infty} \frac{1}{T} \int_0^T x^2(t)dt} \qquad (3.49)$$

Expected Value
(or ensemble average)

The expected value =

$$E[x(t)] = \lim_{T\to\infty}\int_0^T \frac{x(t)}{T}\,dt = \overline{x} \qquad (3.63)$$

The Probability Density Function, *p*(*x*), is the probability that *x* lies in a given interval (e.g. Gaussian Distribution)

The expected value is also given by

$$E[x] = \int_{-\infty}^{\infty} xp(x)dx \qquad (3.64)$$

Recall the Basic Relationships for Transforms:

Recall for SDOF

$$\text{transfer function}: \; G(s) = \frac{1}{ms^2 + cs + k}$$

$$\text{frequency response function}: G(j\omega) = H(\omega) = \frac{1}{k - m\omega^2 + c\omega j}$$

$$\text{unit impulse response function}: h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t$$

$$L[h(t)] = \frac{1}{ms^2 + cs + k} = G(s)$$

And the Fourier Transform of $h(t)$ is $H(\omega)$

What can you predict?

The response of SDOF with *f*(*t*) as input:

Deterministic Input:

$$X(s) = G(s)F(s)$$

$$x(t) = \int_0^t h(t-\tau) f(\tau) d\tau$$

Random Input:

$$S_{xx}(\omega) = |H(\omega)|^2 S_{ff}(\omega)$$

$$E[x^2] = \int_{-\infty}^{\infty} |H(\omega)|^2 S_{ff}(\omega) d\omega$$

In a Lab, the PSD function of a random input and the output can be measured simply in one experiment. So the FRF can be computed as their ratio by a single test, instead of performing several tests at various constant frequencies.

Here we get an exact time record of the output given an exact record of the input.

Here we get an expected value of the output given a statistical record of the input.

Example 3.5.1 PSD Calculation

Consider $m\ddot{x}+c\dot{x}+kx=F(t)$, where the PSD of $F(t)$ is constant S_0

The corresponding frequency response function is:

$$H(\omega)=\frac{1}{k-m\omega^2+c\omega j} \qquad (2.59)$$

$$\Rightarrow \left|H(\omega)\right|^2=\left|\frac{1}{k-m\omega^2+c\omega j}\right|^2=\frac{1}{k-m\omega^2+c\omega j}\bullet\frac{1}{k-m\omega^2-c\omega j}$$

$$=\frac{1}{(k-m\omega^2)^2+c^2\omega^2}$$

From equation (3.62) the PSD of the response becomes:

$$S_{xx}=\left|H(\omega)\right|^2 S_{ff}=\frac{S_0}{(k-m\omega^2)^2+(c\omega)^2}$$

Example 3.5.2 Mean Square Calculation

Consider the system of Example 3.5.1 and compute:

$$E\left[x^2\right] = S_0 \int_{-\infty}^{\infty} \left| \frac{1}{k - m\omega_n^2 + c\omega j} \right|^2 d\omega$$

$$= S_0 \frac{\pi m}{kcm} = \frac{\pi S_0}{kc}$$

Here the evaluation of the integral is from a tabulated value
See equation (3.70).

Section 3.6 Shock Spectrum

Arbitrary forms of shock are probable (earthquakes, …)

The spectrum of a given shock is a plot of the maximum response quantity (*x*) against the ratio of the forcing characteristic (such as rise time) to the natural period.

Maximum response gives maximum stress.

$$x(t) = \int_0^t F(\tau)h(t-\tau)d\tau \qquad (3.71)$$

Using the convolution equation as a tool, compute the maximum value of the response

Recall the impulse response function undamped system:

$$h(t-\tau) = \frac{1}{m\omega_n}\sin\omega_n(t-\tau) \qquad (3.73)$$

$$\Rightarrow$$

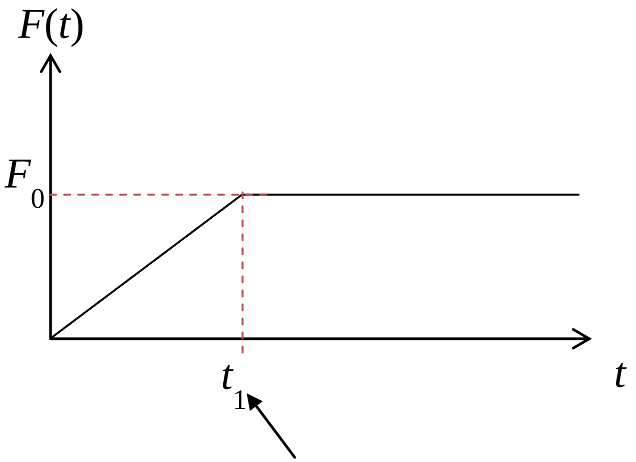
$$x(t)_{\max} = \frac{1}{m\omega_n}\left|\int_0^t F(\tau)\sin\left(\omega_n(t-\tau)\right)d\tau\right|_{\max} \qquad (3.74)$$

Such integrals usually have to be computed numerically

Example 3.6.1 Compute the response spectrum for gradual application of a constant force F_0. Assume zero initial conditions

$$m\ddot{x}(t) + kx(t) = F(t)$$

t_1=infinity, means static loading



The characteristic time of the input

$$F(t) = F_1(t) + F_2(t)$$

$$F_1(t) = \frac{t}{t_1} F_0$$

time shift and negative, like half sine problem

$$F_2(t) = \begin{cases} 0 & 0 < t < t_1 \\ -\left(\frac{t - t_1}{t_1}\right) F_0 & t \ge t_1 \end{cases}$$

Split the solution into two parts and use the convolution integral

$$x_1(t) = \frac{\omega_n}{k}\int_0^t \frac{F_0\tau}{t_1}\sin\omega_n(t-\tau)d\tau = \frac{F_0}{k}\left(\frac{t}{t_1} - \frac{\sin\omega_n t}{\omega_n t_1}\right) \qquad 0 < t < t_1 \tag{3.75}$$

For x_2 apply time shift t_1

$$x_2(t) = -\frac{F_0}{k}\left(\frac{t-t_1}{t_1} - \frac{\sin\omega_n(t-t_1)}{\omega_n t_1}\right), \quad t \geq t_1 \tag{3.76}$$

$$x(t) = x_1(t) + x_2(t) = \frac{F_0}{k}\left(\frac{t}{t_1} - \frac{\sin\omega_n t}{\omega_n t_1}\right) - \frac{F_0}{k}\left(\frac{t-t_1}{t_1} - \frac{\sin\omega_n(t-t_1)}{\omega_n t_1}\right)\Phi(t-t_1) \tag{3.77}$$

Next find the maximum value of this response

To get max response, differentiate *x*(*t*).

In the case of a harmonic input (Chapter 2) we computed this by looking at the coefficient of the steady state response, giving rise to the Magnitude plots of figures 2.8, 2.9, 2.14.

Need to look at two cases 1) $t < t_1$ and 2) $t \geq t_1$

For case 2) solve: (what about case 1? Its max is *X*static)

$$\frac{d}{dt}\left[\frac{F_0}{k\omega_n t_1}\left(\omega_n t_1 - \sin\omega_n t + \sin\omega_n (t - t_1)\right)\right] = 0 \Rightarrow$$

Solve for *t* at xmax, denoted t_p

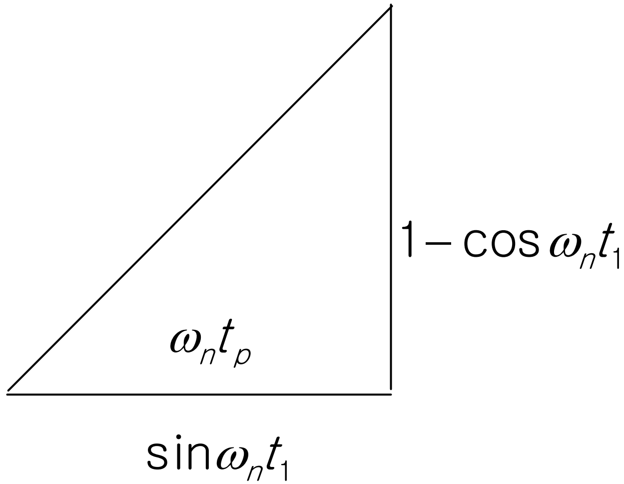
$$-\cos\omega_n t + \cos\omega_n (t - t_1)\Big|_{t=t_p} = 0$$

$$\cos\omega_n t_p = \cos\omega_n t_p \cos\omega_n t_1 + \sin\omega_n t_p \sin\omega_n t_1$$

$$\Rightarrow \omega_n t_p = \tan^{-1}\left(\frac{1-\cos\omega_n t_1}{\sin\omega_n t_1}\right)$$

$$\sqrt{\sin^2\omega_n t_1 + (1-\cos\omega_n t_1)^2}$$

$$= \sqrt{2\left(1-\cos\omega_n t_1\right)}$$



From the triangle:

$$\sin\omega_n t_p = -\sqrt{\frac{1}{2}(1-\cos\omega_n t_1)}$$

$$\cos\omega_n t_p = \frac{-\sin\omega_n t_1}{\sqrt{2(1-\cos\omega_n t_1)}}$$

Minus sq root taken as + gives a negative magnitude

Substitute into $x(t_p)$ to get nondimensional X_{max} :

$$\frac{x_{max}k}{F_0} = 1 + \frac{1}{\omega_n t_1}\sqrt{2\left(1-\cos\omega_n t_1\right)}$$

1st term is static, 2nd is dynamic. Plot versus:

$$\frac{t_1}{T} = \frac{\omega_n t_1}{2\pi}$$

Input characteristic time / System period

Response Spectrum

$\omega n := 2 \cdot \pi \qquad T := \frac{\omega n}{2 \cdot \pi}$

- **Indicates how normalized max output**

$X(t1) := 1 + \frac{1}{\omega n \cdot t1} \cdot \sqrt{2 \cdot (1 - \cos(\omega n \cdot t1))}$

changes as the input pulse width increases.

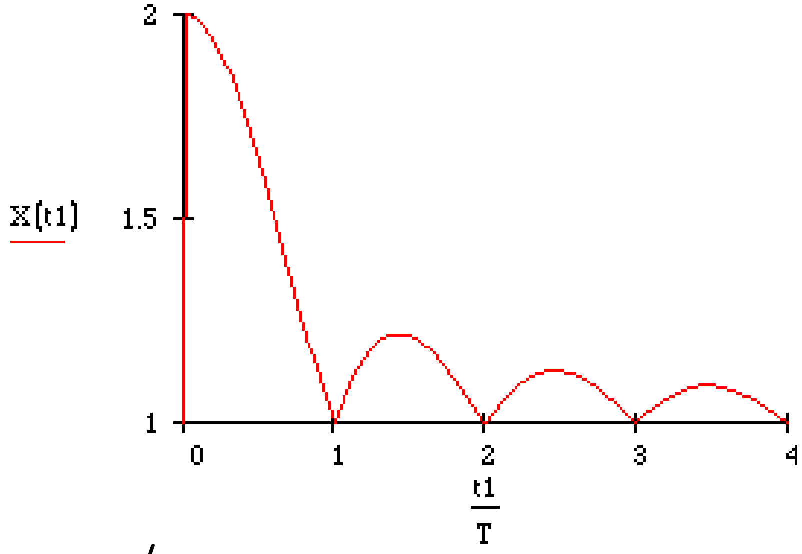
- **Very much like a magnitude plot.**
- **Shows very small t1 can increase**

the response significantly: impact, rather than
smooth force application

- **The larger the rise time, the smaller the peaks**
- **The maximum displacement is minimized if**

rise time is a multiple of natural period

- **Design by MiniMax idea**



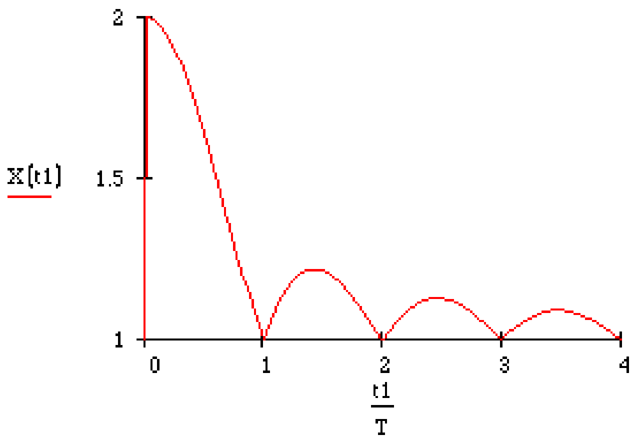
$$X = \frac{x_{max} k}{F_0}$$

Comparison between impulse and harmonic inputs

Impulse Input
Transient Output
Max amplitude versus normalized pulse "frequency"

$\omega n := 2\cdot\pi \qquad T := \frac{\omega n}{2\cdot\pi}$

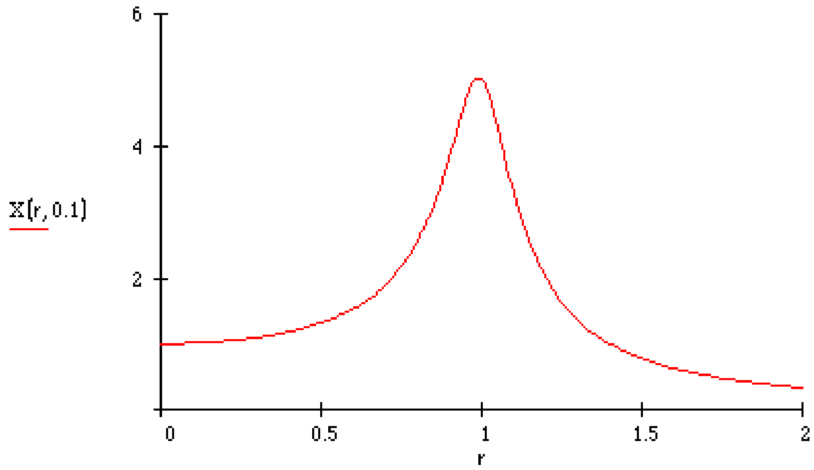
$$X(t1) := 1 + \frac{1}{\omega n\cdot t1}\cdot\sqrt{2\cdot\left(1-\cos\left(\omega n\cdot t1\right)\right)}$$



Harmonic Input
Harmonic Output
Max amplitude versus normalized driving frequency

$$X(r,\zeta) := \frac{1}{\sqrt{\left(1-r^2\right)^2+\left(2\cdot\zeta\cdot r\right)^2}}$$

$r := 0, 0.01 .. 2$



Review of The Procedure for Shock Spectrum

1. Find $x(t)$ using convolution integral
2. Compute its time derivative
3. Set it equal to zero
4. Find the corresponding time
5. Evaluate the max possible value of x (be careful about points where the function does not have derivative!!)
6. Plot it for different input shocks

3.7 Measurement via Transfer Functions

- **Apply a sinusoidal input and measure the response**
- **Do this at small frequency steps**
- **The ratio of the Laplace transform of these to signals then gives and experiment transfer function of the system**

Several different signals can be measured and these are named

TABLE 3.2 TRANSFER FUNCTIONS USED IN VIBRATION MEASUREMENT

Response Measurement	Transfer Function	Inverse Transfer Function
Acceleration	Accelerance	Apparent mass
Velocity	Mobility	Impedance
Displacement	Receptance	Dynamic stiffness

receptance: $$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} \qquad (3.86)$$

mobility: $$\frac{sX(s)}{F(s)} = \frac{s}{ms^2 + cs + k} \qquad (3.87)$$

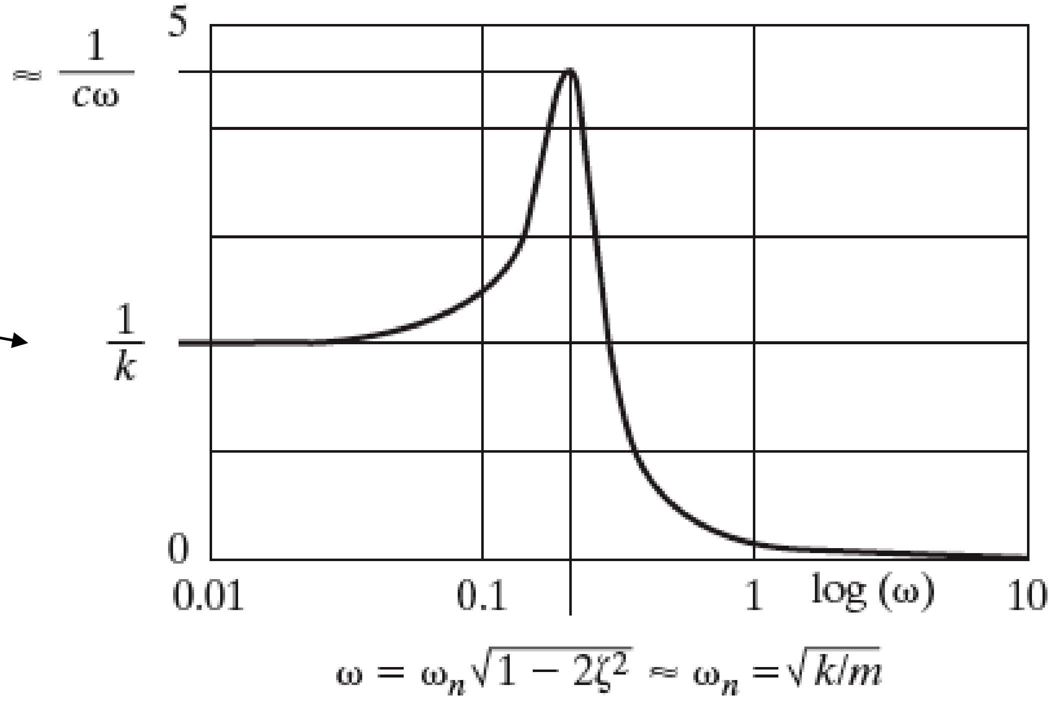
inertance: $$\frac{s^2X(s)}{F(s)} = \frac{s^2}{ms^2 + cs + k} \qquad (3.87)$$

The magnitude of the compliance transfer function yields information about the systems parameters

$$|H(j\omega)| = \frac{1}{\sqrt{(k - m\omega)^2 + (c\omega)^2}} \quad (3.89)$$

$$\left|H\left(j\sqrt{\frac{k}{m}}\right)\right| = \frac{1}{c\omega_n} \quad (3.90)$$

$$|H(0)| = \frac{1}{k} \quad (3.91)$$



3.8 Stability

Stability is *defined* for the solution of free response case:

Stable: $\left|x(t)\right| < M, \quad \forall\, t > 0$

Asymptotically Stable: $\lim_{t\to\infty} x(t) = 0$

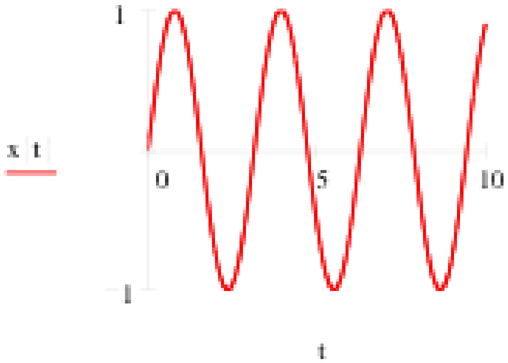
Unstable:

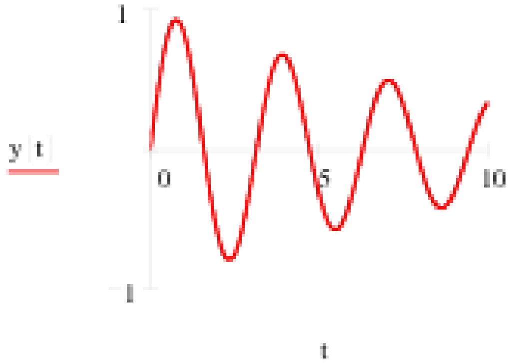
if it is not stable or asymptotically stable

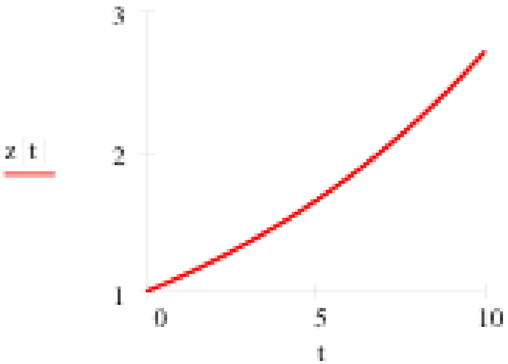
Recall these stability definitions for the free response

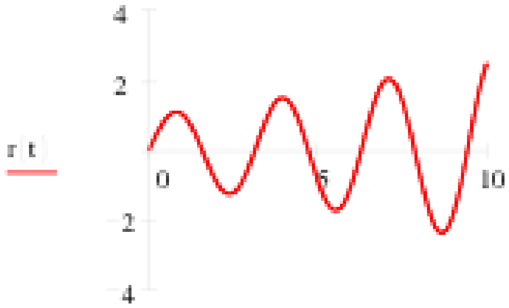
Stable **Asymptotically Stable**

$x(t) := \sin(2 \cdot t) \quad y(t) := e^{-0.1 \cdot t} \cdot x(t) \quad z(t) := e^{0.1 \cdot t} \quad r(t) := z(t) \cdot x(t)$









Divergent instability **Flutter instability**

Stability for the forced response:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)$$

- Bounded Input-Bounded Output Stable
  - ✓ $x(t)$ bounded for ANY bounded $F(t)$
- Lagrange Stable with respect to $F(t)$
  - ✓ If $x(t)$ is bounded for THE given $F(t)$

Relationship between stability of the homogeneous system and the force response

- **If x_{homo} is Asymptotically stable then the forced response is BIBO stable (Bounded input, bounded output)**
- **If x_{homo} is Stable then the forced response MAY be Lagrange Stable or Unstable**

Stability for Harmonic Excitations

The solution to:

$$m\ddot{x}(t) + kx(t) = F_0 \cos \omega t$$

is:

$$x(t) = \frac{v_0}{\omega_n} \sin \omega_n t + \left(x_0 - \frac{f_0}{\omega_n^2 - \omega^2} \right) \cos \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2} \cos \omega t$$

As long as ω_n is not equal to ω this is Lagrange Stable, if the frequencies are equal it is Unstable

For underdamped systems:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F_0 \cos \omega t$$

$$x_p(t) = \underbrace{\frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}}_{X} \cos(\omega t - \underbrace{\tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right)}_{\theta})$$

Add homogeneous and particular to get total solution:

$$x(t) = \underbrace{Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)}_{\text{homogeneous or transient solution}} + \underbrace{X\cos(\omega t - \theta)}_{\substack{\text{particular or} \\ \text{steady state solution}}}$$

Bounded Input-Bounded Output Stable

Example 3.8.1

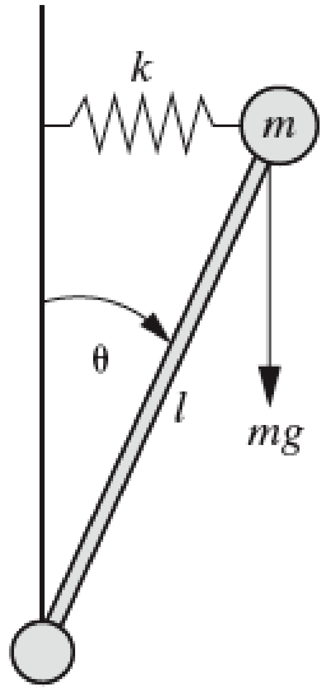
$$\sum M_0 = m\ell^2\ddot{\theta} = -\underbrace{(k\ell\sin\theta)}_{\text{Force from Spring}}\underbrace{(\ell\cos\theta)}_{\text{moment arm}} + \underset{\text{force}}{mg}\underbrace{(\ell\sin\theta)}_{\text{moment arm}}$$

The equation of motion after a small angle approximation is given becomes:

$$m\ell^2\ddot{\theta}(t) + k\ell^2\theta(t) = mg\ell\theta(t)$$

$$\Rightarrow m\ell^2\ddot{\theta}(t) + (k\ell^2 - mg\ell)\theta(t) = 0$$

This will be stable if and only if the coefficent of θ is positive or if $k\ell > mg$



- The system is thus Lagrange stable.
- Physically this tells us the spring must be large enough to overcome

Find a force of the form $F(t) = -a\theta - b\dot{\theta}$

to make the system asymptotically stable (BIBO)

$$m\ell^2\ddot{\theta} + (k\ell^2 - mg\ell)\theta = -a\theta - b\dot{\theta}$$

$$\Rightarrow m\ell^2\ddot{\theta} + b\dot{\theta} + (k\ell^2 - mg\ell + a))\theta = 0$$

Choose $b > 0$ and $a = mg\ell$

$$\Rightarrow m\ell^2\ddot{\theta} + b\dot{\theta} + k\ell^2\theta = 0$$

Then the system is asymptotically stable and BIBO

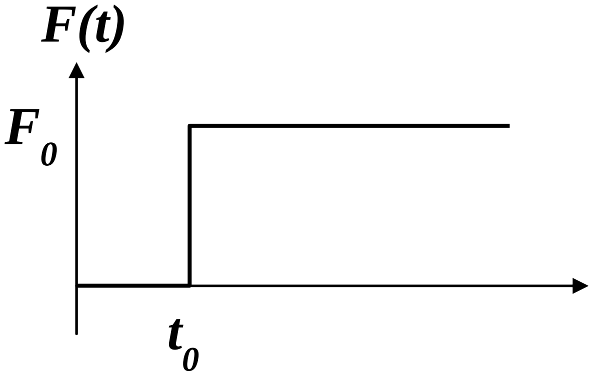
3.9 Numerical Simulation of the response

- **As before in Section 2.8 write equations of motion as state space equations**
- **The Euler integration is just**

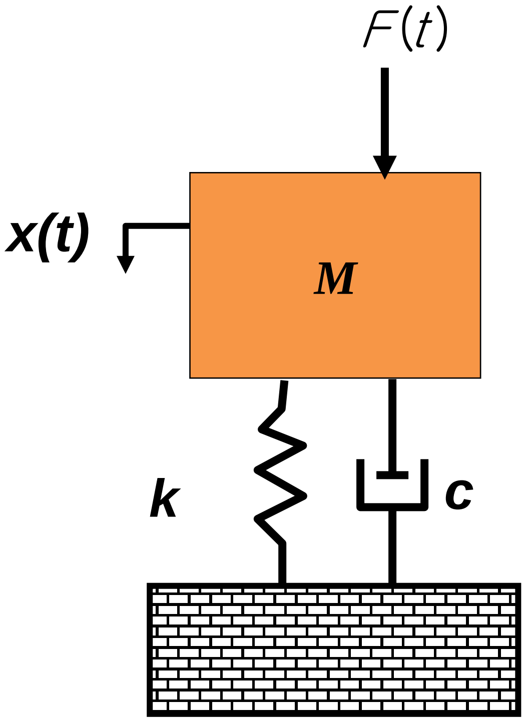
$$\mathbf{x}(t_{i+1}) = \mathbf{x}(t_i) + A\mathbf{x}(t_i)\Delta t + \mathbf{F}(t_i)\Delta t$$

Example 3.9.1 with delay

Let the input force be a step function a $t=t_0$



$$F_0 = 30 \text{ N}$$
$$k = 1000 \text{ N/m}$$
$$\zeta = 0.1$$
$$\omega_n = 3.6$$
$$t_0 = 2 \text{ s}$$



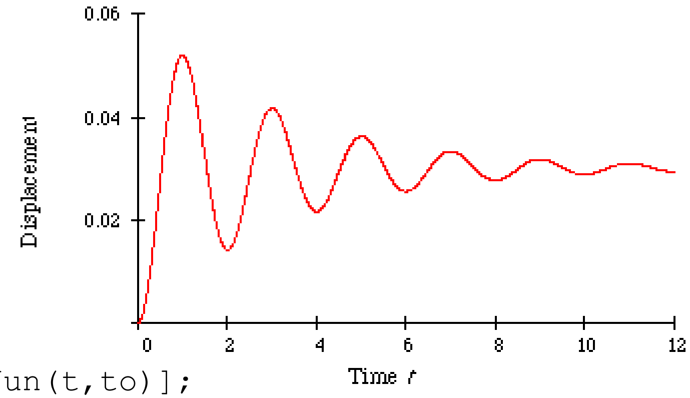
Example 3.9.1 Analytical versus numerical

$$x(t) = \left(0.03 - 0.03e^{-0.316(t-t_0)}\cos[3.144(t-t_0) - 0.101]\right)\Phi(t-t_0)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{F_0}{m}\Phi(t-t_0) \end{bmatrix}$$

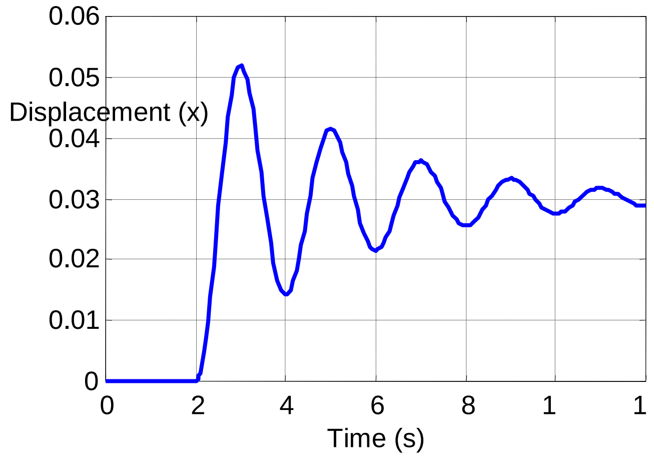
Response to step input

```
clear all
%% Analytical solution (example 3.2.1)
Fo=30; k=1000; wn=3.16; zeta=0.1; to=0;
theta=atan(zeta/(1-zeta^2));
wd=wn*sqrt(1-zeta^2);
t=0:0.01:12;
Heaviside=stepfun(t,to);% define Heaviside Step function for 0<t<12
xt = (Fo/k - Fo/(k*sqrt(1-zeta^2)) * exp(-zeta*wn*(t-to)) *
 cos (wd*(t-to)- theta))*Heaviside(t-to);
plot(t,xt); hold on
%% Numerical Solution
xo=[0; 0];
ts=[0 12];
[t,x]=ode45('f',ts,xo);
plot(t,x(:,1),'r'); hold off
%-------------------------------------------
function v=f(t,x)
Fo=30; k=1000; wn=3.16; zeta=0.1; to=0; m=k/wn^2;
v=[x(2); x(2).*-2*zeta*wn + x(1).*-wn^2 + Fo/m*stepfun(t,to)];
```



Matlab Code

```
x0=[0;0];
ts=[0 12];
[t,x]=ode45('funct',ts,x0);
plot(t,x(:,1))
```




```
function v=funct(t,x)
F0=30;
k=1000;
wn=3.16;
z=0.1;
t0=2;
m=k/(wn^2);
v=[x(2); x(2).*-2*z*wn+x(1).*-wn^2+F0/m*stepfun(t,t0)];
```

Problem 3.22

A wave consisting of the wake from a passing boat impacts a seawall. It is desired to calculate the resulting vibration. Figure P3.22 illustrates the situation and suggests a model. Calculate the resulting response.

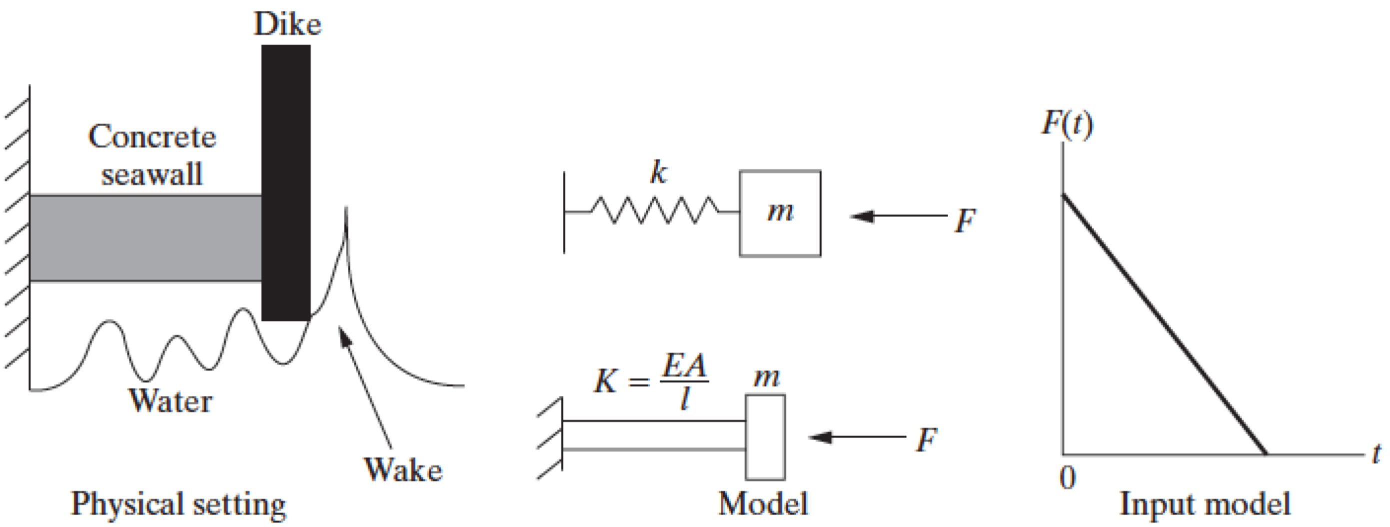
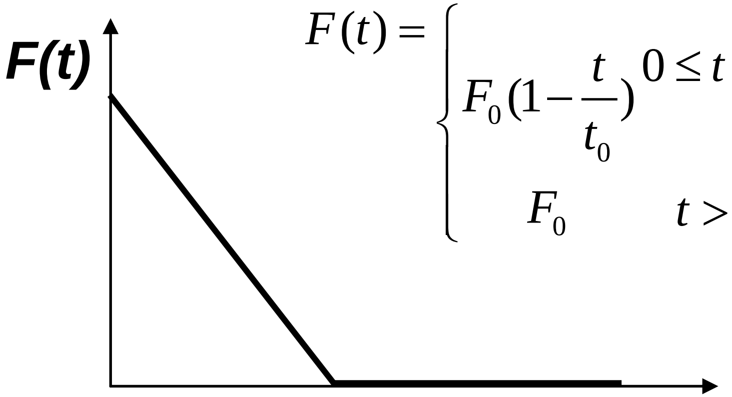


Figure P3.22 A wave hitting a seawall modeled as a nonperiodic force exciting an undamped single-degree-of-freedom, spring–mass system.

Numerical solution of Problem 3.22

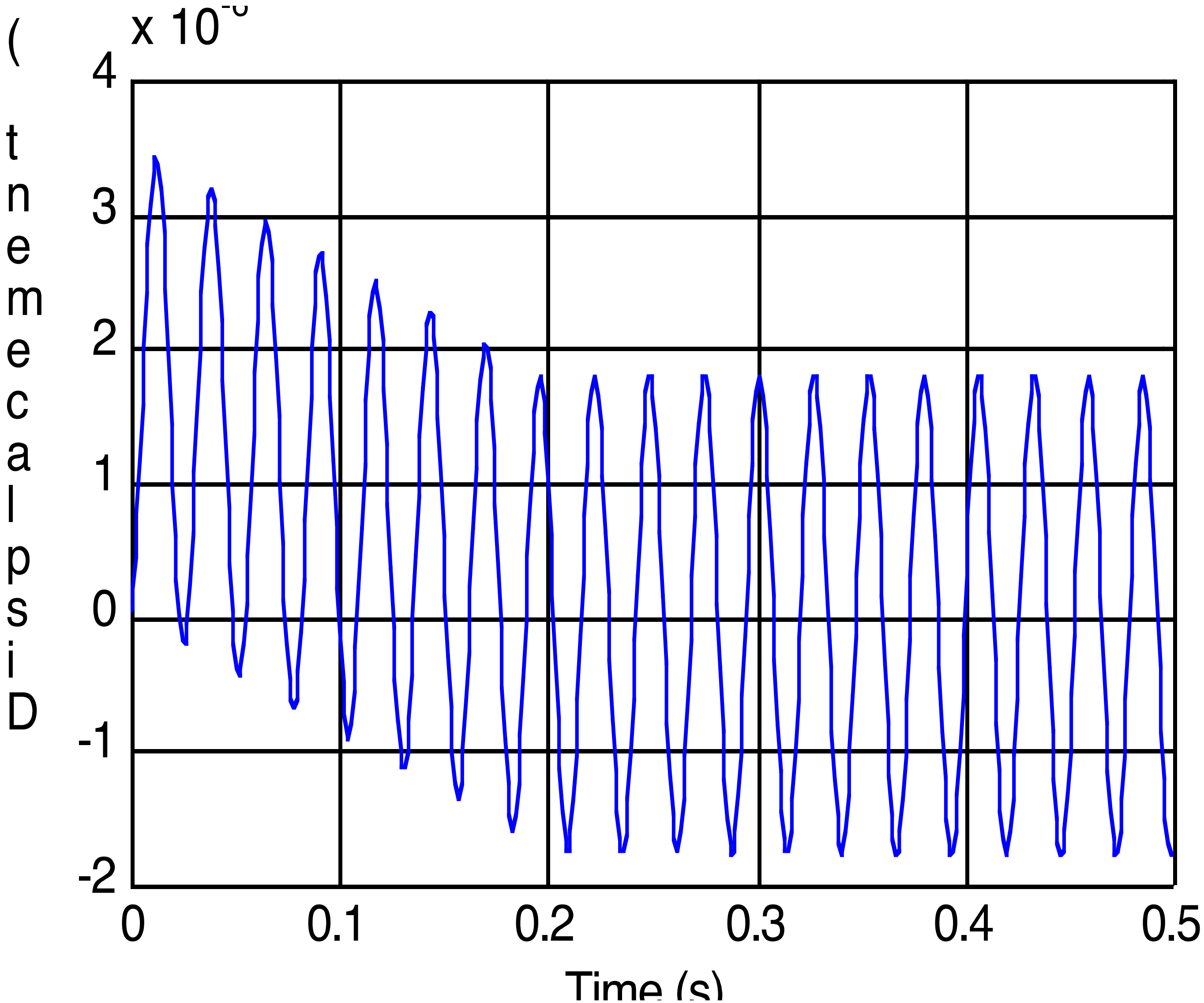
```
%problem 3.19
m=1000;
E=3.8e9;
A=0.03;
L=2;
k=E*A/L;
t0=0.2;
F0=100;
global F0 k m t0
%numerical solution
x0=[0;0];
ts=[0 0.5];
[t,x]=ode45('f_3_19',ts,x0);
plot(t,x(:,1))
```



$$F(t) = \begin{cases} F_0(1-\frac{t}{t_0}) & 0 \le t \le t_0 \\ F_0 & t > t_0 \end{cases}$$

```
function v=f_3_19(t,x)
global F0 k m t0
A=x(2);
F=(((1-t./t0).*stepfun(t,0))-((1-t./t0).*stepfun(t,t0)))*F0/m;
B=(-k/m)*x(1)+F;
v=[A; B];
```

P3.22



3.9 Nonlinear Response Properties

Euler integration formula:

$$\mathsf{x}(t_{i+1}) = \mathsf{x}(t_i) + \mathsf{F}(\mathsf{x}(t_i))\Delta t + \mathsf{f}(t_i)\Delta t$$

Nonlinear term

Analytical solutions not available so we must interrogate the numerical solution

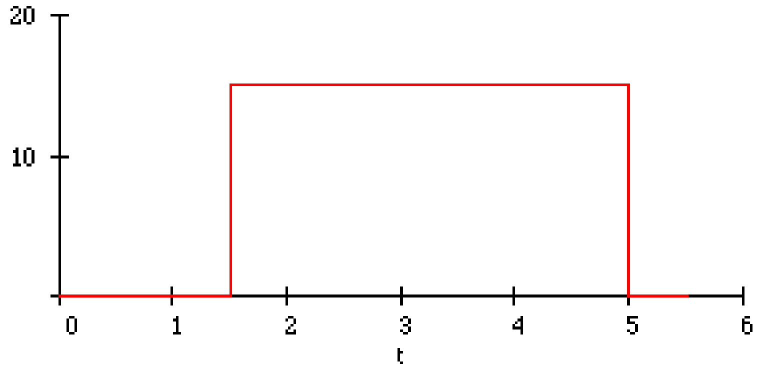
Example 3.10.1 cubic spring subject to pulse input

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)+k_1x^3(t)=1500\left[\Phi(t-t_1)-\Phi(t-t_2)\right]$$

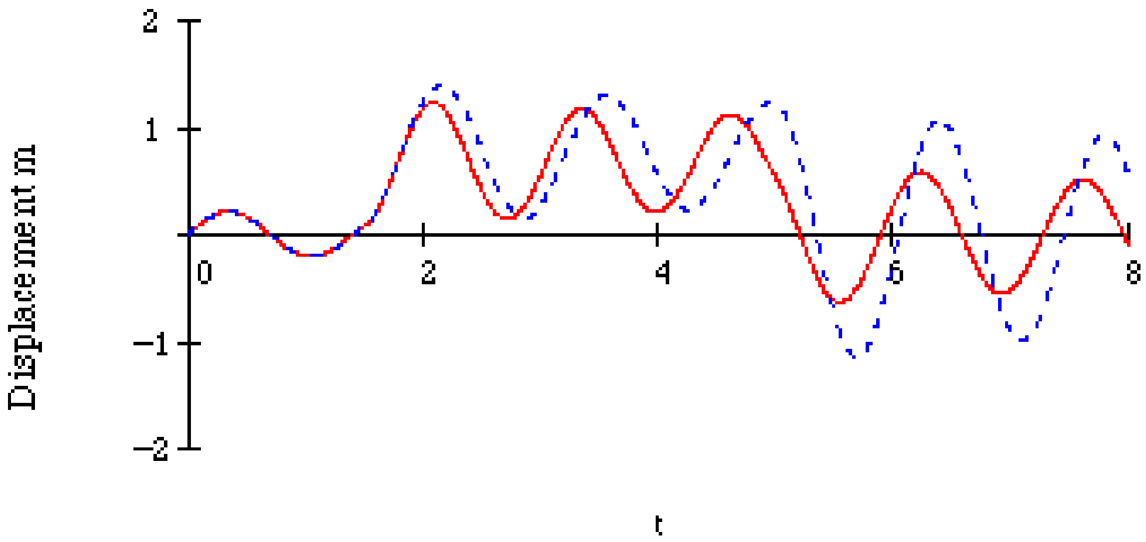
The state space form is:

$$\dot{x}_1(t)=x_2(t)$$

$$\dot{x}_2(t)=-2\zeta\omega_n x_2(t)-\omega_n^2 x_1(t)-\alpha x_1^3(t)+15\left[\Phi(t-t_1)-\Phi(t-t_2)\right]$$



Nature of Response



Red (solid) is nonlinear response.   Blue (dashed) is linear response

Is there any justification? Yes, hardening nonlinear spring.

The first part is due to IC.

Matlab Code

```
clear all
xo=[0.01; 1];
ts=[0 8];
[t,x]=ode45('f',ts,xo);
plot(t,x(:,1)); hold on         % The response of nonlinear system
[t,x]=ode45('f1',ts,xo);
plot(t,x(:,1),'--'); hold off    % The response of linear system
%---------------------------------------------
function v=f(t,x)
m=100; k=2000; c=20; wn=sqrt(k/m); zeta=c/2/sqrt(m*k); Fo=1500; alpha=3;
t1=1.5; t2=5;
v=[x(2); x(2).*-2*zeta*wn + x(1).*-wn^2 - x(1)^3.*alpha+ Fo/m*(stepfun(t,t1)-
stepfun(t,t2))];
%---------------------------------------------
function v=f1(t,x)
m=100; k=2000; c=20; wn=sqrt(k/m); zeta=c/2/sqrt(m*k); Fo=1500; alpha=0; t1=1;
t2=5;
v=[x(2); x(2).*-2*zeta*wn + x(1).*-wn^2 - x(1)^3.*alpha+ Fo/m*(stepfun(t,t1)-
stepfun(t,t2))];
```