Analytical Dynamics

with an Introduction to Dynamical Systems

~ Advanced dynamics or Classical dynamics

(> Relativity ! : speed of light :

Modernmechanics, Quantum mechanics ~ Einstein)

o<u>Newtonian mechanics</u>(F) : A

->each particle : vector :

o<u>Analytical dynamics</u>(Energy) : B

->system as a whole :scalar !

(T , V ; Conservative system) Non-conservative system

% A > B or A = B

Merit or Dis-advantage ?

Vector ? ~ Scalar ?

6 Home works

Mid-term, Final tests

Conservative Systems

Systems of Particles

Motion in Non-inertial Reference Frames

Planar Motion of Rigid Bodies

Virtual Work

*** Problems**

2. Lagrangian Dynamics

Generalized Coordinates

Constraints

Holonomic Systems ~

Kinetic Energy and Generalized Momenta

Generalized Force

Lagrange's Equations of Motion

Consevative Systems

Lagrangian Systems

Dissipative Systems

Forces of Constraint

4. Dynamics of Rotating Bodies

5. Hamiltonian Systems

6. Stability Theory

Preface:

Principles of analytical mechanics

~mechanicalengr.physics and applied math.

- Topics in Advanced dynamics and elegant variational approachto formulating problems in mechanics.

Basically, time t is involved !

vSolid mechanics: Stiffness (PDE)

vDynamics: Mass(ODE)

- **Solid Mechanics + Dynamics = Real system !**

- **Nonlinear Partial Differential Equationsincludingt !**

Pre-requisites :basic dynamics, differential equations

Practice !!!

Chapter 1 PRINCIPLESOF DYNAMICS

F ~a: Inertia property ?

F = ma: Newton law(Principle) :

(Definition?) : why ? God only knows !

Truth - Experiment ; Verification, Reproduce!

~Stem cell ?

Logical path – theoretical + experimental works

SungsooBridge ?

Work ~ Force ;Scala~Vector : Dimension ?

Consuming a Force without doing a work ?

F *dr = 0 ?

Can the potential energy be defined without any restriction ?

<u>Goal for Analytical Mechanics</u>

Systematicmethod to derive exact T and V expressions for general system, and obtain the governing equations as PDE or ODE.

Solution ? : another problem..

We live in constantly changing universe. Why ?

Deep impact ,Tornado, Earthquake...

Atomic particle -- > Comet : Motion

The science of changing systems is known as dynamics

Change was passively accepted and used as a barometer to life.

In order to precise describe and predict the motion of bodies, mathematical technique were invented and increasingly used to model the observed changes.

In fact, the developments of dynamics and mathematics runs parallel.

Through the constant interaction, new mathematical

techniques and principles were discovered invented. : Developments based on mutual interaction !

Theory ~ Experiment..

18 th : Laplace:

19 th : Mathematics and Physics ....

20 th : Modern Technologies Age:

:Relativity theory and Quantum mechanics(Einstein..)

Mechanics

oClassical (Advanced) Mechanics :

Macroscopic world : Scientist and Engineers

oModern (Quantum) Mechanics :

Atomic size, Speed of light(Einstein) - Nanoscale...

But the Classical Mechanics is by no means invalidated

by these elegant and newer theories.

Macroscopic world around us is still extremely well modeled by the Newton's law.

Mechanics

Vector Mechanics ;Analytical(Lagrangian) Mechanics

(Force) ~(Scalar : Using T, V expressions)

free-body diagram~ system as a whole

Total Energy of a System

Generalized coordinate system

Complimentary

Basic Principles of Mechanics

Three basic laws,

1. Static equilibrium 2. Newton's law of motion

3. Action and reaction

Earth-fixed Coordinate : Inertial or Non-inertial ?

<u>KINEMATICS</u> :

Geometric expression only ~coordinate systems

Choice of the reference frame: x , dx/dt , d^2x/dt^2

Absolute velocity, acceleration

Trajectory ~ path

Speed

Global ~ Local coordinate

Local moving frame of reference

- **Normal and tangential coordinates ~ bi-normal**
- **Cylindrical (polar) coordinates**
- **Spherical coordinates**

COORDINATE TRANSFORMATIONS

TIME RATE OF CHANGE OF A UNIT VECTOR

Rotating coordinate system with angular velocity:

KINETICS :

Force ~ origin of motion !

WORK AND ENERGY

CONSERVATIVE SYSTEMS

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Local moving frame of reference

Rectangular coordinate system ~ Moving coordinate system ;orthogonality !

o Normal and tangential coordinates ~ bi-normal

r , v, a

o Cylindrical (polar) coordinates

r , v, a

o Spherical coordinates

r, v, a

COORDINATE TRANSFORMATIONS

(x, y, z)

(r, θ, z)

(r, θ, ϕ)

TIME RATE OF CHANGE OF A UNIT VECTOR

Rotating coordinate system with angular velocity:

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WORK AND ENERGY

CONSERVATIVE SYSTEMS

SYSTEMS OF PARTICLES

Single Particle ?~ Satellite about the Earth ?

All matter is in constant interaction with other matter.

Newton's law : originally formulated for a single particle !

~ > ? multiple

System of N particles each of mass m_j(j= 1,...,N)

***Definition: mass center* of a system of particles**

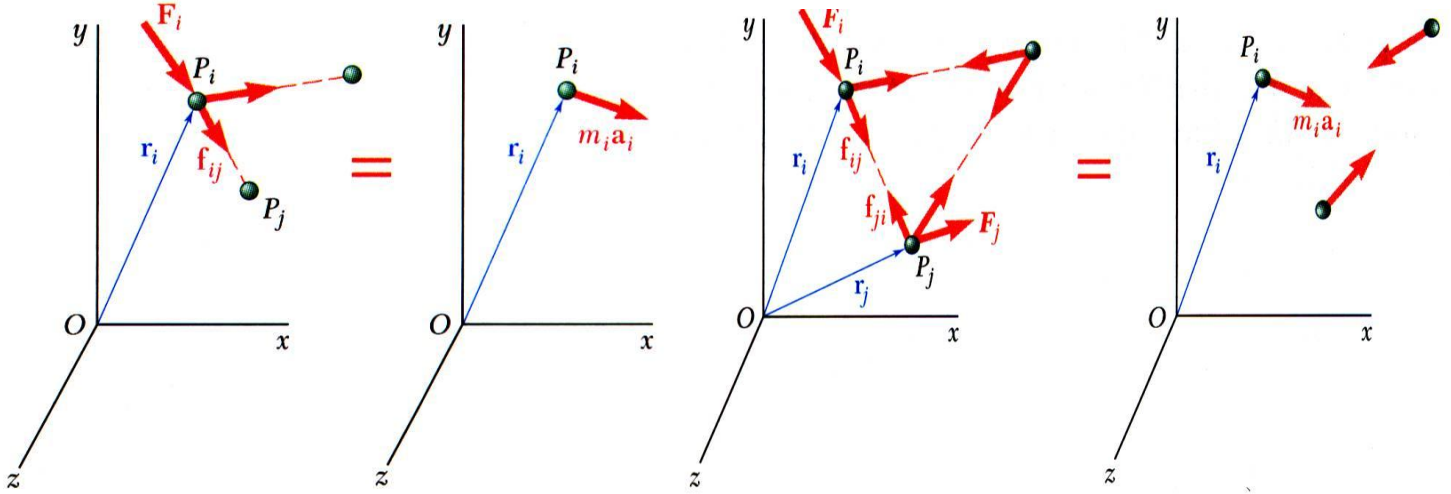
Global axes X-Y-Z-O

Total mass *M*~ for each m_j

Center of mass :Rc: Weighted average

Rc?

Newton's 2nd law



Newton's second law for each particle P_i in a system of *n* particles,

$$\vec{F}_i + \sum_{j=1}^{n} \vec{f}_{ij} = m_i \vec{a}_i ; \vec{r}_i \times \vec{F}_i + \sum_{j=1}^{n} \left(\vec{r}_i \times \vec{f}_{ij} \right) = \vec{r}_i \times m_i \vec{a}_i$$

$\vec{F}_i =$ external force $\quad \vec{f}_{ij} =$ internal forces $\quad m_i \vec{a}_i =$ effective force

- **Summing over all the elements,**

$$\sum_{i=1}^{n}\vec{F}_i+\sum_{i=1}^{n}\sum_{j=1}^{n}\vec{f}_{ij}=\sum_{i=1}^{n}m_i\vec{a}_i;\sum_{i=1}^{n}\left(\vec{r}_i\times\vec{F}_i\right)+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\vec{r}_i\times\vec{f}_{ij}\right)=\sum_{i=1}^{n}\left(\vec{r}_i\times m_i\vec{a}_i\right)$$

- **Since the internal forces occur in equal and opposite collinear pairs, the resultant force and couple due to the internal forces are zero,**

$$\sum\vec{F}_i=\sum m_i\vec{a}_i:\sum\left(\vec{r}_i\times\vec{F}_i\right)=\sum\left(\vec{r}_i\times m_i\vec{a}_i\right)$$

= > (1.50) !

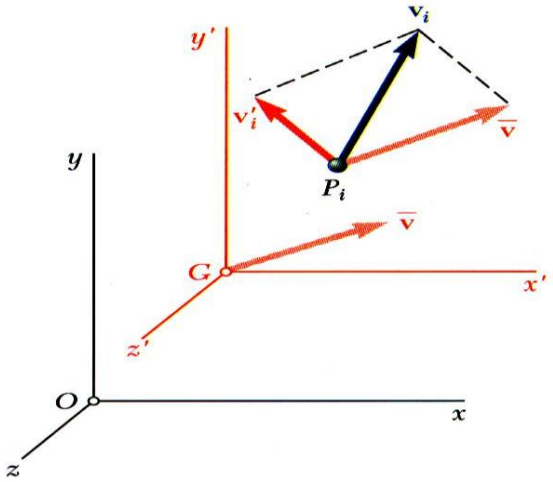
: Absolute coordinate system (X,Y)

Moving reference frame attached at Rc

R_j= Rc + r_j

Angular momentum ..

Kinetic Energy



- **Kinetic energy of a system of particles,**

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i\left(\vec{v}_i \bullet \vec{v}_i\right) = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2$$

Expressing the velocity in terms of the centroidal reference frame,

$$T = \frac{1}{2}\sum_{i=1}^{n}\left[m_i\left(\vec{v}_G + \vec{v}_i'\right)\bullet\left(\vec{v}_G + \vec{v}_i'\right)\right] = \frac{1}{2}\left(\sum_{i=1}^{n} m_i\right)v_G^2 + \vec{v}_G \bullet \sum_{i=1}^{n} m_i\vec{v}_i' + \frac{1}{2}\sum_{i=1}^{n} m_i v_i'^2 = \frac{1}{2}m\vec{v}_G^2 + \frac{1}{2}\sum_{i=1}^{n} m_i v_i'^2$$

- **Kinetic energy is equal to kinetic energy of mass center plus kinetic energy relative to the centroidal frame.**

- **Although $\vec{f}_{ij}$ and $\vec{f}_{ji}$ are equal and opposite, the work of these forces will not, in general, cancel out.**

MOTION IN NONINERTIAL REFERENTIAL FRAME

$$T = \frac{1}{2}MV_c^2 + \frac{1}{2}\sum_{j=1}^{N} m_j \dot{r}_j^2$$

Total Kinetic Energy=

Energy associated with the motion as a single entity (orbital kinetic energy)

+Energy of motion of theindividual particle about the mass center (spin kinetic energy)

Only for $\sum_{j=1}^{N} m_j \dot{r}_j = 0$

MOTION IN NONINERTIAL REFERENC FRAMES

Newton's law is based on the inertial reference frame:

~ Allow moving frame with constant (uniform) velocity

A noninertial reference frame:

With linear acceleration wrt some I.R.F.

or

With some angular velocity wrt an I.R.F.

~ Both will occur in various cases.

Any reference frame attached to the Earth:

Inertial reference frame ?

Acceptable assumption ? : negligible errors

Gun ? I.C.B.M. ?

: Most practical problems involve reference frames ~

~Non-inertial frame !

Objective of this section :
How dynamics can be formulated and analyzed using moving reference frame ?

Reference frame :

- oInertial reference : OXYZ with unit vector I,J,K.
- oAmoving (ω)reference frame :
  - oxyzwith unit vector i,j,k

Refer to Fig.1.28,

$r_p(t) = r_B(t) + r_{rel}(t)$

Then

$v_p = v_B + (\varpi \times r_{rel}) + v_{rel}$

~ Motion of a Particle P in a Box !

= ?

Acceleration:

$$\frac{d}{dt}\mathbf{v}_P = \frac{d}{dt}\mathbf{v}_B + \frac{d}{dt}(\boldsymbol{\omega} \times \mathbf{r}_{rel}) + \frac{d}{dt}\mathbf{v}_{rel}$$

$$\mathbf{a}_P = \mathbf{a}_B + (\dot{\boldsymbol{\omega}} \times \mathbf{r}_{rel}) + \boldsymbol{\omega} \times \frac{d}{dt}\mathbf{r}_{rel} + \frac{d}{dt}\mathbf{v}_{rel}$$

Remember

$$\frac{d}{dt}\mathbf{r}_{rel} = (\boldsymbol{\varpi} \times \mathbf{r}_{rel}) + \mathbf{v}_{rel}$$

$$\frac{d}{dt}\mathbf{v}_{rel} = (\boldsymbol{\varpi} \times \mathbf{v}_{rel}) + \mathbf{a}_{rel}$$

Then,

$$\mathbf{a}_p = \mathbf{a}_B + (\dot{\boldsymbol{\varpi}} \times \mathbf{r}_{rel}) + \boldsymbol{\varpi} \times \frac{d}{dt}\mathbf{r}_{rel} + \frac{d}{dt}\mathbf{v}_{rel}$$

Finally,

$$\mathbf{a}_p = \mathbf{a}_B + (\dot{\boldsymbol{\varpi}} \times \mathbf{r}_{rel}) + \boldsymbol{\varpi} \times (\boldsymbol{\varpi} \times \mathbf{r}_{rel}) + 2(\boldsymbol{\varpi} \times \mathbf{v}_{rel}) + \mathbf{a}_{rel}$$

This is based only on kinematics !

Absolute frame (Inertial frame : OXYZ)

~ Moving frame(Non-inertal frame : oxyz)

Keep in mind : Both observing the same particle (Remenber Fig.1.28)

Coriolisforce :

(PLANA MOTION OF RIGID BODY)

VIRTUAL WORK

Method of Virtual Work :

What is virtual(δ) ? What is actual (d) ?

What is statics ? What is dynamics ?

Time is involved !

Mechanics:

o Vector Mechanics : Free body diagram for isolated body

~ Reaction should be involved !Force !~ Vector !

o Analytical Mechanics , Computational ??

~ System as a whole !

Kinetic Energy and Generalized Momenta

- **Motion of a dynamical system:**

  Evolution of a single point in the configuration space.

- **Physical coordinate of the system vary with time:**

Generalized(~Mathematical) coordinates using

the transformation equations

! Space fixed and Body fixed ; Final goal ?

Up to now,

$(r,\theta,z) and (r,\theta,\phi)$ **: orthogonal coordinate system**

- Time rate of change of the generalized coordinates characterize the motion of a point along a curve in the configuration space.

-Generalizedvelocities of the system: dq_i/dt

~ Change of generalized coordinates w.r.t time:

: Car pendulumsystem (2 D model)

(q_1)distance, +(q_2)angular rotation angle.

$(x, \theta) \sim (q_1, q_2)$ **: body fixed ? orthogonal ?**

: Generalized coordinate !

➔ Generalized velocity are <u>not necessarily</u>

absolute velocity:Depend on ref..

Firstly,single particle in 3-D space : up to Eqn(2.17)

$$\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) \equiv> \frac{1}{2}m(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2)$$

Using index notation(without $\sum$),

$$\frac{1}{2}m(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2) = \frac{1}{2}m\dot{x}_i \bullet \dot{x}_i \quad (\text{for } i=1..3)$$: **Orthogonal system**

Physical coordinate:

$$x_i=x_i(q_1,q_2,q_3,t) \quad (2.15)$$

$q_i(i=1..3)$: **Generalized coordinate**

Applying chain rule for total derivatives to

the transformations (2.15)

Total derivative for *i= 1,,3*

For *i* = 1

$$\frac{dx_1}{dt} = \sum_{j=1}^{3} \frac{\partial x_1}{\partial q_j} \frac{\partial q_j}{\partial t} + \frac{\partial x_1}{\partial t} = \sum_{j=1}^{3} \frac{\partial x_1}{\partial q_j} \dot{q}_j + \frac{\partial x_1}{\partial t}$$

Naturally, (using summation convention, index notation)

$$\frac{dx_i}{dt} = \sum_{j=1}^{3} \frac{\partial x_i}{\partial q_j} \dot{q}_j + \frac{\partial x_i}{\partial t} \equiv \frac{\partial x_i}{\partial q_j} \dot{q}_j + \frac{\partial x_i}{\partial t} = x_{i,j} \bullet \dot{q}_j + x_{i,t} \quad \textbf{(2.16)}$$

If x_i is not an explicit function of time,

what happens?

Next, T = $\frac{1}{2} m \frac{dx_i}{dt} \frac{dx_i}{dt} = \frac{1}{2} m (x_{i,j} \bullet \dot{q}_j + x_{i,t})(x_{i,k} \bullet \dot{q}_k + x_{i,t})$

$$= \frac{1}{2} m x_{i,j} \bullet x_{i,k} \dot{q}_j \bullet \dot{q}_k + \frac{1}{2} m (x_{i,j} \bullet x_{i,t} \bullet \dot{q}_j + x_{i,k} \bullet x_{i,t} \bullet \dot{q}_k) + \frac{1}{2} m x_{i,t} \bullet x_{i,t})$$

2^{nd}term : index k -> j

Why ?

$$= \frac{1}{2}\alpha_{jk}\dot{q}_j \bullet \dot{q}_k + \frac{1}{2} \times 2 \bullet m \bullet x_{i,j} \bullet x_{i,t} \bullet \dot{q}_j + \frac{1}{2} m x_{i,t} \bullet x_{i,t})$$

$$= \frac{1}{2}\alpha_{jk}\dot{q}_j \bullet \dot{q}_k + \beta_j \dot{q}_j + \gamma \qquad \textbf{(2.17)}$$

For single particle !<~>Eqn(2.18) for N

Among the *N* particles, choose *i*-th particle, and then dq_i/dt can be expressed in terms of generalized coordinates ! ->

Just through a coordinate transformation by using a

Chain rule !!wrtx - >*q*

: Rate of change of physical coordinate x_i depends on the rate of change of the generalized coordinates. It may also depend on time *t* if the change of coordinates contain *t*explicitly. (* Moving frame)

Kinetic energy of N-particles :

➔ **In terms of generalized coordinate :**

Eqn(2.17) <~>Eqn(2.19)

T=T(q, dq/dt , *t*)

Actually depend on generalized coordinates ~

determined by the nature of transformation:

$$T = T_2 + T_1 + T_0$$

Homogeneous Quadratic + Linear + Constant

: Coordinate transformation does not

dependent on t → $T_1=T_0=0$

Generalized momentum

= Partial differentiationof T**wrt** dq_i/dt**:**

Kinetic energy/ Generalized velocity

Physical interpretation of a particular component of a generalized momentum p_i **dependson the nature of**

the corresponding generalized coordinates.

In 3–dimensional space :orthogonal bases

$\frac{1}{2}m(v_x^2+v_y^2+v_z^2)$ **~ quardratic function !** $(r,\theta,z),(r,\theta,\varphi).$

→linear momentum

Ex) Earth surface!

Generalized coordinates may be actual *x*-,*y*- and *z*-components of position

Using the definition, $p_x = mdx/dt$,p_y=.., p_z=..

In spherical coordinates, the kinetic energy is

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right.$$

Generalized coordinates : distance, two angles

Generalized momenta conjugate to these coordinates :

$$p_r = m\dot{r} \quad (linear..momentum)$$

$$p_\theta = mr^2 \cos^2 \phi \dot{\theta} \quad (angular..momentum)$$

$$p_\phi = mr^2 \dot{\phi} \quad (angular..momentum)$$

- **Based on geometric configuration ~**

→independent of the type of generalized coordinates !

Generalized Force

- **Vector mechanics :**

  Time rate of change of the momenta of a system ~ force, moment

- **Analytical mechanics:**

  Geometric relationships between generalized coordinates obscure the distinction between the two momentum !

***Energy*concept !**

Virtual Work due to the actual forces is defined as

$$\delta W = \Sigma F_i \bullet \delta r_i (i = 1...N) \qquad \textbf{(2.28)}$$

where $F_i = F_{ix} i$

Applying the chain rule,

$$\delta x_i = \Sigma \frac{\partial x_i}{\partial q_j} \delta q_j (j = 1...n) \qquad \textbf{(2.29)}$$

Virtual displacement is defined for *time is fixed*

$$\delta t = 0$$

The objective is to examine the instantaneous effect of a force on the generalized coordinates.

Each actual force drives the physical coordinates, which in turn will induce some resulting action on the generalized coordinates

Kinetic Energy and Generalized Momenta

- **Physical coordinate :Generalized(~Mathematical) !**

Space fixed and Body fixed ; Final goal ?

EX:

$(r,\theta,z) and (r,\theta,\phi)$ **: orthogonal coordinate system**

Eqn(2.17) <~>Eqn(2.19)

$$T=T(\ q,\ dq/dt\ ,\ t)$$

$$T = T_2 + T_1 + T_0$$

Homogeneous Quadratic + Linear + Constant

: Coordinate transformation does not dependent on t
→$T_1=T_0=0$

Generalized Force ~Work

***Energy*concept !**

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Virtual displacement :*time is fixed*

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Objective

Examine the instantaneous effect of a force on the generalized coordinates.

~Each actual force drives the physical coordinates -

Some resulting action on the generalized coordinates.

$$\delta x_i = \Sigma \frac{\partial x_i}{\partial q_j} \delta q_j \, (j = 1 \ldots n)$$

:virtual displacement of the physical coordinate δx_i

~simultaneous virtual displacements of the generalized coordinates δq_j

- **A virtual displacement of a single coordinate, δx_i,**

  ~ produce a simultaneous virtual displacement in some or all of the generalized coordinates.

As a result:

Virtual work done by a physical component of force F_{ix} under a virtual displacement δx_i ~virtual work done under the simultaneous combinations of virtual displacements δq_j :

$$F_{(i)x}\delta x_{(i)} = F_{(i)x}\frac{\partial x_{(i)}}{\partial q_j}\delta q_j \quad \textbf{(2.30)}$$

Individual terms in the summation ~

Contribution of the physical component F_{ix} along the directionof the generalized coordinate q_j.

Rearranging the terms and factoring out δq_j:

Total virtual work :

$$\delta W = \sum_{j=1}^{n} \left[\sum_{i=1}^{N} \left(F_{ix} \frac{\partial x_i}{\partial q_j} + F_{iy} \frac{\partial y_i}{\partial q_j} + F_{iz} \frac{\partial z_i}{\partial q_j} \right) \right] \delta q_j \quad \textbf{(2.31)}$$

N = ? n = ? : N = n ?

In terms of the simultaneous virtual displacements of the generalized coordinates, can be written as

$$\delta W = Q_j \delta q_j \quad \textbf{for} \quad j=1..n$$

What is Q_j **?**

~ This means what?

- **Generalized force Q_j is determined by computing the**

virtual work done under an infinitesimal change q_i whileleaving the other independent generalized coordinates fixed.

Imagine the system frozen at an arbitrary instant.

: Consider an arbitrary configuration of the system.

Rewriting the virtual as

$$\delta W = Q_i \delta q_i .(i = 1..n)$$

A generalized force Q_j contributes to δW only if the corresponding generalized coordinate q_j is given a virtual displacement. (independent !)

:Virtual work δW of the actual forces for each individual variation of only one of the generalized coordinates at a time.

Since the transformations are <u>invertible</u>, a single

variation δq_j will induce a simultaneous variation of one or more of the physical coordinates.

A virtual displacement of a generalized coordinate inphysical space ~

A combination of virtual displacements subjected to the constraints of the system.

Generally, the corresponding virtual work done by the physicalcomponents of the forces can be computed and set equal to $Q_{(i)} \delta q_{(i)}$.

Example: Figs.2.9, 2.10

Consider a spring-loaded cart with a swinging pendulum attached to it.

This system has two degrees of freedom.

Chosen x and θ as generalized coordinates.

Since x and θ are independent variables,

$$\delta x = \delta x, \qquad \delta\theta = 0 \quad \textbf{(2.34)}$$

And $\delta x = 0, \delta\theta = \delta\theta$ **(2.35)**

are two sets of admissible virtual displacements.

Now compute the corresponding virtual work done by the external forces under each of the designated virtual displacements:

If $\delta x \neq 0$ **and** $\delta\theta = 0$

$$\delta W = -F_s \delta x \quad \textbf{(2.36)}$$

If $\delta x = 0$ **and** $\delta\theta \neq 0$ **(Fig. 2.10),**

$$\delta W = -mgl \sin\theta \delta\theta \quad \textbf{(2.38)}$$

$$\therefore Q_\theta = -mgl \sin\theta \textbf{ (torque)} \qquad \textbf{(2.39)}$$

Then for an arbitrary combination of virtual

displacements, the total virtual work is

$$\delta W = -kx\delta x - mgl\sin\theta\delta\theta \quad \textbf{(2.40)}$$

Note:

Physical interpretation of a generalized force depends on the significance of the related generalized coordinate.

Once a given set of generalized coordinates are specified, the generalized forces can in principle always be

determined, regardless of the physical interpretation of the generalized coordinates.

~Holonomic systems, the computation of generalized forces is very simple. :Virtual work done byholonomic constraint forces under a set of arbitrary virtual displacements compatible with the constraints is equal to zero.

Therefore, in the computation of generalized forces, only the applied forces need to be considered.

This results in a considerable benefit in the formulation of the equations of motion in terms of the generalized coordinates.

Special consideration may be given to conservative forces. Suppose that all the forces acting on a system of *N* particles are conservative. Each physical force is

derivable from a potential function.

Suppose : a single potential function :

$$V = V(x_1, y_1, z_1, x_2, y_2, z_2, \cdots, x_N, y_{N,} z_N)$$

The force on the *i*th particle may be obtained as

$$F_i = -\nabla_i V$$

Where the gradient ∇_i denote the operator

$$\nabla_i = \frac{\partial}{\partial x_i}\hat{\mathrm{i}} + \frac{\partial}{\partial y_i}\hat{\mathrm{j}} + \frac{\partial}{\partial z_i}\hat{\mathrm{k}}$$

Substituting the physical components of the forces (2.42) into Equation (2.31) results in the characterization of the virtual work as the negative of the variation of the potential function :

$$\delta W = -\sum_{i=1}^{N}\left(\frac{\partial V}{\partial x_i}\delta x_i + \frac{\partial V}{\partial y_i}\delta y_i \frac{\partial V}{\partial z_i}\delta z_i\right) = -\delta V \quad \textbf{(2.44)}$$

Thus the virtual work done by a collection of

conservative forces, under specified virtual displacements, is given as the negative of the variation of potential energy.

Principle of Virtual Work:

A conservative system is in static equilibrium iffthe totalpotential energy of the system is stationary

$$\delta V = 0$$

Suppose :Single potential function

$$V = V(x_1, y_1, z_1, x_2, y_2, z_2, \cdots, x_N, y_{N,} z_N)$$

Force on *i*-th particle may be obtained as

$$F_i = -\nabla_i V \quad \textbf{(2.42)}$$

in here

$$\nabla_i = \frac{\partial}{\partial x_i}\hat{\mathbf{i}} + \frac{\partial}{\partial y_i}\hat{\mathbf{j}} + \frac{\partial}{\partial z_i}\hat{\mathbf{k}} \equiv$$

SubstituteEqn.(2.42) into δW : Eqn.(2.31) ,

$$\delta W = -\sum_{i=1}^{N}\left(\frac{\partial V}{\partial x_i}\delta x_i + \frac{\partial V}{\partial y_i}\delta y_i \frac{\partial V}{\partial z_i}\delta z_i\right) = -\delta V \quad \textbf{(2.44)}$$

: Virtual work done by conservative forces ~

Negative of the variation of potential energy.

<u>Principle of Virtual Work:</u>

A conservative system is in static equilibrium iffthe

totalpotential energy of the system is stationary

(MINI ?MAX ?)

$$\delta V = 0$$

Principle of Stationary Potential Energy~

Necessary and sufficient condition for static equilibrium of a conservative system.

Transforming to generalized coordinates :

Total P.E. of a conservative system as

$$V = V(q_1, q_2, \cdots, q_n) \quad \textbf{(2.46)}$$

Consequently, the variation of the P.E. function in terms of δq_j **is: Eqn(2.47)**

or

$$\delta W = -\sum_{j=1}^{n} \frac{\partial V}{\partial q_j} \delta q_j \equiv -V, q_j \delta q_j \quad (\textit{for } j=1..n) \quad \textbf{(2.48)}$$

For a conservative system:

Generalized forces ~ also derivable from a potential function in terms of the generalized coordinates q_j.

That is,

$$Q_j = -\frac{\partial V}{\partial q_j} \quad \textbf{(2.49)}$$

Therefore the determination of generalized forces for conservative systems is very easy (?)

Usingtransformation of coordinates

~As a final step in the derivation of equations of motion

After the change of variables has been consummated, we will only need to keep the final result.

LAGRANGE'S EQUATIONS OF MOTION

Up to now, we consider the connection between physical variable and generalized coordinates based on the

geometric configuration of a system(admissible !).

Especially, generalized coordinates compatible with the constraints make the kinematics much more manageable forholonomic systems

We are now in a position to make the transition between vector mechanics and analytical mechanics.

Instead of using free-body diagrams :

Based on the variation of energy and the minimum number of coordinates needed to characterize the dynamics of the system (always possible ?).

: Lagrangiandynamics !!

Kinetic energy, potential energy, and virtual work are all scalar quantities. Thus, the transformation of these

quantities is rather straightforward.

Based on a system q_j instead of the physical coordinates r_i.

-A unified approach in a way that is *independent* of any particular coordinate system or set of generalized coordinates.

For a system of N particles subjected to only holonomic constraints. The more general case will be considered

later.

Assume a system with *n* degrees of freedom and that there is a transformation :

For the *i* th particle ina vector form as

$$m_i \mathbf{a}_i = \mathbf{F}_i \quad \textbf{(2.50)}$$

or

$$\frac{d\mathrm{p}_i}{dt} = F_i \quad \textbf{(2.51)}$$

: linear momentum of the *i*-th particle as

$$\mathrm{p}_i = m_i \dot{\mathrm{r}}_i \qquad \textbf{(2.52)}$$

Find out how the equations of motiontransform under the transformation to generalized coordinates.

$\frac{d}{dt}(..)$**? :**

Generalized momentum corresponding to the *k* th generalized coordinate is given by

$$\dot{p}_k = \frac{d}{dt}(p_k) = \frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_k}) \quad \textbf{(2.53)}$$

By definition, the total kinetic energy of the system is

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) \quad \textbf{(2.54)}$$

then the generalized momentum p_k as

$$p_k = \frac{\partial T}{\partial \dot{q}_k} = \sum_{i=1}^{N} m_i(\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_k} + \dot{y}_i \frac{\partial \dot{y}_i}{\partial \dot{q}_k} + \dot{z}_i \frac{\partial \dot{z}_i}{\partial \dot{q}_k}) \quad \textbf{(2.55)}$$

Remember the chain rule :

$$\dot{x}_i = \sum_{j=1}^{N} \frac{\partial x_i}{\partial q_j} \dot{q}_j + \frac{\partial x_i}{\partial t}$$

Then take derivative wrt $\dot{q}_k$:

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_k} = \frac{\partial x_i}{\partial q_k} \quad \textbf{(2.57)}$$

Thus, each component p_k can be expressed as Eqn.(2.58)

Taking the total time derivative of Eqn (2.58) and

applying the product rule to the terms in the summation

(Remember : $\frac{d}{dt}(x\dot{y}) = \dot{x}\dot{y} + x\ddot{y}$ **)**

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \sum_{i=1}^{N} m_i \left(\ddot{x}_i \frac{\partial x_i}{\partial q_k} + \ddot{y}_i \frac{\partial y_i}{\partial q_k} + \ddot{z}_i \frac{\partial z_i}{\partial q_k}\right)$$

$$+\sum_{i=1}^{N} m_i \left[\dot{x}_i \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_k}\right) + \dot{y}_i \frac{d}{dt}\left(\frac{\partial y_i}{\partial q_k}\right) + \dot{z}_i \frac{d}{dt}\left(\frac{\partial z_i}{\partial q_k}\right)\right]$$

Remind the terms in the first summation as

the Newton's Second Law

$$m_i \ddot{x}_i = F_{ix} \; m_i \ddot{y}_i = F_{iy} \; m_i \ddot{z}_i = F_{iz}$$

Thus the terms can be rewritten as

$$\sum_{i=1}^{N} m_i (\ddot{x}_i \frac{\partial x_i}{\partial q_k} + \ddot{y}_i \frac{\partial y_i}{\partial q_k} + \ddot{z}_i \frac{\partial z_i}{\partial q_k}) = \sum_{i=1}^{N} (F_{ix} \frac{\partial x_i}{\partial q_k} + F_{iy} \frac{\partial y_i}{\partial q_k} + F_{iz} \frac{\partial z_i}{\partial q_k})$$

where the right-hand side ~ generalized force Q_k given by the transformation equations.

To interpret the second summation terms in Eqn(2.59), note that

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_k}\right) = \sum_{j=1}^{n} \frac{\partial^2 x_i}{\partial q_j \partial q_k}\dot{q}_j + \frac{\partial^2 x_i}{\partial t \partial q_k}$$

$$= \frac{\partial}{\partial q_k}\left[\sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j}\dot{q}_j + \frac{\partial x_i}{\partial t}\right] = \frac{\partial}{\partial q_k}\left[\dot{x}_i\right] \equiv \dot{x}_{i,k}$$

Thus the time rate of change of the *k*th generalized

momentum is given byEqn(2.60)

Finally, the equations of motion in terms of q_k:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k \quad k = 1, 2, \ldots, n \quad \textbf{(2.61)}$$

: General form of Lagrange's Equations of Motion

There is one equation corresponding to each q_k.

The system of equations represents a coupled system of

Thusthe time rate of change of the *k*th generalized momentum is given byEqn(2.60)

Finally, the equations of motion in terms of q_k:

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: General form of Lagrange's Equations of Motion

There is one equation corresponding to each q_k.

The system of equations represents a coupled system of ordinary equations governing the evolution of the dynamical system in terms of the *n* generalized coordinates. ~ Finite D.O.F !

Continuous system (such as beam, plate and shell): PDE !

Alternatively, Lagrange's equations of motion may be

written in terms of the generalized momenta as

$$\frac{d}{dt}(p_k) - \frac{\partial T}{\partial q_k} = Q_k, \qquad k = 1, 2, \ldots, n$$

This means that Newton's Second Law (2.51) is transformed under a change of variables to generalized coordinates $q_1, q_2, \ldots q_n$ **.**

Hence Newton's Second Law is *not* invariant under an

arbitrary change of variables. The extra term represents inertial effects induced by the coordinate transformations.

Lagrange's equations allow the formulation of the equations of motion, independent of the physical significance of the variables.

Note that the dynamics of the system is thus

characterized by the kinetic energy and the virtual work done by generalized forces.

The hallmark of the Lagrangian formulation is that the energy contains the dynamic information.

The use of generalized coordinates, compatible with the constraints, results in the minimum number of variables needed to completely describe the motion.

Furthermore, for generalized coordinates adopted to the constraints, the forces of constraint do not contribute to the virtual work.

Hence the reactions do not appear in the resulting equations of motion.

Ex: a simple pendulum (Fig 2.11).

Assume that a particle of mass *m* is attached to a

massless rod that is free to rotate in a vertical plane about a frictionless pin.

The motion of this single-degree-of-freedom system may be described by the generalized coordinate θ.

The Kinetic energy of the system is given in terms of the generalized velocity $\dot{\theta}$ as

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m l^2 \dot{\theta}^2$$

From a previous example, the generalized force associated with the rotational coordinate of a pendulum was derived, based on virtual work, as

$$Q_\theta = -mgl\sin\theta$$

The equation of motion based on the Lagrangian formulation is therefore represented by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta$$

That is,

$$\frac{d}{dt}\left(ml^2\dot{\theta}\right) - 0 = -mgl\sin\theta$$

which can be set into the more familiar form

$$\dot{\theta} + \frac{g}{l}\sin\theta = 0$$

The systematic approach of the Lagrangian formulation

is evident in this example.

The formulation is based on the Kinetic energy and the virtual work.

Since the variable θ is adopted to the constraint of circular motion, the equation of motion has been set up without need to consider the force of constraint acting on the particle.

The constraint force is in fact the tension in the cable.

Ex :

Consider the two-degree-of-freedom system consisting of two carts coupled by linear elastic springs. (Fig. 2.12) The generalized coordinates q_1 and q_2 represent the displacements of the carts from the unstretched configurations of the springs. The kinetic energy is

readily formulated as

$$T = \frac{1}{2} m_1 \dot{q}_1 + \frac{1}{2} m_2 \dot{q}_2$$

The generalized forces can be deduced by the method of virtual work.

Then

$$Q_1 = -k_1 q_1 + k_2 (q_2 - q_1), Q_2 = -k_2 (q_2 - q_1)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right)-\frac{\partial T}{\partial q_1}=Q_1, \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right)-\frac{\partial T}{\partial q_2}=Q_2 \quad \textbf{(2.62)}$$

The equations of motion (2.62) may be simplified and put in standard form as

$$m_1\ddot{q}_1+(k_1+k_2)q_1-k_2q_2=0$$

$$m_2\ddot{q}_2-k_2q_1+k_2q_2=0$$

In a Matrix Form ?

CONSERVATIVE SYSTEMS

Lagrange's equations of motion represent a unified approach to deriving the governing equations of a dynamical system.

Equations (2.61) are completely general, in that they apply generically to all mechanical systems.

The governing equations are based on the total Kinetic

energy of a system and the generalized forces derived by the method of virtual work.

Only generalized forces directly affecting the generalized coordinates contribute to the virtual work.

Lagrange's equations of motion may also be expressed in several alternate forms, depending on the nature of the generalized forces.

For a *conservative system*, there exists a potential function in terms of the generalized coordinates

$$V = V(q_1, q_2, \ldots, q_n)$$

from which the generalized forces can be derived as

$$Q_k = -\frac{\partial V}{\partial q_k} \quad \textbf{(2.63)}$$

Substituting the generalized force (2.63) into

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = \frac{\partial V}{\partial q_k} \quad \textbf{(2.64)}$$

Since the potential function only depends on the generalized coordinates,…

Thus

$$\frac{\partial T}{\partial \dot{q}_k} = \frac{\partial (T - V)}{\partial \dot{q}_k}$$

Rewriting Lagrange's equations (2.64) results in

$$\frac{d}{dt}\left[\frac{\partial(T-V)}{\partial \dot{q}_k}\right]-\frac{\partial(T-V)}{\partial q_k}=0$$

This version of the equation has a particularly simple form.

The scalar quantity in the parentheses is defined as the Lagrangian function:

$$L(q,\dot{q},t)=T(q,\dot{q},t)-V(q)$$

It is a function of the generalized coordinates and velocities.

The Lagrangianrepresents the difference between the totalKinetic energy and the total Potential energy of a conservative system.

The equations of motion (2.61) can thus be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

which is the standard form of Lagrange's equations of motion for conservative systems.

A formulation based on the Lagrangian is convenience that allows by-passing the determination of generalized forces from the method of virtual work.

It is interesting to note that for a conservative system all the dynamics are characterized by a single scalar function, the Lagrangian of the system.

The Lagrangian function simplifies the equations of motion and often aids in the understanding of the dynamics of the system.

Practices!

1. **A particle of mass m is suspended by a massless wire of length** $r = a + b\cos\omega t..(a > b > 0)$ **to form a spherical pendulum. Find the equation of motion.**

   Sol) **T ~ p.102,Eqn.(2.24), V =** $mgr\cos\theta$

2. **A particle of mass m can slide without friction on the inside of a small tube which is bent in the form**

of a circle of radius *r*. The tube rotate about a vertical diameter with a constant angular velocity ω.

Write the equation of motion.

Sol) T= $\frac{1}{2}mr^2(\dot{\theta}^2+\omega^2\sin^2\theta)$**, V=** $mgr\cos\theta$**, L=T-V**

3. A particle of mass m can slide on a smooth wire having the form $y=3x^2$**, wherethe gravity acts in the**

direction of the negative y-axis.

Obtain the equations of motion.

Sol) T= $\frac{1}{2}m(\dot{x}^2+\dot{y}^2)$**, V=** mgy **with** $y=3x^2$

Insert : y ~ Finally,

4. Text : p.120

LAGRANGIAN SYSTEMS

Most dynamics problems ~ Holonomic !

(:Not all systems are conservative)

Aconservative force -Derivable from a potential energy

(Depending only on the spatial coordinates of a system)

- **Lagrangian can be constructed and the dynamics of the system is contained in the Lagrangian.**

But there may still be a scalar function from which the generalized components of a force may be derived.

Suppose :

A scalar function $V(q,\dot{q},t)$ **for a generalized force** Q_k **as in Text !**

~ We call $V(q,\dot{q},t)$ **as a generalized potential function.**

Substituting the generalized force (2.65), then

Lagrange's equations (2.61) results in

$$\frac{d}{dt}\left[\frac{\partial T(q,\dot{q},t)}{\partial \dot{q}_k}\right]-\frac{\partial T(q,\dot{q},t)}{\partial q_k}=\frac{d}{dt}\left[\frac{\partial V(q,\dot{q},t)}{\partial \dot{q}_k}\right]-\frac{\partial V(q,\dot{q},t)}{\partial q_k} \quad \textbf{(2.66)}$$

Now, we can still define a Lagrangian function in terms of the kinetic energy of the system and the *generalized* potential function as $L(q,\dot{q},t)=T(q,\dot{q},t)-V(q,\dot{q},t)$

By setting all terms to the left-hand side,

$$\frac{d}{dt}\left[\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}_k}\right]-\frac{\partial L(q,\dot{q},t)}{\partial q_k}=0 \quad \textbf{(2.67)}$$

:Identical to Lagrange's eqns for conservative systems.

Note:Unless the potential function depends *only*on the generalized coordinates, the system governed by Equation (2.67) is *not*conservative.

Holonomic systemsderivable from a generalized potential function $V(q,\dot{q},t)$ **are known as Lagrangian systems.**

A well-known exampleof a velocity-dependent potential

-A charged particle in an electromagnetic field.

The force on the particle is given by

$$-e\nabla\phi-\frac{e}{c}\{\dot{A}-v\times curlA\}$$

: e - charge carried by the particle,

ϕ scalar potential,

A vector potential of the field.

The electromagnetic force field is derivable from the generalized potential

$$V(r,\dot{r}) = e\phi(r) - \frac{ev \cdot A}{c}$$

Not all systems are Lagranian, although all generalized forces~Conservative or Non-conservative

: Depending on the nature of the actual forces acting on a system.

The virtual work done by a generalized force Q_k under a virtual displacement δ_{q_k} can be considered

$$Q_k \delta_{q_k} = \delta W_k^{cons} + \delta W_k^{nc}$$

Resultant generalized force associated with a generalized coordinate q_k can thus be spilt into two contributions:

…

Using conservative component such aspotential function,

Then, each generalized force may be decomposed as

$$Q_k = -\frac{\partial V}{\partial q_k} + Q_k^{nc}$$

Construct the Lagrangian function $L = T - V$ **and formulate Lagrange's equations of motion, in hybrid form, as**

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k^{nc}, \qquad k = 1, 2, \ldots, n$$

Here Q_k^{nc} **represent generalized forces *not derivable* from a potential function.**

DISSIPATIVE SYSTEMS

Are all forces derivable from a potential function ?

~ Forces due to dissipation of energy

: frictionforce isnon-conservative,

Nevertheless, some non-conservative generalized forces may still be derivable from yet another scalar function.

- **Components proportional to the velocities of the particles**

$$F_{ix} = -c_{x_i}\dot{x}_i, \qquad F_{iy} = -c_{y_i}\dot{y}_i, \qquad F_{iz} = -c_{z_i}\dot{z}_i$$

The virtual work done by these dissipative forces under a set of virtual displacements is

$$\delta W = \sum_i \mathrm{F}\bullet\delta\mathrm{r} = -\sum_{i=1}^{N}(c_{x_i}\dot{x}_i\delta x_i + c_{y_i}\dot{y}_i\delta y_i + c_{z_i}\dot{z}_i\delta z_i)$$

$$= -\sum_{i=1}^{N}\left[\sum_{k=1}^{n}(c_{x_i}\dot{x}_i\frac{\partial x_i}{\partial q_k}+c_{y_i}\dot{y}_i\frac{\partial y_i}{\partial q_k}+c_{z_i}\dot{z}_i\frac{\partial z_i}{\partial q_k})\delta q_k\right]$$

$$= -\sum_{k=1}^{N}\left[\frac{1}{2}\sum_{k=1}^{N}\frac{\partial}{\partial \dot{q}_k}(c_{x_i}\dot{x}_i^{\,2}+c_{y_i}\dot{y}_i^{\,2}+c_{z_i}\dot{z}_i^{\,2})\right]\delta q_k$$

Generalized forces associated with the dissipation forces

$$Q_k^{nc} = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial}{\partial \dot{q}_k}(c_{x_i}\dot{x}_i^2+c_{y_i}\dot{y}_i^2+c_{z_i}\dot{z}_i^2)$$

…

Now, define a scalar function for*generalized velocities*

$$D = \frac{1}{2}\sum_{i=1}^{N}(c_{x_i}\dot{x}_i^2 + c_{y_i}\dot{y}_i^2 + c_{z_i}\dot{z}_i^2)$$

Thus the dissipative generalized forcesin terms of *D* :

$$\delta W = \sum_{k=1}^{n} Q_k^{nc}\delta q_k = -\sum_{k=1}^{n}\frac{\partial D}{\partial \dot{q}_k}\delta q_k$$

***D*: Rayleigh's Dissipation Function**

Finally,the most general form of Lagrange's equations of motionas

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} + \frac{\partial D}{\partial \dot{q}_k} = Q_k^* : (2.68)$$

where $L = T - V$ **:Lagrangian,*D*:Dissipation function**

Q_k^* **:Generalized force** ***not derivable from a potential function or a dissipation function*.**

Note : Rayleigh's dissipation function ~one-half the rate at dissipated energy : average loss of power in a non-conservative system.

Ex :A simple spring-mass system as in Fig.2.13

Additional loading with a viscous damper,a harmonically applied forcing function

Lagrangian of the system? $L = T - V$

$$L = \frac{1}{2} m\dot{q}^2 - \frac{1}{2} kq^2$$

The dissipation function for viscous damper :

$$D = \frac{1}{2} c\dot{q}^2$$

Another generalized force ~ applied harmonic force

Substituting into Eqn (2.68) :

$$\frac{d}{dt}(m\dot{q}) - (-kq) + c\dot{q} = A\cos\omega_f t$$

which can be put into standard form as

$$m\ddot{q} + c\dot{q} + kq = A\cos\omega_f t$$

Ex:Simple pendulum with pin friction as inFig 2.14

Assume :Pin exerts a *resisting* moment proportional to the angular velocity of the pendulum:

$$M_f = -\nu\dot{\theta}$$

Instantaneous rate of energy loss

$$P = M_f \dot{\theta}$$

Dissipation function (average power lost) is

$$D = \frac{1}{2} \nu \dot{\theta}^2$$

Since $Q_\theta^* = 0$**, the equation of motion is**

$$\frac{d}{dt}(ml^2\dot{\theta}) - (-mgl\sin\theta) + \nu\dot{\theta} = 0$$

FORCES OF CONSTRAINT

Lagrangianformalism:Highlighted by two main features

It has been demonstrated !

Part of the advantage:Constraint forces do no virtual work under a set of virtual displacements compatible with the constraints.

Constraints reduce the number of degrees of freedom.

The constraint forces themselves do not appear in the equations of motion : Symmetry of a system !

Holonomic systems can be described by a set of independent generalized coordinates free of constraints.

Systems with non-holonomic constraints cannot be reduced to independent generalized coordinates.

The equations of motion must be augmented by the Constraints ~>Forces of constraint are also established. Constraint forces in holonomic systemsmay also be analyzed.

Only realize that constraints are enforced by reacting forcesin the directions normal to the constraint surfaces

Physically, a constraint must be imposed in the form of forces or moments. Thus we associate constraints with additional generalized forces acting on the system. These forces depend on the motion and cannot be found priorto solving the equations of motion.

Each holonomic constraint can in principle be replaced by a *reacting constraint force*. Additional degrees of

freedom may be introduced onto the problem by adding generalized coordinates corresponding to the violation of the constraints.

These additional coordinates are called superfluous coordinates. The generalized forces associated with the superfluous coordinates are the forces of constraint. If the original coordinates and the extra coordinates are

FORCES OF CONSTRAINT

Lagrangianformalism:

Generalized coordinateMinimum set of Eqns

Part of the advantage:Constraint forces do no virtual work under a set of virtual displacements compatible with the constraints. Generally,

Constraints reduce the number of degrees of freedom.

The constraint forces do not appear in theeqns of motion : Symmetry of a system ?

~Holonomic systems can be describedin terms of independent generalized coordinates free of constraints.

~Non-holonomic constraintscannot be reduced to independent generalized coordinates.

Eqns of motion must be augmented by theConstraints

~>Forces of constraint are also established.

***Constraint forces in holonomic systemsmay also be analyzed.**

: Constraints are enforced by reacting forces in the directions normal to the constraint surfaces

Physically, a constraint must be imposed in the form of forces or moments. ~>Constraints with additional generalized forces acting on the system. These forces depend on the motion and cannot be found prior to solving the eqns of motion.

: Should be solved simultaneously

Problems with or without constraint ?

~Holonomic constraint can in principle be replaced by a *reacting constraint force*. - Additional dof may be introduced onto the problem by adding generalized coordinates(superfluouscoordinates)corresponding to theviolation of the constraints.

The generalized forces associated with the superfluous coordinates are the forces of constraint.

In case, original coordinates and the extra coordinates are considered as independent, then the resulting eqns of motion will contain the constraint forces.

These forces will only be in the eqns associated with the superfluous coordinates. After the eqns of motion are set up, the superfluous coordinates are set to constant values.

Setting up the problem this way results in eqns involving the constraint forces and also gives the values of these forces necessary to enforce the given constraints.

For non-holonomic constraints, the eqns of motion are formulatedusing Lagrange multiplier method.

Suppose: *n* generalized coordinates $q_1, q_2, \ldots, q_n$ **is restricted(?) by anon-holonomic constraint:**

$$A_1 dq_1 + A_2 dq_2 + \cdots + A_n dq_n + A_0 dt = 0$$

Since the variations take place without increment in time, $\delta t = 0$, the resulting eqn of constraint for the virtual displacements becomes

$$A_1 \delta q_1 + A_2 \delta q_2 + \cdots + A_n \delta q_n = 0 \quad \textbf{(2.69)}$$

Geometrically, Eqn (2.69) defines a direction orthogonal

to the virtual displacement δq **. (Vector form ?)**

Thus the constraint force is a *scalar multiple* of the vector

$(A_1, A_2, \ldots, A_n)$ **. This scalar is a function of time** $\lambda(t)$ **.**

Total generalized force acting on the generalized

coordinate q_k**, including applied and reacting forces, is**

$$Q_k + \lambda A_k$$

The resulting eqns of motion for non-

holonomicsystemsare:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k + \lambda A_k, \qquad k = 1, 2, \ldots, n \qquad \textbf{(2.70)}$$

Eqns (2.70) together with (2.69) represent *n+1* equations in*n+1* unknowns, including the *Lagrange multiplier* **. These eqns are solved simultaneously. In addition to solving for the generalized coordinates, the solution gives the component of the reacting *constraint force* **.

Generalization: System is subjected to *J*non-holonomic constraints given by

$$A_{j1}\dot{q}_1 + A_{j2}\dot{q}_2 + \cdots + A_{jn}\dot{q}_n + A_{j0} = 0 \quad \textbf{(2.71)}$$

or equivalently as

$$A_{j1}dq_1 + A_{j2}dq_2 + \cdots + A_{jn}dq_n + A_{j0}dt = 0$$

where*j* ranges from 1 to the number of such constraints *J*.Coefficients A_{jk} may be functions of the generalized

coordinates and time. Introduce *J* Lagrange multipliers, $\lambda_j(t)$, one for each constraint eqn (2.71).

~Total generalized force driving the *k*-th generalized coordinate is

$$Q_k + \sum_{j=1}^{J} \lambda_j A_{jk}$$

Thus,eqn of motion for each generalized coordinate q_k:

$$\int \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} = Q_k + \sum_{j=1}^{J} \lambda_j A_{jk} \right] \delta q_k \quad \textbf{(2.72)}$$

The set of eqns (2.72) together with the *J* eqns of constraint (2.71) constitute *n* + *J*eqns in *n* + *J*unknowns. These eqns must be solved *simultaneously* for the generalized coordinates and the *J* Lagrange multipliers $\lambda_j(t)$**. The generalized constraint force reacting on the coordinateq_k:**

$$R_k = \sum_{j=1}^{J} \lambda_j A_{jk}$$

Method of Lagrange multipliers may also be applied to systems with holonomic constraints. Recall that a holonomic constraint $f(q_1,....q_n,t) = const$ **may be converted to differential form as**

$$\frac{df}{dt} = \frac{\partial f}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial f}{\partial q_2}\frac{dq_2}{dt} + \cdots + \frac{\partial f}{\partial q_n}\frac{dq_n}{dt} + \frac{\partial f}{\partial t} = 0$$

This is the same form as a non-holonomic constraint

(2.71), with the coefficients

$$A_{jk} = \frac{\partial f}{\partial q_k} \quad A_{j0} = \frac{\partial f}{\partial t}$$

Thus holonomic systems with constraints can also be analyzed, as well as systems having constraints of *both* types.

As an example, (p.120):Consider the dynamics of a

particle constrained to slide on a frictionless wire. This wire is in the shape of a parabola that is rotating about its axis of symmetry with constant angular velocity

---

Cylindrical coordinates are intrinsic to this problem. There is only one degree of freedom, namely the position of the mass on the wire. The two constraints are $z = br^2$ **,** **where** ***b*** **is some constant, and** $\dot{\theta} = \omega$ **. These constraints are**

holonomic, which imply constraints of the form (2.71)

as

$$\lambda_1(\delta z - 2br\delta r) = 0 \text{ and } \lambda_2\delta\theta = 0$$

The coefficients in (2.71) are seen in matrix form as

$$\begin{bmatrix} -2br & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} \delta r \\ \delta\theta \\ \delta z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence there will be two Lagrange multipliers – one for

each constraint. The Lagrangian of the system is

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - mgz$$

The equations of motion are

$$m\ddot{r} - mr\dot{\theta}^2 = -2b\lambda_1 r$$

$$\frac{d}{dt}(mr\dot{\theta}^2) = \lambda_2$$

$$m\ddot{z} + mg = \lambda_1$$

Now since $\dot{\theta} = \omega$**, there are four unknowns** $r(t), z(t), \lambda_1(t)$**, and**

$\lambda_2(t)$. :**Three eqns of motion and theconstraint eqn** $z = br^2$.

Eliminating the multiplier

$\lambda_1(t)$ **results in the equation**

$$\ddot{r} - r\omega^2 = -2br(\ddot{z} + g)$$

Differentiation of the constraint $z = br^2$ **results in**

$$\ddot{z} = 2b\dot{r}^2 + 2br\ddot{r}$$

and so we end up with the single differential equation for

$r(t)$ **as**

$$\ddot{r}(1+4b^2r^2)+4b^2r\dot{r}^2=r(\omega^2+2bmg)-(2.73)$$

The entire analysis reduces to the solution of Eqn (2.73).

The coordinate $z(t)$ **is obtained from the constraint eqn. The two Lagrange multipliers are also given in terms of *r* and *z* from the eqs of motion.**

Finally, the *torque* required to maintain the uniform rotation is

$$\lambda_2 = 2m\omega r\dot{r}$$

And the components of the reacting *constraint force* exerted by the wire on the mass are

$$R_r = -2b\lambda_1(t)r(t) \quad \text{and} \quad R_z = \lambda_1(t)$$

Practice !

As an example, (p.120):Consider the dynamics of a particle constrained to slide on a frictionless wire. This wire is in the shape of a parabola that is rotating about its axis of symmetry with constant angular velocity

---

Cylindrical coordinates are intrinsic to this problem.

There is only one degree of freedom, namely the position of the mass on the wire. The two constraintsare $z = br^2$ **,**

where *b* is some constant, and $\dot{\theta} = \omega$. These constraints are holonomic, which imply constraints of the form (2.71)

as

$$\lambda_1(\delta z - 2br\delta r) = 0 \text{ and } \lambda_2\delta\theta = 0$$

The coefficients in (2.71) are seen in matrix form as

$$\begin{bmatrix} -2br & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} \delta r \\ \delta\theta \\ \delta z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence there will be two Lagrange multipliers – one for each constraint. The Lagrangian of the system is

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - mgz$$

The equations of motion are

$$m\ddot{r} - mr\dot{\theta}^2 = -2b\lambda_1 r$$

$$\frac{d}{dt}(mr\dot{\theta}^2) = \lambda_2$$

$$m\ddot{z} + mg = \lambda_1$$

Now since $\dot{\theta} = \omega$**, there are four unknowns** $r(t), z(t), \lambda_1(t)$**, and** $\lambda_2(t)$**. :Three eqns of motion and theconstraint eqn** $z = br^2$**.**

Eliminating the multiplier $\lambda_1(t)$ **results in the equation**

$$\ddot{r} - r\omega^2 = -2br(\ddot{z} + g)$$

Differentiation of the constraint $z = br^2$ **results in**

$$\ddot{z} = 2b\dot{r}^2 + 2br\ddot{r}$$

and so we end up with the single differential equation for

$r(t)$ **as**

$$\ddot{r}(1+4b^2r^2)+4b^2r\dot{r}^2=r(\omega^2+2bmg)-(2.73)$$

Solution of Eqn (2.73) !

The coordinate $z(t)$ **is obtained from the constraint**

eqn. The two Lagrange multipliers are also given in

terms of *r* and *z*from the eqs of motion.
Finally, the *torque* required to maintain the uniform

rotation is

$$\lambda_2 = 2m\omega r\dot{r}$$

And the components of the reacting *constraint force*

exerted by the wire on the mass are

$$R_r = -2b\lambda_1(t)r(t) \quad \text{and} \quad R_z = \lambda_1(t)$$

Practice!

INTEGRALS OF MOTION

Up to now: **Concern on formulating the eqs of motion.**

Lagragian formalism for a systematic way to apply

Newton's laws of motion using generalizedcoordinates.

What is the next step ?Actually analyzethe dynamics

based on the eqns of motion.

~Eqns consist of a system of *n* ode, each of the 2^{nd} order!

~Typically nonlinear.

Except, the eqnsof motion are linear.

Eqns of motions are sometimeslinearizedbased on the small displacements assumptions.

This may have some utility in stability analysis,but linearization typically destroys the applicability of the eqns of motion.

Generally, eqns of motion are too complicated !

➔ by integration based on elementary methods.

For specified initial conditions, the eqns of motion are usually integrated numerically.

Example, Runge-Kuttaalgorithms : good accuracy.

Drawbackis that the resultant numerical solution is only valid for *one* set of initial conditions.

Aim of analytical mechanics ?

~Analysis of the eqns of motion themselves,

without actually solving the system of eqns.

Such qualitative analysis was introduced in Chapter 1

with the energy analysis of conservative systems.

Conservative systems are distinguished by conservation

of total mechanical energy.

- Allowed the partial integration of eqn of motion.

This concept is readily extended to general systems.

Suppose that a certain combination of the generalized coordinates and velocities remains *invariant* during the evolution of the system. : If there exists some function $G(q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t)$ **thatremains constant over time, then**

$$G(q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t) = C \quad \textbf{(2.74)}$$

or equivalently

$$\frac{dG}{dt} = 0$$

The relation (2.74) is called an integral of the motion.

***C* is called a *constant of motion*. ~ An integral of motion**

represents a quantity that is conserved during the motion.

There are only first derivatives in an integral of motion,

so each integral of motion ~ a partial integration of the original system : used as reduction of the order of the system.

Lagrange's eqns represent *n* second-order (partial) diff. eqns. ~ Ideally, the solution of Lagrange's equations consists of finding 2*n* integral of motion (2.74), each containing *only* the generalized coordinates.(?)

This is typically not possible,but certain systems do admit some integrals of motion.

For example, in a *conservative system*, the total mechanical energy is an *invariant* of the system.

$$T(\mathrm{q},\dot{\mathrm{q}},t) \;+\; V(\mathrm{q}) \;=\; \mathrm{const}$$

isan integral of the motion. Value of the constant of motion is determined by initial conditions.

A conservative system is a special case of a Lagrangian system.

~ Eqs of motion for a Lagrangiansystem :

$$\frac{d}{dt}\left[\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}_k}\right]-\frac{\partial L(q,\dot{q},t)}{\partial q_k}=0 \qquad \textbf{(2.75)}$$

Total time derivative of $L(\mathrm{q},\dot{\mathrm{q}},t)$ **is**

$$\frac{dL}{dt} = \sum_{k=1}^{n} \frac{\partial L}{\partial q_k} \dot{q}_k + \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k + \frac{\partial L}{\partial t}$$

$$(\sim L, q_k \bullet \dot{q}_k + L, \dot{q}_k \bullet \ddot{q}_k + \frac{\partial L}{\partial t} . for.k = 1..n)$$

***Lagrange's Eqns (2.75) we have**

$$\frac{\partial L}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right)$$

this means

$$\frac{dL}{dt} = \sum_{k=1}^{n} \frac{\partial L}{\partial q_k} \dot{q}_k + \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k + \frac{\partial L}{\partial t}$$

$$= \sum_{k=1}^{n} \frac{d}{dt}\left(\dot{q}_k \frac{\partial L}{\partial \dot{q}_k} \right) + \frac{\partial L}{\partial t}$$

Therefore

$$\frac{d}{dt}\left(\sum_{k=1}^{n} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L \right) = -\frac{\partial L}{\partial t} \text{..(2.76)}$$

Dimension of *L* is energy, the quantity in the parentheses

is known as the Jacobi energy function

$$h(\mathrm{q},\dot{\mathrm{q}},t) = \sum_{k=1}^{n} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L$$

Eqn (2.76) can be written :

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t}$$

~ If the Lagrangian does not contain time *t* explicitly, then the Jacobi energy function is *invariant* during the motion. ~ Energy function is an integral of the motion

with

$$\sum_{k=1}^{n} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L = h = \text{cosnt}$$

~ If the Lagrangian does not contain time *t* explicitly,

then the Jacobi energy function is *invariant* during the

motion. ~ Energy function = an integral of the motion

$$\sum_{k=1}^{n} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L = h = \text{cosnt}$$

Generally :Energy integral into a more familiar form

by referring to the kinetic energy expressionas

$$L = T_2 + T_1 + T_0 - V$$

If *V* depends only on the generalized coordinates, then

$$\sum_{k=1}^{n} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} = 2T_2 + T_1$$

the Jacobi energy integral has

$$T_2 - T_0 + V = h \quad \textbf{(2.77)}$$

It is important to note here that the Jacobi energy

integral is not in general the *total energy*, since the term T_1 is missing. It is still a constant of motion.

Without moving coordinates, $T_1 = T_0 = 0$ **,energy integral *is* the total energy ~ Conservation of total mechanical energy:**

$$T + V = h$$

~Kinetic energy is purely quadratic in the generalized

coordinates are called *natural systems*.

The cart-pendulum system is a natural system, ~

Jacobi energy integral is the total energy of the system.

<u>As a modification of this example</u>,

Suppose :

Motion of the cart ~Aconstant speed $\dot{x} = v_0$
for1 DOF for the pendulum, θ **.**

Kinetic energy :

$$T = \frac{1}{2}(m+M)v_0^2 + \frac{1}{2}ml^2\dot{\theta}^2 + mv_0\dot{\theta}l\cos\theta$$

Potential energy :

$$V = -mgl\cos\theta\,.$$

Then, Lagrangian :

$$L = \frac{1}{2}(m+M)v_0^2 + \frac{1}{2}ml^2\dot{\theta}^2 + mv_0\dot{\theta}l\cos\theta + mgl\cos\theta$$

- **Kinetic energy is not purely quadratic, but**

  Eqn (2.77) still gives the Jacobi energy integral as

$$\frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}(m+M)v_0^2 + mgl\cos\theta = h : (2.78)$$

Constant *h* is specified with initial conditions !

Setting $t = t_0$ **in Eqn (2.78), then**

$$h = \frac{1}{2}ml^2\dot{\theta}_o{}^2 - \frac{1}{2}(m+M)v_0^2 + mgl\cos\theta_o$$

Hence Eqn (2.78) may also be written as

$$\frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta = \frac{1}{2}ml^2\dot{\theta}_0^{\,2} - mgl\cos\theta_0$$

Eqns (2.78) and (2.79) : Equivalent forms of the energy integral for the system.

It should be note:

This system is *not* conservative, since work must be done in order to maintain the constant speed of the cart.

Hence the total mechanical energy is not conserved. The integral of motion (2.79) represents conservation of the energy as computed by an observer riding on the cart. The Jacobi energy integral is one type of invariant of motion associated with conservative systems. Certain forms of the Lagrangian admit other integrals of motion. These results when the Lagrangian does not contain

some of the generalized coordinates.

IGNORABLE COORDINATES

Lagrangian system (*n* dof) and generalized coordinates

$q_1, q_2, \ldots, q_n$ **.**

Suppose:There are *m* coordinates $q_{n-m+1}, \ldots, q_n$ **,**

do*not* appear in the Lagrangian, but the corresponding generalized velocities do.

$$L = L(q_1, q_2, \ldots, q_{n-m}; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t)$$

Eqns of motion for the first $n-m$ **coordinates are**

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0, \qquad k = 1, 2, \ldots, n-m$$

and the eqns for the remaining *m* coordinates are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \qquad i = n-m+1, \ldots, n \qquad \textbf{(2.80)}$$

Eqn (2.80) :Last *m*coordinates $q_{n-m+1},\ldots,q_n$ do not appearin the Lagrangian.

Define it as ignorable coordinates or cyclic coordinates.

Or inactive coordinates.

Anyway,for $i=n-m+1,\ldots,n$, eqns (2.80) can be as

$$\frac{\partial L}{\partial \dot{q}_i}=C_i \qquad \textbf{(2.81)}$$

~ Generalized coordinates and velocities :*conserved*,

➔ Eqns (2.81) are also referred to as conservationeqns.

Potential function *V* does depend on generalized velocities,

$$\frac{\partial V}{\partial \dot{q}_i} = 0$$

then,

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i}$$

Thus the integrals of motion (2.81) can be

$$p_i = C_i \ldots (2.82)$$

:Generalized momenta conjugate to the ignorable coordinates are conserved. ~ The individual conservationeqns may be physically interpreted based on the physical significance of each ignorable

coordinate.

The striking result :Eqns of motion corresponding to the ignorable coordinates have been partially integrated.

→ $n-m$ **eqns remain to be analyzed.**

Moreover, Eqns (2.81) do not contain any ignorable coordinates. So (2.81) or (2.82) can be solved for the generalized velocities of the ignorable coordinates

$\dot{q}_{n-m+1}, \ldots, \dot{q}_n$ **withremaining coordinates.**

: For only $n-m$ **eqns of motion in the non-ignorable generalized coordinates** $q_1, q_2, \ldots, q_{n-m}$ **.**

Remaining eqns of motion contain the *constants* C_i **, but these are determined from initial conditions.**

~ Analysis of the system reduces to the analysis of only

$n-m$ **degrees of freedom.**

A more systematic approach for the elimination of ignorable coordinates is to eliminate the ignorable variables*before* the eqns of motion are formulated.

Introduce a new function of the generalized coordinates and velocities.

As above, the *m* conservation eqns associated with each

of the ignorable coordinates,

$$\frac{\partial L}{\partial \dot{q}_i} = C_i, \qquad i = n-m+1, \ldots, n \tag{2.83}$$

are solved for $\dot{q}_{n-m+1}, \ldots, \dot{q}_n$ **in terms of the remaining coordinates and the constants** C_i**.**

Routhian function is defined as,

$$R = \sum_{i=n-m+1}^{n} C_i \dot{q}_i - L$$

: Generalized velocities $\dot{q}_i$ are replaced by the expressions obtained by solving Eqns (2.83) for $\dot{q}_i$.

The result is a function in the non-ignorable coordinates

??

:Partial derivatives of the Routhian function w.r.t the Non-ignorable coordinates and velocities, then

$$\frac{\partial R}{\partial q_k} = -\frac{\partial L}{\partial q_k}, \qquad k = 1, 2, \ldots, n-m$$

$$\frac{\partial R}{\partial \dot{q}_k} = -\frac{\partial L}{\partial \dot{q}_k}, \qquad k = 1, 2, \ldots, n-m \qquad \textbf{(2.85)}$$

Substitution eqn(2.85) into Lagrange's eqns for non-ignorable coordinates results in the $n-m$ **eqns of motion**

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_k}\right) - \frac{\partial R}{\partial q_k} = 0, \qquad k = 1, 2, \ldots, n-m \qquad \textbf{(2.86)}$$

Once again, ignorable coordinates have been effectively

eliminated to reduce the problem to a mere $n-m$ d.o.f

-Reduced system of $n-m$ eqns contains the *m* constants of motion $C_{n-m+1},\ldots,C_n$.

Finally, the ignorable coordinates of Routhian

$$\dot{q}_i = \frac{\partial R}{\partial C_i} \ldots (2.87)$$

: Constant C_i in (2.87) is considered arbitrary until

the initial conditions are invoked. ~ Eqn (2.87) can be integratedas

$$q_i(t) = \int_{t_0}^{t} \frac{\partial R}{\partial C_i} d\tau, \qquad i = n - m + 1, \ldots, n, , , , (2.88)$$

RouthianFunction :

A particle moving in a plane under to a central force

derivable from apotential function $V(r)$.

~ Conservative!and a Lagrangian expression

in polar coordinates as

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

- θ **is ignorable; conjugate momentum is constant,**

$$p_\theta = mr^2\dot{\theta} = C_\theta$$

>>**Angular momentum of the particle is conserved.**

Furthermore, the Routhian function:

$$R(r,\dot{r},C_\theta) = \dot{\theta}C_\theta - L = \frac{C_\theta}{mr^2}\times C_\theta - \{\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2(\frac{C_\theta}{mr^2})^2 - V(r)\}$$

>>

$$R(r,\dot{r},C_\theta) = -\frac{1}{2}m\dot{r}^2 + \frac{C_\theta^2}{2mr^2} + V(r)$$

: Cyclic Coordinate is Removed !: Single DOF !!

Thus, Eqn becomes (2.86) for k=1…n-m as

$$m\ddot{r} - \frac{C_\theta^2}{mr^3} + V'(r) = 0....(2.89)$$

Note :Eqn (2.89) denotes an entire family!ofdesparameterized by the constant C_θ **.**

C_θ **, : Conserved angular momentum>>Eqn(2.89) for**

$r(t)$ **: Non-linear >>Numerical solution!**

***Jacobi energy function:Additionalintegral of motion.**

- **Energy integralfrom the Routhian function *R***

Eq.(2.89) * dr ~

$$(m\ddot{r} - \frac{C_\theta^2}{mr^3} + V'(r) = 0) \bullet dr \ldots . (2.89)'$$

>>

$$\frac{1}{2}m\dot{r}^2 + \frac{C_\theta^2}{2mr^2} + V(r) = E_0 \quad \textbf{(2.90)}$$

In this case, denote conservation of total mechanical energy.

Furthermore, since the ignorable coordinate θ has been suppressed, the KE associated with θ can be combined with the actual PE, $V(r)$, to define an effective potential:

$$V_{\text{eff}} = \frac{C_\theta^2}{2mr^2} + V(r)$$

Hence, construct the phase curves based onEqn (2.90)

$$\frac{1}{2}m\dot{r}^2 + V_{eff}(r) = E_0 -> (\dot{r} = \frac{dr}{dt} = \sqrt{2(E_0 - V_{eff}(r))} / \sqrt{m} :$$

$$-> dt = \ldots\ldots!$$

andthen the solution

$$t - t_0 = \sqrt{\frac{m}{2}} \int_{r_0}^{r} \frac{dr}{\sqrt{E_0 - V_{\text{eff}}}}$$

Also, $\dot{\theta} = \frac{C_\theta}{mr^2}$

thus

$$\theta(t) = \int_{t_0}^{t} \frac{C_\theta}{mr^2} d\tau + \theta_0$$

:Motion of the system has been entirelysolved !

STEADY MOTION

An important and interesting class of motion :

Ignorable coordinate related to steady motion.

This type of motion :when the generalized velocities and conjugate momenta of the *non-ignorable* coordinates are zero.

That is, $\dot{q}_k = \dot{p}_k = 0 \ldots . (2.91)$

>>

for the $k = 1, 2, \ldots, n-m$ **of non-ignorable coordinates.**

This means that each of the non-ignorable coordinates has a *constant value*.

>>Routhian becomes only a function of the constants of motion $C_{n-m+1},\ldots,C_n$ and does not depend on time *t*.

Hence, generalized velocities of the ignorable coordinates are constant. : solution for the ignorable coordinate results in

$$q_i(t) = v_i t + \text{const}, \qquad i = n - m + 1, \ldots, n$$

Hence the characterization of steady motion. These constant values of the non-ignorable coordinates are not completely arbitrary. Conditions on the non-ignorable coordinates q_k are obtained from the equations of motion (2.86). Conditions for steady motion are obtained by substituting

$$\dot{q}_k = 0 \quad \text{and} \quad \ddot{q}_k = 0$$

into the eqns of motion (2.86). It is actually more convenient to first insert the conditions (2.91) into the Routhian.Eqns of motion, and hence the conditions for steady motion, become

$$\frac{\partial R}{\partial q_k} = 0, \qquad k = 1, 2, \ldots, n-m \ldots . (2.92)$$

Eqns (2.92) are solved for the constant values q_{k0} corresponding to steady motion.

One way to consider the situation is that

Non-ignorable coordinates are effectively in equilibrium, whilethe motion is maintained in a steady manner by conservation of momenta of the ignorable coordinates.

Once the conditions for steady motion are established,

the next important consideration is the stability of these motions. That is, what happens to the steady solutions under small disturbances? The nature of the motion near the steady solutions is analyzed by setting

$$q_k(t) = q_{k0} + s_k(t)$$

These expressions are substituted into the Routhian (2.84), which gives

$$\tilde{R} = \tilde{R}(s_1, s_2, \ldots, s_{n-m}, \dot{s}_1, \dot{s}_2, \ldots, \dot{s}_{n-m})$$

Localized eqns of motion *about* the steady motion are

$$\frac{d}{dt}\left(\frac{\partial \tilde{R}}{\partial \dot{s}_k}\right) - \frac{\partial \tilde{R}}{\partial s_k} = 0, \qquad k = 1, 2, \ldots, n-m$$

For small disturbances about steady motion, these

Eqn may be linearized, and then using standard methods

tocharacterize the stability of steady solution

Ex:A spherical pendulum (Fig. 2.16).

Using the spherical angles ϕ and θ, the Lagrangian:

$$L = \frac{1}{2} ml^2 (\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2) + mgl \cos\theta$$

>>Coordinate ϕ is ignorable ~ conjugate momentum

$$p_\phi = ml^2 \dot{\phi} \sin^2 \theta = C \ldots . (2.93)$$

: An integral of motion.

The analysis is reduced to a single degree of freedom

with the Routhian function

$$R(\theta,\dot{\theta},C)=\frac{C^2}{2ml^2\sin^2\theta}-\frac{1}{2}ml^2\dot{\theta}^2-mgl\cos\theta$$

and therefore the eqn of motion,

$$ml^2\ddot{\theta}-\frac{C^2\cos\theta}{ml^2\sin^3\theta}+mgl\sin\theta=0 \quad \textbf{(2.94)}$$

Spherical pendulum is a conservative system, and then

an effective potential:

$$V_{\text{eff}} = \frac{C^2}{2ml^2 \sin^2\theta} - mgl\cos\theta ....(2.95)$$

Eqn of motion (2.94) can be equivalentlywritten as

$$ml^2\ddot{\theta} + \frac{dV_{\text{eff}}}{d\theta} = 0$$

Condition for steady motion :

$$\frac{C^2\cos\theta}{ml^2\sin^3\theta} - mgl\sin\theta = 0$$

or

$$C^2 \cos\theta = m^2 g l^3 \sin^4\theta$$

From the conservation of angular momentum (2.93), the condition for steady motion reduces to

$$l\dot{\phi}_0^2 = g \sec\theta_0 \quad \textbf{(2.96)}$$

So if the initial conditions ϕ_0 **and** θ_0 **satisfy (2.96), the**

angle θ **and the angular velocity** $\dot{\phi}$ **will remain constant and the tip of the pendulum will execute uniform circular motion. To investigate the stability of perturbations from this steady motion, we set**

$$\theta = \theta_0 + s(t)$$

andsubstitute into the eqn of motion (2.94). After linearization based on small values of $s(t)$**, we obtain the**

DE for the perturbation:

$$\ddot{s} + \frac{g}{l}(3\cos\theta_0 + \sec\theta_0)s = 0$$

The stability may also be determined by analyzing the effective potential V_{eff} (2.95) in the neighborhood of $\theta = \theta_0$.

LAGRANGE'S EQUATIONS FOR IMPULSIVE

FORCES

Principle of Impulse and Momentum >>

Generalized in the Lagrangian formalism.

During impact :Very large forces are generated

over a very small time interval. ~ Not a practical matter

to record these forces over the very small time

>>>Instantaneous form of Newton's Second Law is of little use in impact problems.

>>>Eqns of motion are integrated over the time interval of impact.

$$\hat{\mathrm{F}} = \int_{t_0}^{t_0+\Delta t} \sum \mathrm{F}(t)\,dt$$

By the Principle of Impulse and Momentum, velocities change by a finite amount over the time interval Δt**. As long as the time interval is taken**

infinitesimally small, the displacements do not change

and hence remain continuous.

Therefore, Impulsive force ~ Finding velocity changeimmediately after the impact..withoutdisplacement change

Integrating Lagrange's eqns of motion for holomicsystems over the time interval between $t_1 = t_0$ **and**

$t_2 = t_0 + \Delta t$ **, we have**

$$\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) dt - \int_{t_1}^{t_2} \frac{\partial T}{\partial q_k} dt = \int_{t_1}^{t_2} Q_k dt, \qquad k = 1, 2, \ldots, n \quad \textbf{(2.97)}$$

Now letting $\Delta t \to 0$**,**

$$\left.\frac{\partial T}{\partial \dot{q}_k}\right|_2 - \left.\frac{\partial T}{\partial \dot{q}_k}\right|_1 = \hat{Q}_k \qquad k = 1, 2, \ldots, n \quad \textbf{(2.98)}$$

Second term on the left-hand side of Eqn (2.97) vanishes, since the generalized coordinates are continuous and the generalized velocities remain bounded during the impact. The integral on the right-hand side of Eqn (2.97) is the

generalized impulse $\hat{Q}_k$ **.**

The impulsive form of Lagrange's eqns (2.98) can also be

$$\Delta p_k = \hat{Q}_k, \qquad k = 1, 2, \ldots, n \quad \textbf{(2.99)}$$

relating the change in generalized momentum p_k **to the applied generalized impulse** $\hat{Q}_k$ **. Since the generalized momenta are polynomials in the generalized velocities,**

there is no need to solve any differential equations to obtain the velocities immediately after impact. Computation of the generalized impulses is formally identical to finding generalized forces. At any instant, the virtual impulsive energy acquired by the system under virtualdisplacements compatible with the constraints is

$$\delta\hat{W} = \sum_{j=1}^{n} \hat{Q}_j \delta q_j$$

As with generalized forces, the independent degrees of freedom are incremented one at a time to determine the individual contributions to $\delta\hat{W}$.

Ex: A four-bar linkage constrained to slide smoothly along the the *x*-direction

(Fig. 2.17). The system has two degrees of freedom and as generalized coordinates we can take the location of the center of mass, x_1, and the angle θ. We assume that the mechanismis at rest when an impulse $\hat{F}$ is suddenly applied, at point A, in the *x*-direction.

Solving this problem by vector methods involves calculation of the linear and angular momenta of the

system and invoking the momenta are easily derived from the kinetic energy of the system.

The generalized impulses are formally computed as if they were generalized forces.

The kinetic energy of the system is

$$T = 2m\dot{x}_1^2 + \frac{8}{3}mb^2\dot{\theta}^2$$

The generalized momenta conjugate to x_1 **and** θ**,**

respectively, are

$$p_{x1} = 4m\dot{x}_1, \qquad p_\theta = \frac{16}{3} mb^2 \dot{\theta}$$

Similar to computing virtual work, we consider the independent virtual displacements

$$x_1 \to x_1 + \delta x_1, \ \delta\theta = 0 \quad \text{and} \quad \delta x_1 = 0, \theta \to \theta + \delta\theta$$

The virtual impulsive energy becomes

$$\delta\hat{W} = \hat{Q}_{x1}\delta_{x1} + \hat{Q}_{\theta}\delta\theta$$

in which the generalized impulses are

$$\hat{Q}_{x1} = \hat{F}\ , \hat{Q}_{\theta} = 2b\sin\theta\hat{F}$$

Since the system starts from rest, substitution of the above into Lagrange's equations for implusive systems (2.99) results in the acquired generalized velocities

$$\dot{x}_1 = \frac{\hat{F}}{4m}, \dot{\theta} = \frac{3\sin\theta}{8mb}\hat{F}$$

Practice !

A horizontal rod of mass m and length 2L falls under gravity and strakes a knife edge loaded one half of the way from the center to end of the rod. It's velocity just before impact is v . Coefficient of restitution between rod and knife edge is e.

a. **Velocity of the center of mass**
b. **Angular velocity immediately after the rod strikes the ground.**

Sol: Assume the impulse is applied at the impact.

Total enegy at any instant : $T = \frac{1}{2} m(\dot{x}_c^2 + \dot{y}_c^2) + \frac{1}{2} I \dot{\theta}^2$

Virtual work of impulse : $\delta W = \hat{F}(\delta y_c + \frac{1}{2} L\delta\theta) = \hat{Q}_{x_c} \delta x_c + \hat{Q}_{y_c} \delta y_c + \hat{Q}_\theta \delta\theta$

$\sim \quad \hat{Q}_{x_c} = 0, \hat{Q}_{y_c} = \hat{F}, \hat{Q}_\theta = \frac{1}{2} L\hat{F}$

Change of generalize Momentum:

$$\Delta(m\dot{x}_c) = m\dot{x}_c = 0 - - - (1) : \dot{x}_c = 0$$

$$\Delta(m\dot{y}_c) = m(\dot{y} + v) = \hat{F} - (2)$$

$$\Delta(I\dot{\theta}) = I\dot{\theta} = \frac{L}{2}\hat{F} - - - (3) : (I = \frac{1}{3}mL^2)$$

$$And .. \dot{y}_c + \frac{L}{2}\dot{\theta} = ev >>>> \dot{y}_c = ev - \frac{L}{2}\dot{\theta} - - (4)$$

$$(4) : \dot{\theta} = \frac{2}{L}(ev - \dot{y}_c) - - - > (3)\hat{F} = ...$$

$$: \dot{y}_c = \frac{v}{7}(4e - v) .. \dot{\theta} = \frac{6}{7L}(1 + e)v$$

ELECTROMECHANICAL ANALOGIES

The Lagrangian formalism is based on energy and therefore has applicability that goes far beyond simple mechanical systems (Fig. 2.18). A very practical extension of the theory is to electrical circuits and combined electromechanical systems. A direct application of Lagrangian's equations to electrical circuits is based on the parameters given in Table 2.1. Energy carried by an inductor coil is

CHAPTER THREE:

CALCULUS OF VARIATIONS

INTRODUCTION

Lagrangian formulation of eqns of motion:

~ Energy of a system and work done by external forces.

: Dynamical system with inertial properties–

: Kinetic energy

Also,conservative generalized forces ~Derivable from

: Potential energy

***Non-conservative part ~**

Applying virtual workprinciple !

Most systems can be synthesized by the construction ofa Lagrangian ~

Theoretical & Experimentally Verified !

oPrimary feature of Lagrangian dynamics ~

Independent of any coordinate system

Physical coordinates~>Generalized coordinates

:More intrinsic to the constraints

~ Associated analytical approach : Possible to use a system approach in deriving the governing eqns.

Concept of a configuration spaceusing generalized coordinates.

Evolution of a dynamic system :A single point in the configuration space

As in Fig. 3.1,a system of *N* particles moving freely in space is described by *3N* generalized coordinates$q_1 \ldots q_{3N}$.: Configuration space is thus a *3N*-dimensional space.

As the system undergoes its motion between fixed times $t_1$and t_2, the evolution is traced out by a unique

path in the configuration space.

For a conservative holonomic system, the action is defined as

$$I[q] = \int_{to}^{t1} (T - V)dt \ldots\ldots (3.1)$$

Many(?) dynamical systems evolve toextermize the valueof the action integral (3.1).

~ Of all possible ones,anextremum relative to the values for the other paths. Therefore, the action integral assigns a number to each possible path in configuration space.

Evidently, we can followanother way of analyzing the motion of a dynamical system. ~ Advantage to this new

point of view :Formulation is also independent of the particular generalized coordinates used.

Also,the ideas will directly extend to systems of infinite degrees of freedom, such as bodies composed of a continuum of points.

- **This is in contrast to the Lagrangian formulation,**

  which by derivation is restricted to systems with only a finite number of degrees of freedom !

~ Focus on the mathematics involved with finding extrema of integrals that depend in specific ways on functions as *inputs*. This objective is the essence of the area of analysis known as the calculus of variations.

:Differential calculus is full of standard tools available to analyze the extreme values of ordinary functions. Fortunately, there are many parallels between the calculus of variations and the ordinary calculus of functions.

Define some concepts and terminology

:A function is usually taken as an assignment of real values. A function of one variable $f(\cdot)$ **assigns to each** x **a given value** $f(x) \in R.$**. A function of several variables assigns a value to a point given by real-valued coordinates**

$$(x_1, \ldots\ldots x_N) \mapsto f(x_1, \ldots x_N) \in R.$$

Graph of a function of one variable is a curve

~ Graph of a function of two variables is a surface.

Generalization of a function is called a functional.

An assignment of a real value to a point, to avector, or to an entire function.

Now concern here with functionalsthat are defined on some suitable space or set of functions.

: Functionals~Integral functionals.

- Functionals defined by the integration of some expression involving an input function:

$$f(x) \mapsto I[f(x)]$$

Integral functional $I[..]$ **may be of the form (3.1).**

Ex: integral functionals include *area under the graph* of a

function and *arclength of a curve* between two points.

Ex might look like …

where $p(x)$ **and** $q(x)$ **are specified.**

Integral functionals may be defined as integrals over some interval or as integrals over some region in space.

Integration interval is the domain of the input function.

Argument or input of an integral functional may be a single function or several functions.

It all depends on the context of the problem. It is important to distinguish between the domain of the input function and the domain of the functional itself, which is comprised of some class of admissible functions.

Consider afunctionals of the form

$$I[y(x)] = \int_{x0}^{x1} F(x, y(x), y'(x))dx \quad \textbf{(3.2)}$$

and its natural generalizations.

Integrand in (3.2) is called the *Lagrangian* of the integral functional~Fundamental objective of the calculus of variations is to establish conditions under which an integral functional attains an extreme value.

These conditions evidently depend on the form of the Lagrangian of the functional ~ will lead to the conditions on the particular input functions that make the integral a maximum or minimum.

An input function that renders the value of the integral

functional a maximum or minimum is called an *extramal*.

Now many interesting problems can be formulated in terms of integral functionals.

Geometry of curves and surfaces.

Most of the physical applications are based on mechanics. We will begin with several motivating examples.

Fig.3.2: Given two fixed but arbitrary points P_1 and P_2 in a plane, we can connect these two points with a curve that is the graph of a continuous function $y(x)$.

Now if we consider the collection of all continuously differentiable functions passing through the points P_1 and P_2, we can consider the associated arclength of

each curve. The arclength of each curve is given by the integral functional

$$L[y]=\int_{x1}^{x2}\sqrt{(dx)^2+(dy)^2}=\int_{x1}^{x2}\sqrt{1+y'^2}\,dx....(3.3)$$

Ex :Of all the continuously differentiable curves passing through point P_1 and P_2, find the function whose arclength is minimal. That is, of all smooth functions passing through P_1 and P_2, find the one that has the smallest length. We intuitively know the correct answer, but it would be nice to have a formal way to decisively solve this problem.

***Brachistochrone problem*(Fig. 3.3).**

A particle is free to slide down a frictionless wire with fixed endpoints at P_1 and P_2 .

Assuming wire has finite length, then determine the time t* it takes for the particle to slide along the wire. Given that the particle starts from rests, it would be interesting to find the shape of the wire for which the time of travel between the two endpoints is as small as possible.

We can formulate this problem as follows: Applying the Principle of Work and Energy,

$$W_{1\to 2} = T_2 - T_1$$

at any point $(x, y(x))$ **along the curve,**

$$mgy = \frac{1}{2}mv^2 - \frac{1}{2}mv_0{}^2$$

Starting from rest, the speed of the particle is given as

$$v = \sqrt{2gy}$$

Now since

$$v = \frac{ds}{dt}$$

We have the differential relation

$$dt = \frac{ds}{\sqrt{2gy}} \quad \textbf{(3.4)}$$

Integrating both sides of Eqn (3.4),

$$t_2 - t_1 = \int_{x_1}^{x_2} \frac{ds}{\sqrt{2gy}}$$

Thus the total time of travel along the curve is obtained

$$t^* = \int_{x_1}^{x_2} \frac{\sqrt{1+[y'(x)]^2}}{\sqrt{2gy(x)}}\,dx$$

Notice that in each of these problems the quantity to be minimized was formulated in terms of an integral.

Given an appropriate function $y(x)$ **that satisfies certain conditions, the function is entered as an argument**

and the resultant integral evaluated. This assignment of a scalar value to an entire function is the operation assigned to an integral functional. Symbolically, the value of a functional associated with a specified input function $y(x)$ is expressed as $I[y(x)]$. Integral functionals are a special case of such evaluations.

The domain of a functional, that is, the collection of functions satisfying certain conditions, is defined as the associated set of *admissible functions*. In the case of integral functionals, admissibility typically requires certain differentiablility conditions and specified boundary conditions. These conditions are usually specified with the problem.

The problem of finding a function $y(x)$ out a set of

admissible functions that minimized (maximizes) a given functional $I[y]$ is called a variational problem. The actual value of the functional, or even if it is a maximum or a minimum, is of little concern. The important thing is that the value of the **extremal is *stationary*.**

A representative integral functional, as in the examples, has the form

$$I[y(x)] = \int_{x0}^{x1} F(x, y(x), y'(x))dx \quad \text{...(3.2)}$$

in which the Lagrangian $F(x, y, y')$ is a smooth function of three variables. The Lagrangian is the integrand of the functional $I[y]$. The function of three variables $F(\alpha, \beta, \gamma)$ specifies the relationship of all the variables in the

integrand of the functional.

For the minimal length problem :

For the branchistochrone problem:

EXTREMA OF FUNCTIONS

An extremum problem consists of finding the largest or smallest value of a quantity. For functions of one or two variables, the function can be graphed and we immediately see where the function attains its extreme values. These may be inside of a domain, or the extrema may be located at points on the boundary. For functions of more than two variables, graphing is not possible, so we must report to performing a comparison of the values at a point with neighboring

values. That is, we examine the local rate of change of a function. These ideas carry over directly to finding extrema of integral functionals.

To find the extrema of a function inside an interval, we look for *local stationary behavior*. At a point x where a function attains a local extremum, given an infinitesimal change, the value of the function should remain the same; otherwise we do not have an extremum. This is the same as examing the *local linearization* of the function. At an extremum, the function should be flat. The rate of change in every possible direction must be zero. Since there are only two directions, this is easy. Analytically, this means that the differential of of the function is equal to zero.

Hence, a necessary condition that the function $f(x)$ be stationary at a point x_0 is that the derivative $f'(x_0)$ is equal to zero. The location where this happens is called *a critical point*. This condition is only necessary, since the condition implies that at the critical point the function can have a local maximum, a local minimum, or an inflection point. Further examination is required, namely checking out the local curvature at the critical point. This involves the second-derivative test. For functions defined on finite intervals, the values at the endpoints must be checked separately, since the derivates are not defined there.

For functions of two variables, say $f(x,y)$, let us assume that a point (x_0, y_0) is a critical point. In order for the

function to be stationary at this point, we must examine the variation of the function as we move an infinitesimal amount in any possible direction. So let $\mathbf{r} = \Delta x\,\mathbf{i} + \Delta y\,\mathbf{j}$ **be any vector that will denote some fixed but arbitrary direction. We can use a small parameter,** ε**, to test the variation of the function under a infinitesimal displacement:**

Now the function

$$f(\mathbf{r}, \varepsilon) = f(x_0 + \varepsilon\Delta x, y_0 + \varepsilon\Delta y)$$

is a function of a single variable ε**. This can be thought of as cutting a slice through the surface defined by** $f(x, y)$ **along the direction of the vector r. This curve is parameterized by** ε**. Note that at** $\varepsilon = 0$**,** $f(\mathbf{r}, 0) = f(x_0, y_0)$**. Thus we have reduced the analysis to**

searching condition that $f(\mathbf{r},\varepsilon)$ is stationary at $\varepsilon=0$ is that $f'(\mathbf{r},0)=0$. The rate of change with respect to ε is

$$\frac{df}{d\varepsilon}=\frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y \quad \textbf{(3.5)}$$

Setting $\varepsilon=0$ in Eqn (3.5), we find that

$$\frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y=0 \quad \textbf{(3.6)}$$

in which the partial derivatives are evaluated *at the point* (x_0,y_0).The left-hand side of Eqn (3.6) is equal to zero for an arbitrary direction specified by Δx and Δy if and only if the partial derivatives are equal to zero at (x_0,y_0). That is,

****(3.7)**

The condition (3.7) give a necessary and sufficient

condition for local stationary behavior of ** at the point **.

These conditions extend to higher dimensions for functions of *n* variables **, as

****

We now generalize these concepts to establish criteria for stationary values of scalar quantities that depend on entire functions as arguments. Before formally developing the theory, let us consider a simple motivational example.

Remember : Δ, d, δ

Lagrangianeqn in hybrid form : Page.115

$$I[y(x)] = \int_{x0}^{x1} F(x, y(x), y'(x))dx \quad \textbf{(3.2)}$$

Lagrangian **of the integral functional~Fundamental objective of the "calculus of variations" is to establish conditions under which an integral functional attains an "extreme" value.**

: Maximum or minimum.

Ex :Of all the continuously differentiable curves passing through point P_1 **and** P_2**, find the function whose**

arclength is minimal.

Thus the total time of travelalong the curve is obtained

$$t^* = \int_{x_1}^{x_2} \frac{\sqrt{1+[y'(x)]^2}}{\sqrt{2gy(x)}}\,dx$$

: F(y, y`) !

Notice that in each of these problems the quantity "to be Minimized" was formulated in terms of an integral.

EXTREMA OF FUNCTIONS

NECESSARY CONDITIONS FOR AN EXTREMUM

Establishing local stationary behavior of a functional is a generalization of locating the critical points of a function.

:Local maxima and minima are found by setting the derivative of the function w.r.t independent variable equals to 0 ~ slop =0 (Fig.3.4) !

-Value of the function is stationary.

: At a critical point aninfinitesimal variation of the independent variable results in no change in the value of the function. ------ Function space !

For integral functionals, the arguments or inputs are entire functions belonging to a specified admissible set.
~we must rely on the notion of local stationary behavior.

Fundamental Lemma : Calculus of Variations.

$G(x)$ **:Continuous function in** $[x_1, x_2]$

$$\int_{x1}^{x2} G(x)\eta(x)dx = 0$$

for all smooth functions $\eta(x)$ **with** $\eta(x_1) = \eta(x_2) = 0 ... then .. G(x) \equiv 0$
for all points(Goveringeqn ! for example G = T-V)

Focus: Minimizing(Maximizing) the functional

$$I[y(x)] = \int_{x0}^{x1} F(x, y(x), y'(x))dx$$ **with BC**

~Apply the local stationary behavior of a functional.

Let $y*(x)$ **:Admissible function that minimizes** $I[y]$**.**
Suppose $y(x)$ **: Another admissible ftnas in (Fig. 3.5)**
"close" in some sense to $y*(x)$
Admissible~ $y(x)$**: continuously differentiable withBCs.**

Then

$y(x) = y^*(x) + \varepsilon\eta(x)$: $\eta(x)$ **Differentiable**

: **Perturbing function** $\varepsilon\eta(x)$: ***Variation* of function** $y(x)$ **.**

Since $y^*(x)$ **is extremal~ Local minimum value of** $I[y(x)]$ **,**

: $I[y^*(x) + \varepsilon\eta(x)] \geq I[y^*(x)]$ **for all** ε **near 0.**

Once a variation $\varepsilon\eta(x)$ **is fixed, but arbitrary, then**

$I[y^*(x) + \varepsilon\eta(x)]$

actually becomes a function of a single real variable!

: **Reduced the problem to a single real variable for**

local stationary behavior with the zero-slope criterion for functions.

$$[\frac{d}{d\varepsilon} I(y^*+\varepsilon\eta)]_{\varepsilon=0} = 0$$

~>

$$I[y^*+\varepsilon\eta] = \int_{x0}^{x1} F(x, y^*+\varepsilon\eta, y^{*\prime}+\varepsilon\eta')dx \quad \textbf{(3.8)}$$

:Derivative of Eqn (3.8) wrt the ε **is**

(or $I(\varepsilon) = \int_{x0}^{x1} F(x, \bar{y}, \bar{y}')dx$ **,**

$$\frac{dI(\varepsilon)}{d\varepsilon} = \int_{x0}^{x1} [\frac{\partial F}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial \varepsilon} + \frac{\partial F}{\partial \bar{y}'} \frac{\partial \bar{y}'}{\partial \varepsilon}]dx = \int_{x0}^{x1} [\frac{\partial F}{\partial \bar{y}} \eta + \frac{\partial F}{\partial \bar{y}'} \eta']dx$$

$$For..\varepsilon = 0,$$

$$\frac{dI(\varepsilon)}{d\varepsilon}] = \int_{x0}^{x1}[\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta']dx$$

=

$$\frac{dI[y^* + \varepsilon\eta]}{d\varepsilon} = \int_{x0}^{x1}[\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta']dx$$

Integrating by parts and (applying the BCs)

$$\int_{x0}^{x1}[\frac{\partial F}{\partial y}\eta - \frac{d}{dx}(\frac{\partial F}{\partial y'})\eta]dx + \frac{\partial F}{\partial y'}\eta]\!]_{x1}^{x2}$$

For an extreme value of the functional,

$$\int_{x0}^{x1}[\frac{\partial F}{\partial y} - \frac{d}{dx}(\frac{\partial F}{\partial y'})]\eta(x)dx$$

: $\eta(x)$ **is an *arbitrary* function that vanishes at the endpoints of the interval, the Fundamental Lemma of the Calculus of Variations allow us to conclude**

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \ldots . (3.9)$$

~An ordinary differential equation that represents the necessary condition for an admissible $y(x)$ to be a minimizing function.

: Euler-Lagrange equation

Back to the future ! P. 115

$$x -> t$$

$$y, y' -> q, \dot{q} \qquad !$$

Ex : **(3.10) : p.169**

SPECIAL CASES OF THE EULER-LAGRANGE EQUATION

$$F = F(x, y, y')$$
$$F = F(x, y')$$
$$F = F(y, y')$$
$$F = F(x, y)$$
$$F = F(y')$$

Eqn(3.13)! Jacobi energy integral

$$\delta \int \quad = \int \delta \qquad ?$$

THE VARIATIONAL OPERATOR(?)
NATURAL BOUNDARY CONDITIONS(?)
GENERALIZATIONS(?)
SEVERAL INDEPENDENT VARIABLES
VARIATIONAL PROBLEMS WITH CONSTRAINTS

HAMILTON'S PRINCIPLE

THE VARIATIONAL OPERATOR(?)
NATURAL BOUNDARY CONDITIONS

$$\cdots \quad + \frac{\partial F}{\partial y'} \delta y \Big]_{x1}^{x2} = 0 \ldots (3.15)$$

as in Fig.3.6

Then, at x_2 **for** $\delta I = 0$**, Euler-Lagrange Eqn. is satisfied with** $\frac{\partial F}{\partial y'}$ **=0 : Natural BC or Force BC.(ex: ICBM…)**

(δy **=0 :Essential BC or Geometric BC)**

GENERALIZATIONS

$F = F(x, y, y') : most.simple!$

~> For more generality with several functions !

As in Eqn.(3.16)

y =($y_1, \ldots, y_n$): independent Ftns

Ex : !

If $F = F(x, y, y', y'')..?$ **:** $\delta I = ?$ **~ page 172.**

Ordinary Differential Eqn !

Partial Differential Eqn(x,t)

:Beam bending problem: $EI \frac{d^4 y(x,t)}{dx^4} = m \frac{d^2 y(x,t)}{dt^2}$

~ $T = \frac{1}{2} m (\frac{\partial y(x,t)}{\partial t})^2, V = \frac{1}{2} EI (\frac{\partial^2 y(x,t)}{\partial x^2})^2$

SEVERAL INDEPENDENT VARIABLES

VARIATIONAL PROBLEMS WITH CONSTRAINTS

Two Types of Constraint!

~ **Integral constraint**

Eqn(3.20)

Isoperimetric problem

$$(F+\lambda G)_{,y} - [(F+\lambda G)_{,y'}]_{,x} = 0 \ldots\ldots (3.23)$$

- **More general isoperimetric problem**

Eqn(3.24)

~Eqns of constraint

(Non~) holonomic !

$$G(x, y_1 ..., y_n) = 0 .....(3.25)$$

$$or$$

$$Taking.variation,$$

$$G_{,yi}\, \delta y_i = 0$$

THE VARIATIONAL OPERATOR(?)
NATURAL BOUNDARY CONDITIONS
GENERALIZATIONS

Beam bending problem: $EI\frac{\partial^4 y(x,t)}{\partial x^4} = m\frac{\partial^2 y(x,t)}{\partial t^2}$

~ $T = \frac{1}{2}m(\frac{\partial y(x,t)}{\partial t})^2, V = \frac{1}{2}EI(\frac{\partial^2 y(x,t)}{\partial x^2})^2$

SEVERAL INDEPENDENT VARIABLES
VARIATIONAL PROBLEMS WITH CONSTRAINTS

~ Integral constraint :

More generalisoperimetric problem

~Eqns ofconstraint :(Non~) holonomicconstraint ! : Boat :

HAMILTON'S PRINCIPLE

Aim:

? Lagrangian formulation ~ Calculus of Variation?

For a Particle (p.108) :

$$F_i + R_i = m_i \ddot{x}_i$$

Energy concept : T, W

Total virtual (time : fixed) work :

$$\delta W : (F_i + R_i)\delta x_i = m_i \ddot{x}_i \delta x_i .. (i = 1,,,3N)$$

Remember !

$$\frac{d}{dt}(m_i \dot{x}_i \delta x_i) = m_i \ddot{x}_i \delta x_i + m_i \dot{x}_i \delta \dot{x}_i = \delta W + \delta[\frac{1}{2} m_i x_i^2] ..(i = 1,,3N)$$

Eqn(3.28):

$$\delta T + \delta W = (m_i \dot{x}_i \delta x_i)_{,t}$$

Integrating over the time domain and applying BC in time

$$\delta x_i(t_0) = \delta x_i(t_1) = 0$$

Finally,

$$\int (\delta T + \delta W) dt = 0 ...(3.29):$$

‘Hamilton’s Principle’

Advantage of variational point of view:

Hamilton's principle may be extended to continuous systems with infinite number of DOF !

Wave eqn : $\rho u_{,tt} = \mu u_{,xx} + f(x,t)$

Euler beam vibration : $\rho u_{,tt} = EIu_{,xxxx} + f(x,t)$

HAMILTON'S PRINCIPLE