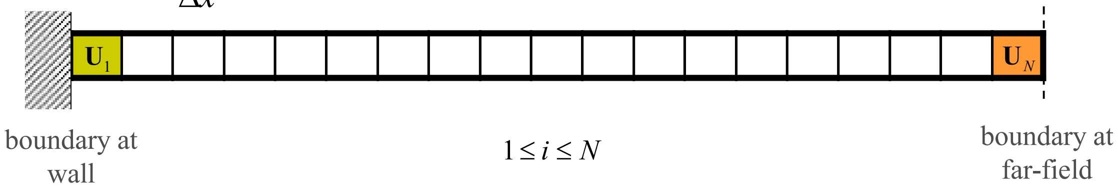
Chap. 4-8. Boundary Conditions

- **Finite computational domain**
  - Evaluation of numerical fluxes at wall and far boundary
  - Ex) 1-D finite computational domain
    - $\mathbf{U}_1$ and $\mathbf{U}_N$ cannot be updated without the knowledge of fluxes at boundaries.

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+1/2}^n - \mathbf{F}_{i-1/2}^n\right) \quad \rightarrow \quad \mathbf{F}_{1/2}^n = \mathbf{F}\left('\mathbf{U}_0', \mathbf{U}_1\right),\ \mathbf{F}_{N+1/2}^n = \mathbf{F}\left(\mathbf{U}_N, '\mathbf{U}_{N+1}'\right)$$



- For subsonic/transonic compressible flows around airfoil, far-field distance (l) can change from *O(10c)* to *O(100c)* depending on the nature of far-field BCs.
- Ex) Velocity correction at far field boundary
  - Steady subsonic/transonic flows around 2-D airfoil
    - the flow in the far field can be modelled by the P-G equation as $(1-M_\infty^2)\phi_{xx} + \phi_{yy} = 0$.
    - the perturbed far field velocity induced by a point vortex approximation of airfoil is given by

$$u(=\phi_x) = \frac{\Gamma}{2\pi}\frac{\beta y}{x^2+\beta^2 y^2},\ v(=\phi_y) = -\frac{\Gamma}{2\pi}\frac{\beta x}{x^2+\beta^2 y^2} \text{ with } \beta = \sqrt{1-M_\infty^2},\ \Gamma = \frac{cc_l}{2}u_\infty.$$

    - the total velocity component at far field can be modelled by

$$u_{far} = (\cos\alpha + F\sin\theta)u_\infty,\ v_{far} = (\sin\alpha - F\sin\theta)u_\infty \text{ with } F = \frac{cc_l\beta}{4\pi r}\frac{1}{[1-M_\infty^2\sin^2(\theta-\alpha)]}.$$

- Improper BCs (wall and far-field) will spoil numerical accuracy and convergence characteristics.

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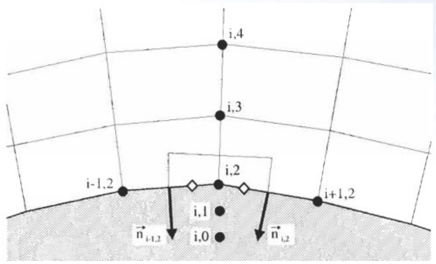
- ***Wall Boundary Condition***
  - **For 2-D Euler eqns.,**



$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{F}}{\partial x}+\frac{\partial \mathbf{G}}{\partial y}=\mathbf{0} \quad \text{or} \quad \frac{\partial}{\partial t}\int \mathbf{U}dS+\int\left(\mathbf{F}dy-\mathbf{G}dx\right)=\mathbf{0}$$

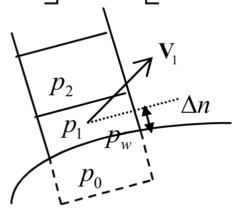
  - **Flux evaluation at a cell *(i, j)* adjacent to the wall**

$$(\mathbf{F}\Delta y-\mathbf{G}\Delta x)_w=(\mathbf{F}n_x+\mathbf{G}n_y)_w\Delta l \quad \text{with } \mathbf{n}=(\Delta y,-\Delta x)/\Delta l=(\cos\theta,\sin\theta) \text{ and } \Delta l=\sqrt{\Delta x^2+\Delta y^2}$$

For (x, y) coordinate, $\left(\mathbf{F}n_x+\mathbf{G}n_y\right)_w \Delta l \ = \begin{bmatrix} \rho(\mathbf{V}\cdot\mathbf{n}) \\ \rho u\left(\mathbf{V}\cdot\mathbf{n}\right)+pn_x \\ \rho v\left(\mathbf{V}\cdot\mathbf{n}\right)+pn_y \\ \rho\left(\mathbf{V}\cdot\mathbf{n}\right)H \end{bmatrix}_w \Delta l = \begin{bmatrix} 0 \\ p_w\Delta y \\ -p_w\Delta x \\ 0 \end{bmatrix}$

For $(\hat{x},\hat{y})$ coordinate, $\mathrm{T}^{-1}\mathbf{F}(\hat{\mathbf{U}})_w\Delta l=\mathrm{T}^{-1}\begin{bmatrix} \rho U \\ \rho U^2+p_w \\ \rho UV \\ \rho UH \end{bmatrix}\Delta l=\mathrm{T}^{-1}\begin{bmatrix} 0 \\ p_w \\ 0 \\ 0 \end{bmatrix}\Delta l=\begin{bmatrix} 0 \\ p_w\cos\theta \\ p_w\sin\theta \\ 0 \end{bmatrix}\Delta l=\begin{bmatrix} 0 \\ p_w\Delta y \\ -p_w\Delta x \\ 0 \end{bmatrix}$

  - **Estimation of wall pressure**



• Direct flux evaluation using ghost cell

- define quantities at the ghost cell $j=0$ by

$\rho_0=\rho_1, U_0=U_1, V_0=-V_1, \left(\rho E\right)_0=\left(\rho E\right)_1$, and compute the interface flux like a normal cell

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- **Estimation of wall pressure (Cont'd)**

• Direct flux evaluation using ghost cell

- O.K. for flat wall or mildly curved wall, but not for wall with steep curvature such as the airfoil leading-edge

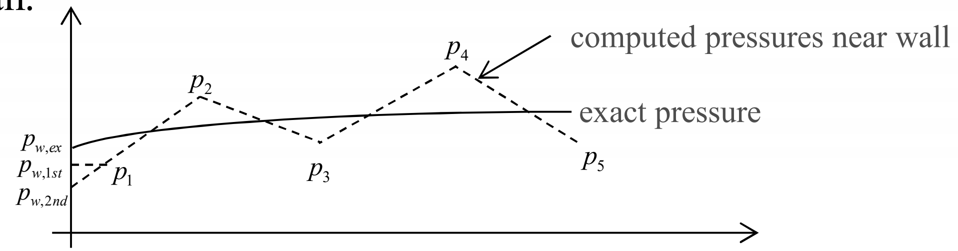
• Extrapolation

- Estimate the wall pressure from pressure distribution inside the computational domain

- First-order approximation: $\partial p / \partial n = 0 \rightarrow p_w = p_1$ with $O(\Delta n)$ error

- Second-order approximation: $p_w = p_1 - \dfrac{\partial p}{\partial n}\Delta n \cong p_1 - \dfrac{p_2 - p_1}{2\Delta n}\Delta n = \dfrac{3p_1 - p_2}{2}$

with $O(\Delta n^2)$ error, but this may not always provide expected accuracy if there is numerical oscillation near wall.



• Apply force equilibrium from the normal momentum eqn.: pressure force = centrifugal force

$$\frac{\partial p}{\partial n} = \frac{\rho|\mathbf{V}|^2}{R} = \frac{\rho\left(u^2+v^2\right)}{R} \rightarrow \frac{p_1 - p_w}{\Delta n} \cong \frac{\rho_1\left(u_1^2+v_1^2\right)}{R}, \quad p_w = p_1 - \frac{\rho_1\left(u_1^2+v_1^2\right)}{R}\Delta n$$

and R is computed using neighboring grids $(i-1, j)$, (i, j), $(i+1, j)$.

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- ***Far-field Boundary Condition***
  - **2-D non-conservative form of Euler eqns. with primitive variables of** (ρ, u, v, s)

$$\frac{\partial \rho}{\partial t} + (\mathbf{V}\cdot\nabla)\rho + \rho\nabla\cdot\mathbf{V} = 0, \quad \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} + \frac{\nabla p}{\rho} = \mathbf{0}, \quad \frac{\partial s}{\partial t} + (\mathbf{V}\cdot\nabla)s = 0$$

  - **Assume waves are propagating normal to the local** $\hat{y}$**-dir. and no shocks**
    - Local 1-D isentropic flow with $p/\rho^\gamma = e^s$

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} + \frac{u}{\rho}\frac{\partial \rho}{\partial x} + \frac{\partial u}{\partial x} = 0 \qquad \text{Eq.(1)} \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \qquad \text{Eq.(2)}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} = 0 \qquad \text{Eq.(3)} \qquad \frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = 0 \qquad \text{Eq.(4)}$$

From $c^2 = \left.\frac{\partial p}{\partial \rho}\right)_s = \gamma\rho^{\gamma-1}e^s, \quad 2\log c = \log\gamma + (\gamma-1)\log\rho + s$

$$\frac{2dc}{c} = (\gamma-1)\frac{d\rho}{\rho} + ds \;\rightarrow\; \frac{d\rho}{\rho} = \frac{2}{\gamma-1}\frac{dc}{c} - \frac{1}{\gamma-1}ds \qquad \text{Eq.(5)}$$

Equation (1) by using Eq. (4) and Eq.(5) becomes

$$\frac{2}{\gamma-1}\left(\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x}\right) - \frac{c}{\gamma-1}\left(\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x}\right) + c\frac{\partial u}{\partial x} = 0 \;\rightarrow$$

$$\frac{2}{\gamma-1}\left(\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x}\right) + c\frac{\partial u}{\partial x} = 0 \;\text{ or }\; \frac{\partial}{\partial t}\left(\frac{2c}{\gamma-1}\right) + u\frac{\partial}{\partial x}\left(\frac{2c}{\gamma-1}\right) + c\frac{\partial u}{\partial x} = 0 \qquad \text{Eq.(6)}$$

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Also from $p/\rho^{\gamma} = e^{s}$, $\dfrac{dp}{p} = \gamma \dfrac{d\rho}{\rho} + ds$ or $\dfrac{dp}{\rho} = c^2 \dfrac{d\rho}{\rho} + \dfrac{c^2}{\gamma} ds \xrightarrow{\text{Eq.(5)}} \dfrac{dp}{\rho} = \dfrac{2c}{\gamma-1} dc - \dfrac{c^2}{\gamma(\gamma-1)} ds$ Eq.(7)

From Eq.(2) with Eq. (7)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{2c}{\gamma-1}\frac{\partial c}{\partial x} - \frac{c^2}{\gamma(\gamma-1)}\frac{\partial s}{\partial x} = 0 \qquad \text{Eq.(8)}$$

From Eqs.(8) $\pm$ (6),

$$\frac{\partial}{\partial t}\left(u + \frac{2c}{\gamma-1}\right) + (u+c)\frac{\partial}{\partial x}\left(u + \frac{2c}{\gamma-1}\right) - \frac{c^2}{\gamma(\gamma-1)}\frac{\partial s}{\partial x} = 0 \rightarrow \frac{\partial R^+}{\partial t} + (u+c)\frac{\partial R^+}{\partial x} - \frac{c^2}{\gamma(\gamma-1)}\frac{\partial s}{\partial x} = 0 \quad \text{Eq.(9)}$$

$$\frac{\partial}{\partial t}\left(u - \frac{2c}{\gamma-1}\right) + (u-c)\frac{\partial}{\partial x}\left(u - \frac{2c}{\gamma-1}\right) - \frac{c^2}{\gamma(\gamma-1)}\frac{\partial s}{\partial x} = 0 \rightarrow \frac{\partial R^-}{\partial t} + (u-c)\frac{\partial R^-}{\partial x} - \frac{c^2}{\gamma(\gamma-1)}\frac{\partial s}{\partial x} = 0 \quad \text{Eq.(10)}$$

We may assume $\dfrac{c^2}{\gamma(\gamma-1)}\dfrac{\partial s}{\partial x} \cong 0$, and Eqs. (3), (4), (9) and (10) can be approximately expressed as

$$\begin{bmatrix} v \\ s \\ R^+ \\ R^- \end{bmatrix}_t + \begin{bmatrix} u & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & u+c & 0 \\ 0 & 0 & 0 & u-c \end{bmatrix} \begin{bmatrix} v \\ s \\ R^+ \\ R^- \end{bmatrix}_x = \mathbf{0} \qquad \text{Eq.(11)} \quad \text{(locally 1-D Riemann-invariant-like expression)}$$

- Note that $\begin{bmatrix} s \\ R^+ \\ R^- \end{bmatrix}_t + \begin{bmatrix} u & 0 & 0 \\ 0 & u+c & 0 \\ 0 & 0 & u-c \end{bmatrix} \begin{bmatrix} s \\ R^+ \\ R^- \end{bmatrix}_x = \mathbf{0}$ for 1-D Euler equation without approximation

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- **With Eq.(11), far-field BC can be specified by the sign of the locally 1-D wave-speed and the corresponding invariant.**
  - Supersonic flow with $|u| > c$
    → pure extrapolation from the inner computational cell, or specify everything as free stream values $(v, s, R^+, R^-)_{N+1} = (v, s, R^+, R^-)_N$ or $(v, s, R^+, R^-)_\infty$
  - Subsonic flow with $|u| < c$
    • $c > u > 0$: extrapolate v, s, R^+ and specifiy R^- : $(v, s, R^+)_{N+1} = (v, s, R^+)_N$ and $R^-_{N+1} = R^-_\infty$
    • $-c < u < 0$: extrapolate v, s, R^- and specifiy R^+ : $(v, s, R^-)_{N+1} = (v, s, R^-)_\infty$ and $R^+_{N+1} = R^+_N$
    $\rightarrow (u, v, c, s)_{N+1}$ are obtained. $\rightarrow$ all other physical variables at far-field boundary can be determined.