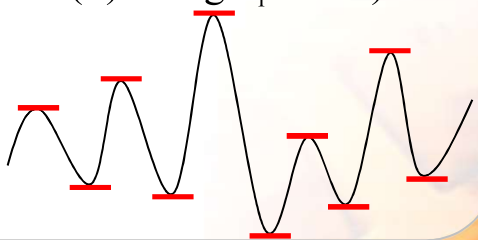
Chap. 2-2. Non-linear Stability and Total Variation

- ***Linear Stability and Non-linear Stability***
  - **Linear stability**
    - Linear schemes with constant coefficients → Fourier error analysis and superposition principle → Restriction on amplification factor to curb the unbounded growth of error, $|g| = |\hat{u}^{n+1}/\hat{u}^n| \le 1$ → boundedness of CFL number or time-step → Convergence to the exact sol. of PDE with some norms (Lax equivalence theorem)
  - **Non-linear stability**
    - Schemes become non-linear with coefficients containing solution → Fourier error analysis superposition principle are no longer available.
    - Treatment of G-W phenomenon/Control of oscillation across discontinuity → Treatment of local extrema and their behavior is essential. → Any useful tool?
- ***Total Variation as a Tool for Stability Criterion***
  - **As a way to connect non-linear stability with a convergence of a computed solution, total variation is considered.**

    $TV\left(u(x,t)\right) \equiv \left\| u'(x,t) \right\|_{L_1} = \int_{-\infty}^{\infty} \left| u'(x) \right| dx$ (total measure of oscillation of $u(x)$ using L_1 norm)
  - **In a discrete form**

    $$TV\left(u_D\right) \equiv \sum_{j=-\infty}^{\infty} \left| u_j - u_{j-1} \right| = 2\left(\sum \text{maxima} - \sum \text{minima} \right)$$

    $\rightarrow$ TV is a useful tool to measure local oscillation in $1-\mathrm{D}$ setting.

Chap. 2-2. Non-linear Stability and Total Variation

- ***Space of 'Total Variation Stable' Functions in*** D=[-M,M]×[0,T]

  Consider $\mathbf{H} = \left\{ u \in L_{1,T} :\ TV_D\left(u\right) \le R \ \text{ and } \ \text{supp}\left(u\left(x,t\right)\right) \subset \left[-M, M\right] \ \text{ for all } t \in \left[0,T\right] \right\}$

  - $u \in L_{1,T} \ \rightarrow \ \left\|u\right\|_{1,T} = \int_0^T \int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial x}\right| dxdt$ is bounded.
  - $TV_D\left(u\right) \le R \ \rightarrow \ TV_D\left(u\right) = \int_0^T \int_{-\infty}^{\infty} \left(\left|\frac{\partial u}{\partial x}\right| + \left|\frac{\partial u}{\partial t}\right| \right) dxdt$ is bounded by R.
  - $\text{supp}\left(u\left(x,t\right)\right) \subset \left[-M, M\right]$ for all $t \in \left[0,T\right] \ \rightarrow$ for all $t \in \left[0,T\right]$, $u\left(x,t\right) = 0$ if $\left|x\right| > M$
  - **H is a 'compact set' in $L_{1,T}$ → Every sequence in H has a 'convergent subsequence' in H → By combining it with the Lax-Wendroff theorem, the convergence to 'a weak solution' of SCL is guaranteed.**
    - (Lax and Wendroff) If a consistent and conservative scheme yields a converged solution, the solution converges to a weak solution of SCL.
    - Compare TV(total variation) stability with Lax equivalence theorem for linear stability
      - To be a conservative scheme is an additional essential element for capturing discontinuity in non-linear problems
      - Difference in guaranteeing the quality of convergence
        - Non-linear stability does not necessarily converge to the physically correct solution → entropy condition
      - Convergence characteristics of linear/non-linear stabilities still depend on norms

Chap. 2-2. Non-linear Stability and Total Variation

- (***Discrete Total Variation Stability in H***) **For a conservative scheme with a Lipschitz-continuous numerical flux** $F^n_{j+1/2}$**,**

$$TV_D(u_D) \le \sum_{n=0}^{N=[T/\Delta t]+1} \sum_{j=-\infty}^{\infty} \left[\Delta t \left| u^n_{j+1} - u^n_j \right| + \Delta x \left| u^{n+1}_j - u^n_j \right| \right] = \sum_{n=0}^{N} \left(\Delta t \cdot TV(u^n) + \left\| u^{n+1} - u^n \right\|_1 \right)$$

$$\le \sum_{n=0}^{N} ('\alpha'\Delta t + '\beta'\Delta t) = (\alpha+\beta)\Delta t N = T(\alpha+\beta) \equiv R$$

- For the conservative finite volume discretization,

$$u^{n+1}_j - u^n_j = \frac{\Delta t}{\Delta x}(F^n_{j+1/2} - F^n_{j-1/2}) \text{ and } \left\| u^{n+1} - u^n \right\|_1 = \Delta t \sum_{j=-\infty}^{\infty} \left| F^n_{j+1/2} - F^n_{j-1/2} \right|$$

From a Lipschitz-continuous numerical flux,

$$\left| F^n_{j+1/2} - F^n_{j-1/2} \right| = \left| F_{j+1/2}\left(u^n_{j-p}, u^n_{j-p+1}, \dots, u^n_{j+q}\right) - F_{j-1/2}\left(u^n_{j-p-1}, u^n_{j-p}, \dots, u^n_{j+q-1}\right) \right|$$

$$\le K \max_{-p \le r \le q} \left| u^n_{j+r} - u^n_{j+r-1} \right| \le K \sum_{r=-p}^{q} \left| u^n_{j+r} - u^n_{j+r-1} \right|$$

Thus, $\left\| u^{n+1} - u^n \right\|_1 \le K\Delta t \sum_{j=-\infty}^{\infty} \sum_{r=-p}^{q} \left| u^n_{j+r} - u^n_{j+r-1} \right| \le K\Delta t \sum_{r=-p}^{q} TV(u^n) \le K\Delta t(q+p+1)\alpha = \beta\Delta t$

- $TV(u^n) \le \alpha$ is realized by *enforcing a strong* TVD(TV diminishing) *condition* that $TV(u^{n+1}) \le TV(u^n)$ for all n and $\Delta t : TV(u^{n+1}) \le TV(u^n) \le \dots TV(u^1) \le TV(u^0) = \alpha$. As a relaxed condition, the TVB(TV bounded) condition can be considered: $TV(u^n) \le M$ for all n and Δt. $\rightarrow$ Since the exact solution of SCL is constant along the characteristic, SCL has a non-increasing TV property.

Chap. 2-3. High Resolution Monotonic Schemes

- ***Flux Corrected Transport (FCT) Method and Flux Limiter***
  - **The first algorithm that recognizes the importance of Godunov's theorem and introduces a way of non-linear limiting the cell-interface flux**
    - See the works by Boris and Book(1973), Zalesak(1979), and others
    - See also the work by V. P. Kolgan(1972) mentioned by Van Leer(2011)
  - **For** $u_t + au_x = 0$ with $u_j^{n+1} = u_j^n - \dfrac{\Delta t}{\Delta x}\left(F_{j+1/2} - F_{j-1/2}\right)$

$$\bullet \text{ Design } F_{j+1/2} \quad s.t. \;\; F_{j+1/2} = \begin{cases} \text{2nd-order (or more) in smooth region} \\ \text{1st-order across local extrema} \end{cases}$$

$$\rightarrow \text{Let } \begin{cases} F_{j+1/2}^H : \text{a 2nd-order 'non-monotonic' flux (say, Lax-Wendroff flux)} \\ F_{j+1/2}^L : \text{a 1st-order 'monotonic' flux (say, upwind flux)} \end{cases}$$

$$F_{j+1/2} = F_{j+1/2}^L + \alpha_{j+1/2}\left(F_{j+1/2}^H - F_{j+1/2}^L\right) \equiv F_{j+1/2}^L + F_{j+1/2}^C \quad \text{with } \alpha_{j+1/2} \begin{cases} \approx 1 \;\; \text{for smooth region} \\ \approx 0 \;\; \text{near local extrema} \end{cases}$$

  - **Two-step procedure**
    - S1) Compute $F_{j+1/2}^L,\ F_{j+1/2}^H$ from u_j^n
    - S2) Define 'anti-diffusive' flux

$$\tilde{F}_{j+1/2} = F_{j+1/2}^H - F_{j+1/2}^L = d_{j+1/2}^L - d_{j+1/2}^H = \varepsilon \Delta u_{j+1/2}^n \;\; (\varepsilon = \varepsilon^L - \varepsilon^H > 0)$$

    - S3) Obtain the intermediate lower-order (or 1st-order) 'monotonic' solution

Chap. 2-3. High Resolution Monotonic Schemes

$$\bar{u}_j = u_j^n - \frac{\Delta t}{\Delta x}\left(F^L_{j+1/2} - F^L_{j-1/2}\right)$$

- S4) Correct $\tilde{F}_{j+1/2}$ s.t. the final updated solution (u_j^{n+1}) is free of extrema not found in $\bar{u}_j$ or u_j^n

$$F^C_{j+1/2} \equiv \alpha_{j+1/2}\tilde{F}_{j+1/2} = \alpha_{j+1/2}\left(d^L_{j+1/2} - d^H_{j+1/2}\right) = \alpha_{j+1/2}\varepsilon\Delta u^n_{j+1/2} = \varepsilon^C\Delta u^n_{j+1/2} \quad \text{with } 0 \le \alpha_{j+1/2} \le 1$$

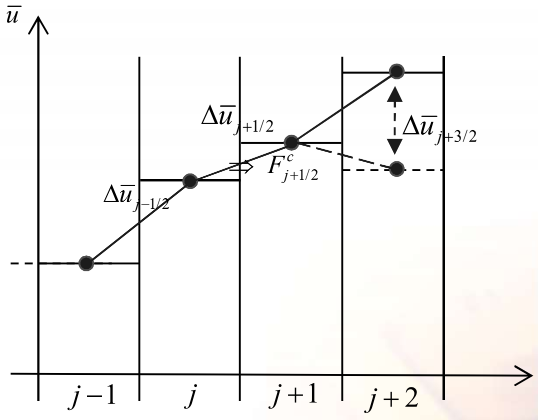
- S5) Update the final solution with the corrected flux $F^C_{j+1/2}$

$$u_j^{n+1} = \bar{u}_j - \frac{\Delta t}{\Delta x}\left(F^C_{j+1/2} - F^C_{j-1/2}\right) = u_j^n - \frac{\Delta t}{\Delta x}\left\{ \left[F^L_{j+1/2} + \alpha_{j+1/2}\left(F^H_{j+1/2} - F^L_{j+1/2}\right)\right]_{j+1/2} - [\ldots]_{j-1/2} \right\}$$

- S6) The corrected flux $F^C_{j+1/2}$ is designed as

$$F^C_{j+1/2} = \min\text{mod}\left(\frac{\Delta x}{\Delta t}\Delta\bar{u}_{j-1/2}, \tilde{F}_{j+1/2}, \frac{\Delta x}{\Delta t}\Delta\bar{u}_{j+3/2}\right) = \min\text{mod}\left(\frac{\Delta x}{\Delta t}\Delta\bar{u}_{j-1/2}, \varepsilon^C\Delta u^n_{j+1/2}, \frac{\Delta x}{\Delta t}\Delta\bar{u}_{j+3/2}\right)$$

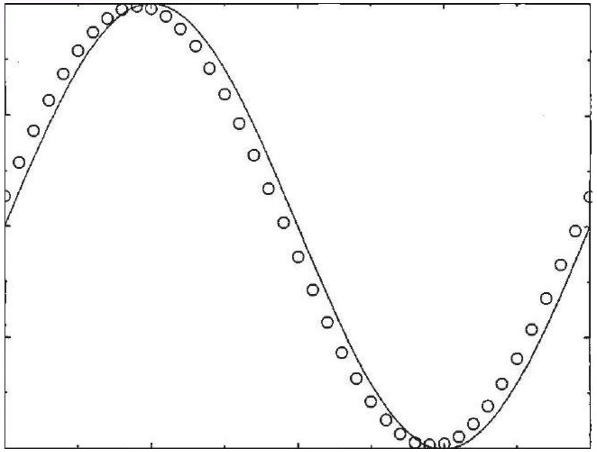
- The anti-diffusive flux is controlled by the intermediate monotonic solution such that it does not create new local extrema.
  - Monitor the intermediate monotonic soln. to check local extrema
  - Updated soln. satisfies the monotonic constraint in terms of the intermediate distribution
    $\min(\bar{u}_{j-1}, \bar{u}_j, \bar{u}_{j+1}) \le u_j^{n+1} \le \max(\bar{u}_{j-1}, \bar{u}_j, \bar{u}_{j+1})$
- Overall construction is largely based on numerical intuition lacking theoretical basis or mathematical rigor.
- Two-step procedure
  → Generalized one-step procedure by Zalesak



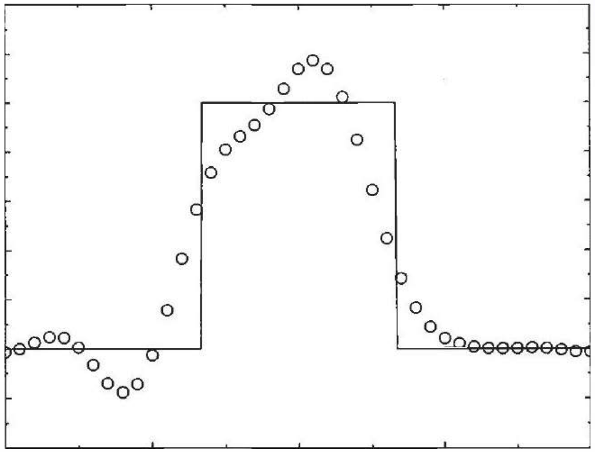
<Determination of the corrected flux>

Chap. 2-3. High Resolution Monotonic Schemes

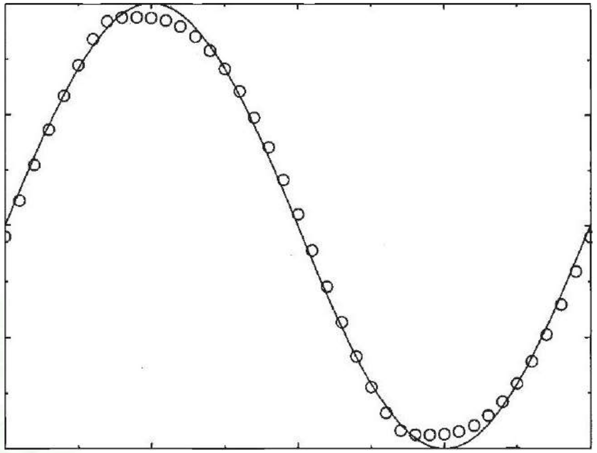
- **Example**
  - Linear advection problem with smooth and discontinuous profiles



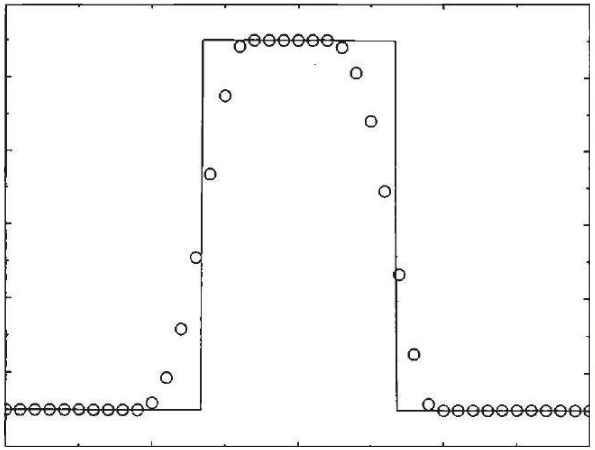
Lax-Wendroff method for smooth profile



Lax-Wendroff method for discontinuous profile



FCT method for smooth profile



FCT method for discontinuous profile

Chap. 2-3. High Resolution Monotonic Schemes

- ***TVD Schemes using Flux Limiters***
  - **A class of one-step monotonic schemes using a refined form of flux limiters**
    - See the works by Harten(1983, 1984), Sweby(1984), Yee(1989), and others
    - Solid mathematical foundation with TVD stability

• Let $F^H_{j+1/2}$ be a 2nd-order non-monotonic flux and $F^L_{j+1/2}$ be a 1st-order monotonic flux,

and the limited flux form is assumed as $F_{j+1/2} = F^L_{j+1/2} + \phi_j \left(F^H_{j+1/2} - F^L_{j+1/2} \right)$.

ϕ_j is a limiter function which monotors the local behavior of solution u^n_j.

• Take $F^H_{j+1/2}$ as the L-W flux, and $F^L_{j+1/2}$ as the upwind flux

For $u_t + au_x = 0$ with $a > 0$, the L-W scheme becomes

$$u^{n+1}_j = u^n_j - \frac{\sigma}{2}\left(u^n_{j+1} - u^n_{j-1}\right) + \frac{\sigma^2}{2}\left(u^n_{j+1} - 2u^n_j + u^n_{j-1}\right)$$

$$= u^n_j - \sigma\left(u^n_j - u^n_{j-1}\right) - \frac{\sigma(1-\sigma)}{2}\left(u^n_{j+1} - 2u^n_j + u^n_{j-1}\right) = u^n_j - \frac{\Delta t}{\Delta x}\left(F_{j+1/2} - F_{j-1/2}\right)$$

Thus, $F_{j+1/2} = \underbrace{au^n_j}_{\text{upwind}} + \underbrace{\frac{a(1-\sigma)}{2}\Delta u^n_{j+1/2}}_{\substack{\text{Lax-Wendroff correction} \\ \text{yielding lagging oscillations}}} \quad \rightarrow$ Consider $F_{j+1/2} = au_j + \phi_j \frac{a(1-\sigma)}{2}\Delta u_{j+1/2}$ with $\phi_j \geq 0$.

• Design the limiter function, $\phi_j = \phi(r_j)$, to meet with the TVD stability by introducing a parameter,

$r_j = \Delta u_{j-1/2} / \Delta u_{j+1/2}$, to measure the change of local slope (or total variation)

Chap. 2-3. High Resolution Monotonic Schemes

- **Three-point TVD schemes**
  - (*Harten*) The three-point scheme of the form $u_j^{n+1} = Ru^n = u_j^n - C_{j-1/2}\Delta u_{j-1/2}^n + D_{j+1/2}\Delta u_{j+1/2}^n$

    is TVD if $C_{j-1/2}$, $D_{j+1/2} \geq 0$, and $C_{j+1/2} + D_{j+1/2} \leq 1$ for all j.

    (pf) $\Delta u_{j+1/2}^{n+1} = u_{j+1}^{n+1} - u_j^{n+1} = \left(1 - C_{j+1/2} - D_{j+1/2}\right)\Delta u_{j+1/2} + D_{j+3/2}\Delta u_{j+3/2} + C_{j-1/2}\Delta u_{j-1/2}$

    $$\left|\Delta u_{j+1/2}^{n+1}\right| \leq \left(1 - C_{j+1/2} - D_{j+1/2}\right)\left|\Delta u_{j+1/2}\right| + D_{j+3/2}\left|\Delta u_{j+3/2}\right| + C_{j-1/2}\left|\Delta u_{j-1/2}\right|$$

    $$\xrightarrow{\sum_{-\infty}^{\infty}[......]} TV\left(u_j^{n+1}\right) = TV\left(Ru^n\right) = \sum_j \left|\Delta u_{j+1/2}^{n+1}\right| \leq \sum_j \left|\Delta u_{j+1/2}^n\right| = TV\left(u_j^n\right)$$

    - Re-interpretation of Godunov's positivity condition using TV or enforcing TVD stability by positivity condition
  - Flux limited form of L-W scheme

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(F_{j+1/2} - F_{j-1/2})$$

$$= u_j^n - \frac{\Delta t}{\Delta x}\left\{a\left(u_j^n - u_{j-1}^n\right) + \frac{a(1-\sigma)}{2}\left(\Delta u_{j+1/2}\phi_j - \Delta u_{j-1/2}\phi_{j-1}\right)\right\}$$

$$= u_j^n - C_{j-1/2}\Delta u_{j-1/2}^n + D_{j+1/2}\Delta u_{j+1/2}^n \quad \rightarrow C_{j-1/2} = \underbrace{\sigma\left[1 - \frac{(1-\sigma)}{2}\phi_{j-1}\right]}_{\text{positive}}, \quad D_{j+1/2} = -\underbrace{\frac{\sigma(1-\sigma)}{2}\phi_j}_{\text{negative}}$$

With $D_{j+1/2}\Delta u_{j+1/2}^n = D_{j+1/2}\dfrac{\Delta u_{j+1/2}^n}{\Delta u_{j-1/2}^n}\Delta u_{j-1/2}^n$, $C_{j-1/2} = \sigma + \dfrac{\sigma(1-\sigma)}{2}\left[\phi_j \dfrac{\Delta u_{j+1/2}^n}{\Delta u_{j-1/2}^n} - \phi_{j-1}\right]$ and $D_{j+1/2} = 0$

• Note that TVD stability is based on linear stability, and does not give time-step information.

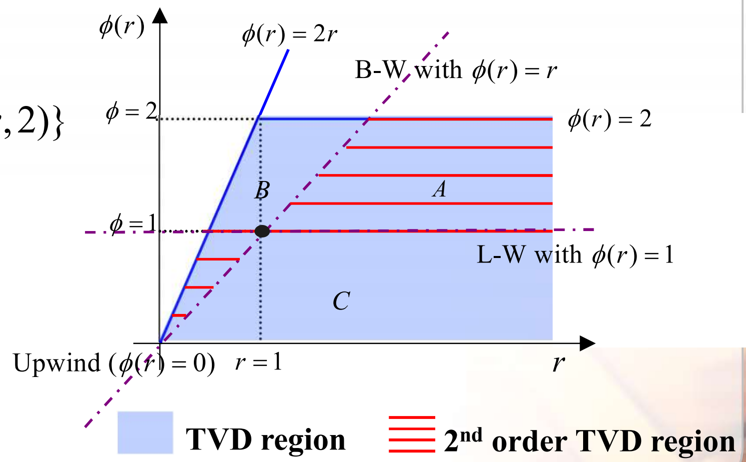
Chap. 2-3. High Resolution Monotonic Schemes

- TVD condition is satisfied if $0 \le C_{j-1/2} \le 1$ for all j.

$$0 \le \sigma + \frac{\sigma(1-\sigma)}{2}\left[\phi_j \frac{\Delta u^n_{j+1/2}}{\Delta u^n_{j-1/2}} - \phi_{j-1}\right] \le 1 \rightarrow \left|\frac{\phi_j}{r_j} - \phi_{j-1}\right| \le 2 \text{ for all } j \text{ or } \left|\frac{\phi(r_1)}{r_1} - \phi(r_2)\right| \le 2$$

$$\text{Thus, we have } \phi(r) \text{ s.t. } \begin{cases} 0 \le \dfrac{\phi(r)}{r} \le 2 \text{ and } 0 \le \phi(r) \le 2, \text{ if } r \ge 0 \\ \phi(r) = 0, \text{ if } r < 0 \text{ (to prevent the accentuation of local extrema)} \end{cases}$$

- 2nd-order TVD Region and TVD limiters
  - $\phi(\mathrm{r})_{r=1}=1$ to get a smooth transition with second-order accuracy
  - Convex combination of L-W and B-W is desirable to avoid too much compression (region B) or too much diffusion (region C). Thus, the TVD region (region A) is preferred.
  - TVD limiters
    - superbee limiter: $\phi(r) = \max\{0,\ \min(1, 2r),\ \min(r, 2)\}$
    - van Leer limiter: $\phi(r) = (|\mathrm{r}|+\mathrm{r})/(1+|\mathrm{r}|)$
    - MC limiter: $\phi(r) = \max\{0,\ \min((1+r)/2, 2, 2r)\}$
    - minmod limiter: $\phi(r) = \min(1, r)$
    - many other limiters are possible.



Chap. 2-3. High Resolution Monotonic Schemes

- **Monotone stability**
  - Alternative (but more restrictive than TVD) approach to realize nonlinear stability (by Harten, Hyman and Lax)

• For $u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}\left(u_{j-p}^n, u_{j-p+1}^n, \ldots, u_{j+q}^n\right) - F_{j-1/2}\left(u_{j-p-1}^n, u_{j-p}^n, \ldots, u_{j+q-1}^n\right)\right)$

$= H\left(u_{j-p-1}^n, u_{j-p}^n, \ldots, u_{j-1}^n, u_j^n, u_{j+1}^n, \ldots, u_{j+q-1}^n, u_{j+q}^n\right)$, the scheme is called monotone

if $\frac{\partial u_j^{n+1}}{\partial u_{j+l}^n} = \frac{\partial H}{\partial u_{j+l}^n} \geq 0$, for all $l \in [-(p+1), q]$. Then, u_j^n converges to a weak solution of SCL.

• Lipschitz-continuous monotne flux

Consider three-point schemes with $u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}(u_j^n, u_{j+1}^n) - F_{j-1/2}(u_{j-1}^n, u_j^n)\right)$,

$$\frac{\partial H}{\partial u_{j+l}^n} \geq 0 \rightarrow \frac{\partial H}{\partial u_{j-1}^n} = \frac{\Delta t}{\Delta x}\frac{\partial F_{j-1/2}}{\partial u_{j-1}^n} \geq 0,\ \frac{\partial H}{\partial u_j^n} = 1 - \frac{\Delta t}{\Delta x}\left(\frac{\partial F_{j+1/2}}{\partial u_j^n} - \frac{\partial F_{j-1/2}}{\partial u_j^n}\right) \geq 0,\ \frac{\partial H}{\partial u_{j+1}^n} = -\frac{\Delta t}{\Delta x}\frac{\partial F_{j+1/2}}{\partial u_{j+1}^n} \geq 0.$$

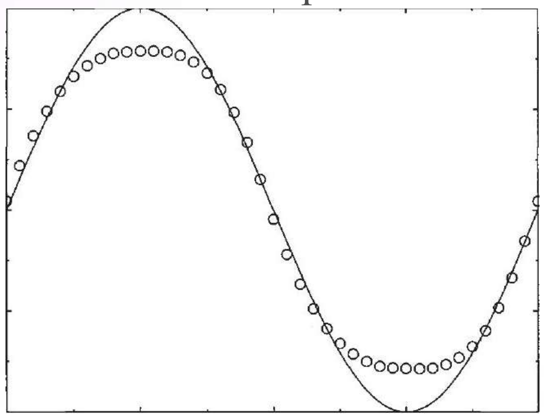
In general, $F_{j+1/2}(u_j, u_{j+1})$ is a Lipschitz-continuous monotne flux, if $\frac{\partial F_{j+1/2}}{\partial u_j} \geq 0$ and $\frac{\partial F_{j+1/2}}{\partial u_{j+1}} \leq 0.$

Several fluxes belong to this category such as

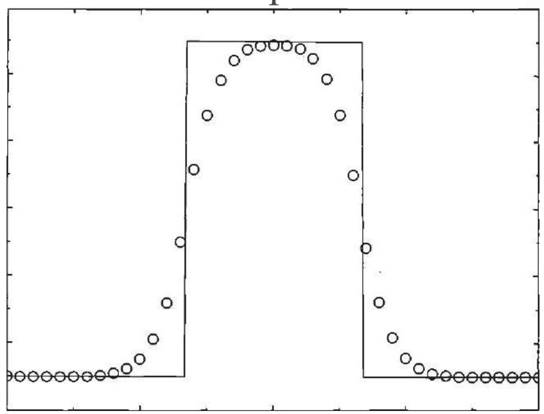
- $F_{j+1/2,up} = \frac{a}{2}\left(u_j^n + u_{j+1}^n\right) - \frac{|a|}{2}\Delta u_{j+1/2}^n$ - $F_{j+1/2,LLF} = \frac{a}{2}\left(u_j^n + u_{j+1}^n\right) - \max_{u\in[u_j, u_{j+1}]}|a(u)|\Delta u_{j+1/2}^n$
- $F_{j+1/2,EO} = \frac{a}{2}\left(u_j^n + u_{j+1}^n\right) - \frac{1}{2}\int_{u_j}^{u_{j+1}}|a(u)|\,dl$

Chap. 2-3. High Resolution Monotonic Schemes

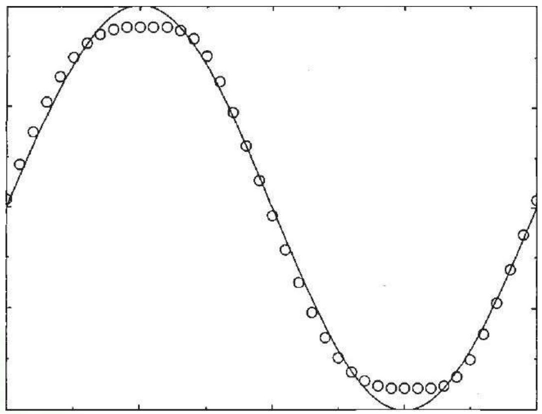
- **Example**
  - Linear advection problem with smooth and discontinuous profiles



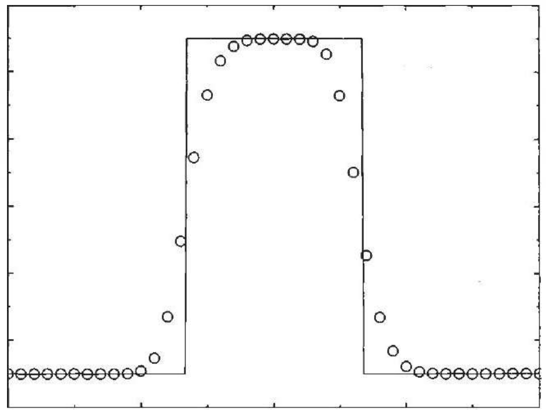
Flux-limited method with minmod limiter



Flux-limited method with minmod limiter

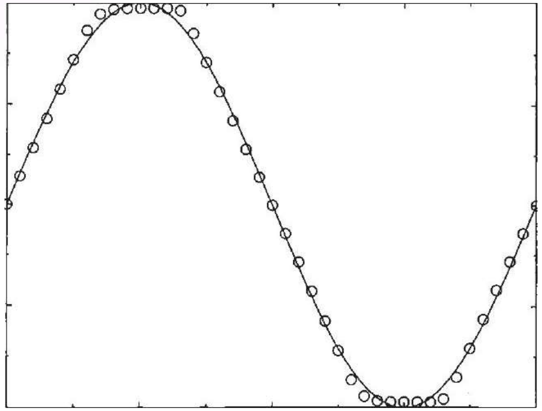


Flux-limited method with van Leer limiter

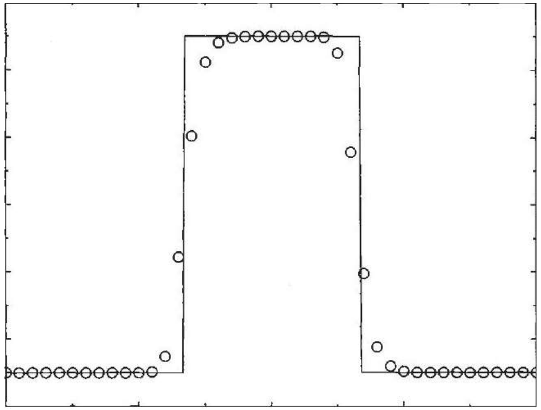


Flux-limited method with van Leer limiter

- **Example**
  - Linear advection problem with smooth and discontinuous profiles



Flux-limited method with superbee limiter



Flux-limited method with superbee limiter

  - Performance of limiters depending on flow regions
    - Sharp capturing of discontinuous region and preservation of smooth extrema
    - Clipping error of extrema : $O(\Delta x) \sim O(\Delta x^2)$

- **Implicit TVD formulation**

(*Harten*) The three-point one-step general implicit formulation, $Lu^{n+1} = Ru^n$, is TVD

$$u_j^{n+1} - \theta(D_{j+1/2}\Delta u_{j+1/2} - C_{j-1/2}\Delta u_{j-1/2})^{n+1} = u_j^n + (1-\theta)(D_{j+1/2}\Delta u_{j+1/2} - C_{j-1/2}\Delta u_{j-1/2})^n \text{ with } 0 \le \theta \le 1,$$

if the following conditions hold for all j.

$$\left.\begin{array}{l} \bullet\ C_{j-1/2},\ D_{j+1/2} \ge 0, \text{ and } (1-\theta)(C_{j+1/2} + D_{j+1/2}) \le 1 \ \rightarrow\ \mathrm{TV}(Ru^n) \le \mathrm{TV}(u^n) \\ \bullet\ K \le -\theta(C_{j-1/2}, D_{j+1/2}) \le 0 \text{ for some constant K.} \ \rightarrow\ \mathrm{TV}(Lu^{n+1}) \ge \mathrm{TV}(u^{n+1}) \end{array}\right\} \Rightarrow\ \mathrm{TV}(u^{n+1}) \le \mathrm{TV}(u^n)$$

- **For arbitrary wave speed *a***

$$u_j^{n+1} = u_j^n - \frac{\sigma}{2}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{\sigma^2}{2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

$$= \begin{cases} u_j^n - \sigma\left(u_j^n - u_{j-1}^n\right) - \dfrac{\sigma(1-\sigma)}{2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) & \text{if } a \geq 0 \\ u_j^n - \sigma\left(u_{j+1}^n - u_j^n\right) + \dfrac{\sigma(1+\sigma)}{2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) & \text{if } a \leq 0 \end{cases}$$

Thus, we have $F_{j+1/2} = \begin{cases} au_j + \dfrac{a(1-\sigma)}{2}\Delta u_{j+1/2} \rightarrow au_j + \dfrac{a(1-\sigma)}{2}\Delta u_{j+1/2}\phi_j & \text{if } a \geq 0, \\ au_{j+1} - \dfrac{a(1+\sigma)}{2}\Delta u_{j+1/2} \rightarrow au_{j+1} - \dfrac{a(1+\sigma)}{2}\Delta u_{j+1/2}\phi_{j+1} & \text{if } a \leq 0. \end{cases}$

It can be shown that $F_{j+1/2}$ satisfies the three-point TVD condition with ϕ_j in the 2nd-order TVD region

s.t. $\phi_j = \begin{cases} \phi(r_j^+) & \text{if } a \geq 0 \\ \phi(r_j^-) & \text{if } a \leq 0 \end{cases}$ with $r_j^+ = \dfrac{\Delta u_{j-1/2}}{\Delta u_{j+1/2}} = \dfrac{1}{r_j^-}$. Finally, we have

$$F_{j+1/2} = \frac{a}{2}\left(u_j^n + u_{j+1}^n\right) - \frac{|a|}{2}\Delta u_{j+1/2}^n + \left[\text{sgn}(a) - \sigma\right]\frac{\phi_{j'}}{2}a\Delta u_{j+1/2}^n$$

$$= \frac{a}{2}\left(u_j^n + u_{j+1}^n\right) + \frac{a}{2}\left[\phi_{j'}\left(\text{sgn}(a) - \sigma\right) - \text{sgn}(a)\right]\Delta u_{j+1/2}^n \quad \text{with } j' = \begin{cases} j & \text{if } a \geq 0 \\ j+1 & \text{if } a \leq 0 \end{cases}$$

Chap. 2-3. High Resolution Monotonic Schemes

- **For non-linear case**
  - Similar formulation for variable wave speed leads to

$$F_{j+1/2} = \begin{cases} f_j^n + \dfrac{a_{j+1/2}}{2}\left(1-\sigma_{j+1/2}\right)\Delta u_{j+1/2}\phi_j & \text{if } a_{j+1/2} \geq 0 \\ f_{j+1}^n - \dfrac{a_{j+1/2}}{2}\left(1+\sigma_{j+1/2}\right)\Delta u_{j+1/2}\phi_{j+1} & \text{if } a_{j+1/2} \leq 0 \end{cases}$$

$$= \frac{1}{2}\left(f_j^n + f_{j+1}^n\right) - \frac{\left|a_{j+1/2}\right|}{2}\Delta u_{j+1/2}^n + \left[\operatorname{sgn}\left(a_{j+1/2}\right) - \sigma_{j+1/2}\right]\frac{\phi_{j'}}{2} a_{j+1/2}\Delta u_{j+1/2}^n$$

$$= \frac{1}{2}\left(f_j^n + f_{j+1}^n\right) + \frac{a_{j+1/2}}{2}\left[\phi_{j'}\left(\operatorname{sgn}\left(a_{j+1/2}\right) - \sigma_{j+1/2}\right) - \operatorname{sgn}\left(a_{j+1/2}\right)\right]\Delta u_{j+1/2}^n$$

$$\text{with } a_{j+1/2} = \begin{cases} \dfrac{f_{j+1}^n - f_j^n}{u_{j+1}^n - u_j^n} & \text{if } u_{j+1}^n \neq u_j^n \\ f'\left(u_j^n\right) & \text{if } u_{j+1}^n = u_j^n \end{cases} \text{ and } \sigma_{j+1/2} = \left|a_{j+1/2}\right|\frac{\Delta t}{\Delta x}, \text{ and } j' = \begin{cases} j & \text{if } a_{j+1/2} \geq 0 \\ j+1 & \text{if } a_{j+1/2} \leq 0 \end{cases}$$

  - Monitor local flow behavior based on solution difference or flux difference

$$r_j^+ \left(= \frac{1}{r_j^-}\right) = \frac{\Delta u_{j-1/2}}{\Delta u_{j+1/2}} \text{ or } \frac{(F^{LW} - F^{UP})_{j-1/2}}{(F^{LW} - F^{UP})_{j+1/2}} = \frac{a_{j-1/2}(1-\sigma_{j-1/2})\Delta u_{j-1/2}}{a_{j+1/2}(1-\sigma_{j+1/2})\Delta u_{j+1/2}}$$