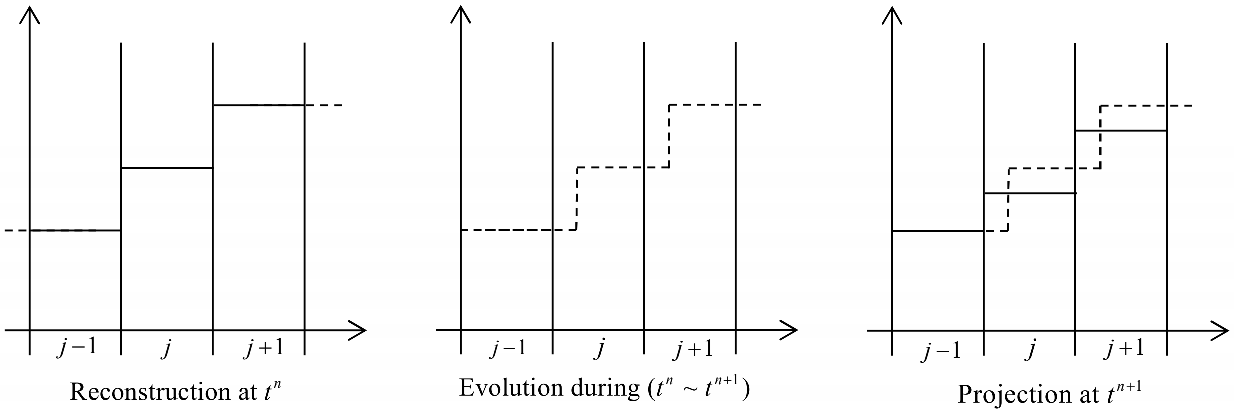
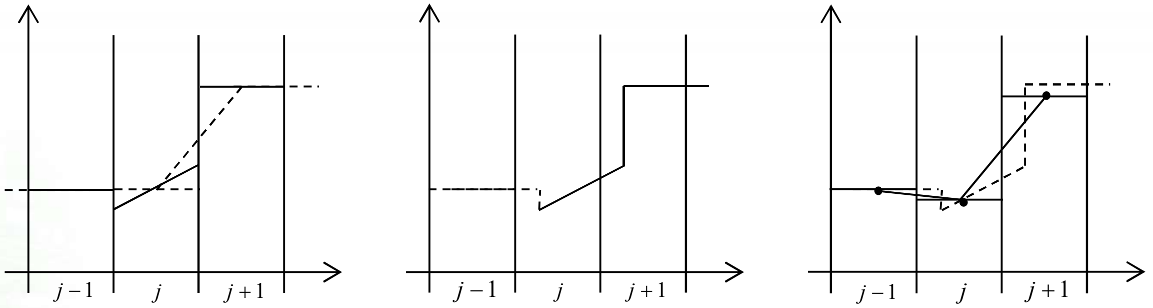
- ***MUSCL Schemes and Slope Limiters***
  - **Geometric interpretation on numerical scheme in terms of reconstruction, evolution and projection step.**
    - See the works by van Leer(1977, 1979), and others
    - Ex) Flow physics of first-order upwind scheme for $u_t + au_x = 0$, $a > 0$



- Reconstruction step: approximation of the exact initial distribution
  - In MUSCL, the initial profile is approximated (or reconstructed) by a linear profile.
- Evolution step: reconstructed initial profile is convected $\Delta s_x = a \cdot \Delta t$ by $u_t + au_x = 0$.
  - The reconstructed profile on Eulerian mesh is drifting in a Lagrangian manner.
  - Flux evaluation by tracing characteristic propagation of a reconstructed profile in *(x,t)* plane
- Projection step: projection of the convected profile to update a new solution at $t^{n+1} = t^n + \Delta t$.
  - The drifted profile is remapped onto Euler mesh to obtain cell-averaged values.

Chap. 2-3. High Resolution Monotonic Schemes

- Extension to 2nd-order reconstruction, evolution and projection step
  - Reconstruct a linear profile from cell-average values
    $u_r(x) = u_j + S_j^n (x - x_j),\ x_{j-1/2} \le x \le x_{j+1/2}$ with a local slope S_j^n
  - Upwinding flux evaluation at evolution time-step, Δt
    $$u_{j+1/2}(t) = u_j + S_j^n \left(\frac{\Delta x}{2} - a(t - t^n) \right) \rightarrow F_{j+1/2} = \int_{t^n}^{t^{n+1}} a u_{j+1/2}(t) dt = a u_j + S_j^n \frac{a \Delta x}{2}(1 - \sigma) \text{ with } a > 0$$
  - Depending on the treatment of slope, three variations at projection step
    · Scheme I: obtain u_j^{n+1} by, say, $S_j^n = \dfrac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$
    · Scheme II, III: obtain both solution(u_j^{n+1}) and slope(S_j^{n+1}), which resembles FEM-type solution approximation.
    $$\left.\begin{array}{l} \text{(II) } S_j^{n+1} = \dfrac{u_{j+1/2}(t^{n+1}) - u_{j-1/2}(t^{n+1})}{2\Delta x} = \dfrac{u_j^n - u_{j-1}^n}{\Delta x} + (S_j^n - S_{j-1}^n)(0.5 - \sigma) \\ \text{(III) } \displaystyle\int_{x_{j-1/2}}^{x_{j+1/2}} u_r^n (x - a\Delta t)(x - x_j) dx = \int_{x_{j-1/2}}^{x_{j+1/2}} u_r^{n+1}(x)(x - x_j) dx \end{array}\right\} \text{ to update } \begin{bmatrix} u_j^{n+1} \\ S_j^{n+1} \end{bmatrix}$$
  - Monotonic treatment of local slope is essential to be consistent with Godunov's constraint.



  - Non-monotonic estimation of the local slope creates a new (artificial) local extrema. → numerical oscillations

Chap. 2-3. High Resolution Monotonic Schemes

- If $TV(u(x)) \le TV(u^n)$ in reconstruction step, the whole steps are TVD, since evolution and projection steps are intrinsically total variation not increasing.
  → a higher-order monotonic solution at $t = t^n + \Delta t$
- Subcell distribution *u(x)* not creating new extrema after solution reconstruction
  - $\min(u_{j-1}, u_j, u_{j+1}) \le u(x) \le \max(u_{j-1}, u_j, u_{j+1})$ with

  $$\tilde{S}_j^n = \min\bmod\left(2\frac{u_j^n - u_{j-1}^n}{\Delta x}, S_j^n, 2\frac{u_{j+1}^n - u_j^n}{\Delta x}\right) \text{ or}$$

  Slope limiters to satisfy monotonic subcell distribution
- 2nd-order MUSCL flux with slope limiter and 1st-order upwind flux
  - 1st-order upwinding using cell-averaged values (u_j, u_{j+1})

  $$F_{j+1/2,\ 1st} = \frac{1}{2}\left(f_j + f_{j+1}\right) - \frac{\left|a_{j+1/2}\right|}{2}\Delta u_{j+1/2} = fn\left(u_j, u_{j+1}\right)$$

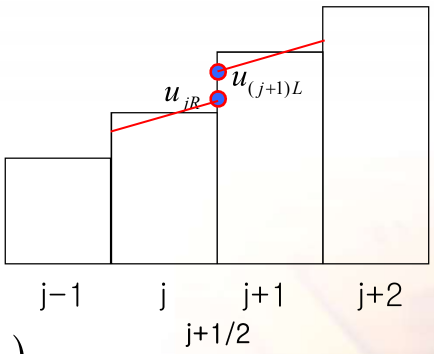
  - Linear distribution with slope limiting

  $$u(x) = u_j + \tilde{S}_j^n (x - x_j), \quad x_{j-1/2} \le x \le x_{j+1/2}$$

  $$u_{jR} = u(x_{j+1/2}) = u_j + \frac{\Delta x}{2}\tilde{S}_j^n \text{ and } u_{jL} = u(x_{j-1/2}) = u_j - \frac{\Delta x}{2}\tilde{S}_j^n$$

  - Replace u_j, u_{j+1} by $u_{jR}, u_{(j+1)L}$

  $$F_{j+1/2,\ 2nd} = \frac{1}{2}\left(f_{jR} + f_{(j+1)L}\right) - \frac{\left|a_{(j+1/2)(L,R)}\right|}{2}\Delta u_{(j+1/2)(L,R)} = fn\left(u_{jR}, u_{(j+1)L}\right)$$



Chap. 2-3. High Resolution Monotonic Schemes

- **PPM – A quadratic extension of MUSCL approach**
  - Initial reconstruction is approximated by a quadratic polynomial.

    Assume $u(x) = a_0 + a_1(x - x_j) + a_2(x - x_j)^2$ with $x_{j-1/2} \le x \le x_{j+1/2}$,

    and determne a_0, a_1 and a_2 from

    $$u_j = \int_{x_{j-1/2}}^{x_{j+1/2}} u(x)dx,\ u_{L,j} = u(x_{j-1/2}) = a_0 - a_1\frac{\Delta x}{2} + a_2\frac{\Delta x^2}{4} \text{ and } u_{R,j} = a_0 + a_1\frac{\Delta x}{2} + a_2\frac{\Delta x^2}{4}$$

    $$\rightarrow\ a_0 = u_j - \frac{\Delta x^2}{12}a_2,\ a_1 = \frac{u_{R,j} - u_{L,j}}{\Delta x} \text{ and } a_2 = \frac{u_{R,j} - u_{L,j}}{\Delta x^2}\left(\frac{u_{R,j} + u_{L,j}}{2} - u_j\right)$$

    $u_{R,j}$ and $u_{L,j}$ are then determined by some complicated blending of limited 4th-order or 2nd-order accurate approximations of $u_{j+1/2}$, with many extra techniques, such as shock sensing and artificial compression.

    - More details can be found in Colella and Woodward (1984).

- ***LED Schemes using Positivity Condition***
  - **Start from a semi-discrete form,** $\Delta x \dfrac{du_j}{dt} + F_{j+1/2} - F_{j-1/2} = 0$
    - For first-order scheme

Take $F_{j+1/2} = \dfrac{1}{2}\left(f_j + f_{j+1}\right) - d_{j+1/2}\left(= \alpha_{j+1/2}\Delta u_{j+1/2}\right)$ and estimate $a_{j+1/2} = \begin{cases} \dfrac{f_{j+1}^n - f_j^n}{u_{j+1}^n - u_j^n} & \text{if } u_{j+1}^n \neq u_j^n \\ f'\left(u_j^n\right) & \text{if } u_{j+1}^n = u_j^n \end{cases}$

$$F_{j+1/2} = \frac{1}{2}\left(f_j + f_{j+1}\right) - \alpha_{j+1/2}\Delta u_{j+1/2} \xrightarrow{f_{j+1} = f_j + a_{j+1/2}\Delta u_{j+1/2}} f_j + \left(\frac{a_{j+1/2}}{2} - \alpha_{j+1/2}\right)\Delta u_{j+1/2}$$

$$F_{j-1/2} = \frac{1}{2}\left(f_j + f_{j-1}\right) - \alpha_{j-1/2}\Delta u_{j-1/2} \xrightarrow{f_{j-1} = f_j - a_{j-1/2}\Delta u_{j-1/2}} f_j - \left(\frac{a_{j-1/2}}{2} + \alpha_{j-1/2}\right)\Delta u_{j-1/2}$$

Thus, $\Delta x \dfrac{du_j}{dt} = -F_{j+1/2} + F_{j-1/2} = \left(\alpha_{j+1/2} - \dfrac{a_{j+1/2}}{2}\right)\Delta u_{j+1/2} - \left(\alpha_{j-1/2} + \dfrac{a_{j-1/2}}{2}\right)\Delta u_{j-1/2}$

$$= C_j^+\left(u_{j+1} - u_j\right) + C_j^-\left(u_{j-1} - u_j\right) \text{ with } C_j^+ = \alpha_{j+1/2} - \frac{a_{j+1/2}}{2},\ C_j^- = \alpha_{j-1/2} + \frac{a_{j-1/2}}{2}$$

• For $\dfrac{du_j}{dt} = \sum_{i \neq j} c_{ij}\left(u_i - u_j\right)$ with the local stencil of u_j, if $c_{ij} \geq 0$ (positivity condition), local maxima do not increase and local minima do not decrease (LED condition).

From $C_j^{\pm} \geq 0$, $\alpha_{j+1/2} \geq \left|a_{j+1/2}\right|/2$ for all j, yielding the minimum amount of numerical diffusion to satisfy the LED condition is upwinding for the first-order scheme.

- **Design of limited anti-diffusion**
  - A simple candidate with arithmetic anti-diffusive flux

$$d_{j+1/2} = d^L_{j+1/2} - d^H_{j+1/2} = \alpha_{j+1/2}\left[\Delta u_{j+1/2} - \underbrace{\frac{1}{2}\left(\Delta u_{j+3/2} + \Delta u_{j-1/2}\right)}_{\text{3rd-order anti-diffusive flux}}\right]$$

    - This violates the positivity condition. → some limited form of anti-diffusion

  - Set $d_{j+1/2} = \alpha_{j+1/2}\left[\Delta u_{j+1/2} - L\left(\Delta u_{j+3/2}, \Delta u_{j-1/2}\right)\right]$ with a limited-average operator *L(u,v)*

• p1. $L(u,v) = L(v,u)$ • p2. $L(\alpha u, \alpha v) = \alpha L(u,v)$

• p3. $L(u,u) = u$ • p4. $L(u,v)\begin{cases} = 0 & \text{if } \operatorname{sgn}(u) \neq \operatorname{sgn}(v) \\ \neq 0 & \text{if } \operatorname{sgn}(u) = \operatorname{sgn}(v) \end{cases}$

Note that p1~p3 are natural properties for any average, and p4 is to impose positivity condition.

By setting $\alpha = 1/u\,(\text{or } 1/v)$,

• p1, p2: $L(1, v/u) = L(u,v)/u$ $\left(or\ L(u/v,1) = L(u,v)/v\right) \rightarrow L(u,v) = uL(1, v/u)$ $\left(\text{or } vL(u/v,1)\right)$

Consider $\varphi(r) = L(1,r) = rL(1/r,1)$ with $r = v/u$, then $\varphi(r) = r\varphi(1/r)$

• p3: $\varphi(1)=1$ • p4: $\varphi(r)\begin{cases} = 0 & \text{if } r < 0 \\ > 0 & \text{if } r > 0 \end{cases}$

Thus, $L(u,v)$ is another form (general form) of limting function.

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- With $d_{j+1/2} = \alpha_{j+1/2}\left[\Delta u_{j+1/2} - L\left(\Delta u_{j+3/2}, \Delta u_{j-1/2}\right)\right]$ and $F_{j+1/2} = \frac{1}{2}\left(f_j + f_{j+1}\right) - d_{j+1/2}$

$$\Delta x \frac{du_j}{dt} = -F_{j+1/2} + F_{j-1/2} = -\left(f_j + \frac{a_{j+1/2}}{2}\Delta u_{j+1/2} - d_{j+1/2}\right) + \left(f_j - \frac{a_{j-1/2}}{2}\Delta u_{j-1/2} - d_{j-1/2}\right)$$

$$= -\frac{a_{j+1/2}}{2}\Delta u_{j+1/2} + \alpha_{j+1/2}\left[\Delta u_{j+1/2} - L\left(\Delta u_{j+3/2}, \Delta u_{j-1/2}\right)\right] - \frac{a_{j-1/2}}{2}\Delta u_{j-1/2} - \alpha_{j-1/2}\left[\Delta u_{j-1/2} - L\left(\Delta u_{j+1/2}, \Delta u_{j-3/2}\right)\right]$$

$$= \left[\alpha_{j+1/2} - \frac{a_{j+1/2}}{2} + \alpha_{j-1/2} L\left(1, \frac{\Delta u_{j-3/2}}{\Delta u_{j+1/2}}\right)\right]\Delta u_{j+1/2} - \left[\alpha_{j-1/2} + \frac{a_{j-1/2}}{2} + \alpha_{j+1/2} L\left(\frac{\Delta u_{j+3/2}}{\Delta u_{j-1/2}}, 1\right)\right]\Delta u_{j-1/2}$$

$$= \left[\alpha_{j+1/2} - \frac{a_{j+1/2}}{2} + \alpha_{j-1/2}\phi\left(r_j^+\right)\right]\Delta u_{j+1/2} - \left[\alpha_{j-1/2} + \frac{a_{j-1/2}}{2} + \alpha_{j+1/2}\phi\left(r_j^-\right)\right]\Delta u_{j-1/2}$$

$$= C_j^+ (u_{j+1} - u_j) + C_j^- (u_{j-1} - u_j)$$

with $C_j^+ = \alpha_{j+1/2} - \frac{a_{j+1/2}}{2} + \alpha_{j-1/2}\phi\left(r_j^+\right), C_j^- = \alpha_{j-1/2} + \frac{a_{j-1/2}}{2} + \alpha_{j+1/2}\phi\left(r_j^-\right)$

$$r_j^+ = \Delta u_{j-3/2} / \Delta u_{j+1/2}, \quad r_j^- = \Delta u_{j+3/2} / \Delta u_{j-1/2}$$

By requiring $C_j^\pm \geq 0,$ the LED scheme is obtained if $\alpha_{j+1/2} \geq \frac{1}{2}\left|a_{j+1/2}\right|$ for all j.

- **Examples of LED limiters**
  - Let $S(u,v)=\frac{1}{2}(\mathrm{sgn}(u)+\mathrm{sgn}(v))=\begin{cases}0 & \text{if } \mathrm{sgn}(u)\neq\mathrm{sgn}(v)\\ \pm 1 & \text{if } \mathrm{sgn}(u)=\mathrm{sgn}(v)\end{cases}$

• Minmod: $L(u,v)=S(u,v)\min(u,v)$

• van Leer: $L(u,v)=S(u,v)\dfrac{2uv}{u+v}$

• superbee: $L(u,v)=S(u,v)\max\left(\min\left(2|u|,|v|\right),\min\left(2|v|,|u|\right)\right)$

• Limited arithmetic mean: $L(u,v)=S(u,v)\min(\dfrac{|u+v|}{2},\alpha|u|,\alpha|v|)$. It recovers the MC limiter if $\alpha=2$.

Or more gnerally,

• Augmented arithmetic average: $L(u,v)=\dfrac{1}{2}D(u,v)(u+v)$ with $D(u,v)=1-\left|\dfrac{u-v}{u+v}\right|^q$

$L(u,v)\big|_{q=1}=\min\mathrm{mod}, L(u,v)\big|_{q=2}=$ van Leer, $L(u,v)\big|_{q\to\infty}=$ arithmetic mean bouned by TVD region

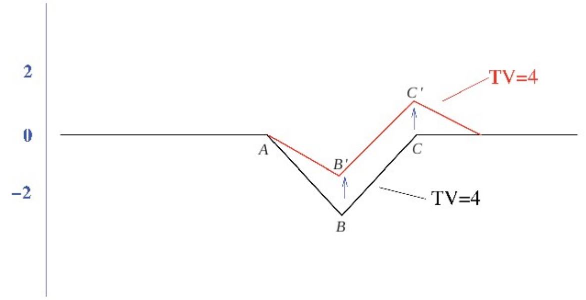
• α-mod: $L(u,v)=S(u,v)\dfrac{(1+\alpha)|u||v|}{\max\left(|u|,|v|\right)+\alpha\min\left(|v|,|u|\right)}=\begin{cases}\text{minmod if } \alpha=0\\ \text{van Leer if } \alpha=1\end{cases}$

• α-β-mod: $L(u,v)=S(u,v)\dfrac{(1+\alpha)|u|^{(1+\beta)/2}|v|^{(1+\beta)/2}}{\max\left(|u|^\beta,|v|^\beta\right)+\alpha\min\left(|v|^\beta,|u|^\beta\right)}$

If $\alpha=\beta=0$, $L(u,v)=\sqrt{uv}$ (limited geometric mean)

LED and TVD schemes, and Stability

- See the works by Jameson(1995), and others
- LED condition does not accentuate local extrema nor create new local extrema.
  - LED schemes $\underset{no}{\overset{yes}{\rightleftarrows}}$ TVD schemes
  - Monotonicity violation of TVD in multi-dimensional situation. $\rightarrow$ MLP condition



- LED schemes and l_∞ stability
  - Linear 1-D case

For $\dfrac{du_j}{dt} = \sum_{i \neq j} c_{ij}\left(u_i - u_j\right)$ with $c_{ij} = c_{ij}(\Delta x, \Delta t, a) > 0$ on a 'local' stencil of u_j,

consider $\dfrac{u_j^{n+1} - u_j^n}{\Delta t} = \sum_{i \neq j} c_{ij}\left(u_i - u_j\right)$

Then, $u_j^{n+1} = u_j^n + \sum_{i \neq j} \tilde{c}_{ij}\left(u_i^n - u_j^n\right) = (1 - \sum_{i \neq j} \tilde{c}_{ij})u_j^n + \sum_{i \neq j} \tilde{c}_{ij} u_i^n$ with $\tilde{c}_{ij} = c_{ij}\Delta t. \rightarrow v_i^{n+1} = \sum_j a_{ij} v_j^n$

From consistency, $\sum_j a_{ij} = 1$, and $\left|v_i^{n+1}\right| \leq \sum_j \left|a_{ij}\right|\left|v_j^n\right| \leq \sum_j \left|a_{ij}\right| \left\|v^n\right\|_\infty \rightarrow \left\|v^{n+1}\right\|_\infty \leq \left(\max_i \sum_j \left|a_{ij}\right|\right)\left\|v^n\right\|_\infty$

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- LED schemes can provide l_∞ stability (Cont'd)

Thus, l_∞ stability is maintained by requiring $\max_i \sum_j |a_{ij}| \le 1$ (or $\|A\|_\infty \le 1$ with $v^{n+1} = Av^n$).

From consistency ($\sum_j a_{ij} = 1$) and positivity ($a_{ij} \ge 0$), $\max_i \sum_j |a_{ij}| \le 1$.

This can be realized by $1 - \sum_{i \ne j} \tilde{c}_{ij}$ and by limiting the time-step as $\Delta t \le \dfrac{1}{\sum_{i \ne j} c_{ij}}$.

- Nonlinear multi-dimensional case

For $\dfrac{du_j}{dt} = \sum_{i \ne j} c_{ij}\left(u_i - u_j\right)$ with $c_{ij} = c_{ij}(u, \Delta x, \Delta t, a) > 0$ on a local stencil of u_j,

similar results can be obtained by assuming a Lipschitz-continuous monotne flux to evaluate $c_{ij}(u)$.

- See the works by Barth (2003), MLP schemes (Yoon and Kim (2008), Park and Kim (2010, 2012))