Chap. 2-3. High Resolution Monotonic Schemes

- ***ENO Schemes***
  - **Essentially-monotonic, arbitrary higher-order interpolation to overcome the shortcomings of TVD (or 2nd-order monotonic) methods**
    - See the works by Harten, Enquist, Osher, Chakravarthy(1987, 1989), and others
  - **TVD vs. ENO**
    - (TVD) Locally first-order accurate across all extrema to strictly enforce monotonicity → accuracy loss across smooth extrema due to excessive diffusion ($\sim O(\Delta x^{1+\varepsilon})$ error, minmod) or clipping ($\sim O(\Delta x^2)$ error, superbee)
    - (TVD) A fixed stencil → 2nd-order monotonic schemes with 5-point flux evaluation in general (*j-2 ~ j ~ j+2*) → difficult to realize higher-order TVD schemes
      - Harten's 3-point explicit/implicit TVD schemes can be generalized into multi-point counterparts (Jameson and Lax, 1986)
    - (ENO) Allow the increase of local extrema up to the order of truncation error to achieve higher-order accuracy across smooth extrema
    - (ENO) Use locally adaptive smooth stencil for higher-order interpolation
  - **Procedure to construct piecewise smooth polynomial**
    - Start from, i) cell-averaged value(u_i) or primitive function(U) with $U(x) = \int_{x_0}^{x} u(x)dx, u = dU/dx$

      ii) flux function$\left(f_i\right)$

- **ENO polynomial reconstruction for u_i**
  - Basic idea: Newton's divided difference and Newton's polynomial

For $u_i, u_{i+1}, u_{i+2}, \ldots, u_{i+n,}$ consider $u[x_i] = u_i,\ u[x_i, x_{i+1}] = \{u[x_{i+1}] - u[x_i]\}/\Delta x = (u_{i+1} - u_i)/\Delta x,$

$u[x_i, x_{i+1}, x_{i+2}] = \{u[x_{i+1}, x_{i+2}] - u[x_i, x_{i+1}]\}/(2\Delta x) = \{(u_{i+2} - 2u_{i+1} + u_i)/\Delta x - (u_{i+1} - u_i)/\Delta x\}/(2\Delta x),$

$\ldots,\ u[x_i, x_{i+1}, \ldots, x_{i+n}] = \{u[x_{i+1}, x_{i+2}, \ldots, x_{i+n}] - u[x_i, x_{i+1}, \ldots, x_{i+n-n}]\}/(n\Delta x)$

With u_l at $(n+1)$ grid points $x_{l,\ i\le l\le i+n}$ and the sequence of divided differences,

we have the $n-th$ order Newton's polynomial of $p_n(x)$ to approximate $u(x)$ as

$$p_n(x) = u[x_i] + u[x_i, x_{i+1}](x - x_i) + u[x_i, x_{i+1}, x_{i+2}](x - x_i)(x - x_{i+1}) + \ldots$$
$$+ u[x_i, x_{i+1} \ldots, x_{i+n}](x - x_i)(x - x_{i+1})\ldots(x - x_{i+n-1}) \text{ with } p_n(x_l) = u_l$$

- If $u(x)$ is sufficiently smooth ($\in C^{n+1}$), $u[x_i, x_{i+1}, \ldots, x_{i+k}] = \left.\frac{1}{k!}\frac{d^k u}{dx^k}\right|_\xi$ and thus

$$e_n(x) = u(x) - p_n(x) = (x - x_i)(x - x_{i+1})\ldots(x - x_{i+n})\frac{1}{(n+1)!}\left.\frac{d^{(n+1)}u}{dx^{(n+1)}}\right|_\xi \text{ with } x_i \le \xi \le x_{i+n}.$$

Thus, the magnitude of $u[x_i, x_{i+1}, \ldots, x_{i+k}]$ can be exploited to measure $e_{k-1}(x)$.

- If the $l-th$ derivative of $u(x)$ has a discontinuity at $x = x_p$, it can be shown that

$$u[x_i, x_{i+1}, \ldots, x_{i+k}] = O\left(\frac{1}{\Delta x^{k-l}}\left(\frac{d^l u(x_p^+)}{dx^l} - \frac{d^l u(x_p^-)}{dx^l}\right)\right) \text{ with } x_i \le x_p \le x_{i+k}.$$

$\rightarrow$ By checking the magintude of $u[x_i, x_{i+1}, \ldots, x_{i+k}]$, stencil can be chosen adaptively to minimize both the interpolation error and to avoid a discontinuity.

- **ENO polynomial reconstruction u_i (cont'd)**
  - Implementation: compare Newton's divided difference for each candidate stencil
    - (S1) Starting from the cell i, take $\begin{cases} i+1 & \text{if } \left|u\left[x_i, x_{i+1}\right]\right| \le \left|u\left[x_{i-1}, x_i\right]\right| \\ i-1 & \text{if } \left|u\left[x_i, x_{i+1}\right]\right| > \left|u\left[x_{i-1}, x_i\right]\right| \end{cases}$

$$\rightarrow\ u(x) = u_i + u\left[x_j, x_{j+1}\right](x - x_i),\quad x_{j-1/2} \le x \le x_{j+1/2} \quad \text{(minmod linear reconstruction)}$$

$$\text{with } j = \begin{cases} i & \text{if } \left|u\left[x_i, x_{i+1}\right]\right| \le \left|u\left[x_{i-1}, x_i\right]\right| \\ i-1 & \text{if } \left|u\left[x_i, x_{i+1}\right]\right| > \left|u\left[x_{i-1}, x_i\right]\right| \end{cases}$$

    - (S2) Repeat (S1) to select a piecewise parabolic and cubic interpolation polynomial

$$u(x) = u_i + u\left[x_j, x_{j+1}\right](x - x_i) + u\left[x_k, x_{k+1}, x_{k+2}\right](x - x_j)(x - x_{j+1}) \quad \text{(parabolic)}$$

$$\text{with } k = \begin{cases} j & \text{if } \left|u\left[x_j, x_{j+1}, x_{j+2}\right]\right| \le \left|u\left[x_{j-1}, x_j, x_{j+1}\right]\right| \\ j-1 & \text{if } \left|u\left[x_j, x_{j+1}, x_{j+2}\right]\right| > \left|u\left[x_{j-1}, x_j, x_{j+1}\right]\right| \end{cases}$$

Similarly for a piecewise-cubic reconstruction,

$$u(x) = u_i + u\left[x_j, x_{j+1}\right](x - x_i) + u\left[x_k, x_{k+1}, x_{k+2}\right](x - x_j)(x - x_{j+1})$$
$$+\, u\left[x_l, x_{l+1}, x_{l+2}, x_{l+3}\right](x - x_k)(x - x_{k+1})(x - x_{k+2})$$

$$\text{with } l = \begin{cases} k & \text{if } \left|u\left[x_k, x_{k+1}, x_{k+2}, x_{k+3}\right]\right| \le \left|u\left[x_{k-1}, x_k, x_{k+1}, x_{k+2}\right]\right| \\ k-1 & \text{if } \left|u\left[x_k, x_{k+1}, x_{k+2}, x_{k+3}\right]\right| > \left|u\left[x_{k-1}, x_k, x_{k+1}, x_{k+2}\right]\right| \end{cases}$$

- Implementation: compare Newton's divided difference for each candidate stencil (cont'd)
  - (S3) ENO stencil obtained by a recursive manner

For $0 \le m \le n-1$ with $l_0(i) = i$, select $l_{m+1}(i)$ as

$$l_{m+1}(i) = \begin{cases} l_m(i) & \text{if } \left| u\left[x_{l_m(i)}, x_{l_m(i)+1}, \ldots, x_{l_m(i)+m+1}\right] \right| \le \left| u\left[x_{l_m(i)-1}, x_{l_m(i)}, \ldots, x_{l_m(i)+m}\right] \right| \\ l_m(i) - 1 & \text{if } \left| u\left[x_{l_m(i)}, x_{l_m(i)+1}, \ldots, x_{l_m(i)+m+1}\right] \right| > \left| u\left[x_{l_m(i)-1}, x_{l_m(i)}, \ldots, x_{l_m(i)+m}\right] \right| \end{cases}$$

  - (S4) ENO polynomial in Newton form

From the ENO stencil of $\left[x_{l_n(i)}, x_{l_n(i)+1}, \ldots, x_{l_n(i)+n}\right]$, construct a $n-th$ order ENO polynomial in Newton form as

$$u_E(x) = u[x_{l_n(i)}] + u[x_{l_n(i)}, x_{l_n(i)+1}]\left(x - x_{l_n(i)}\right) + u\left[x_{l_n(i)}, x_{l_n(i)+1}, x_{l_n(i)+2}\right]\left(x - x_{l_n(i)}\right)\left(x - x_{l_n(i)+1}\right) + \ldots$$

$$+ u[x_{l_n(i)}, x_{l_n(i)+1}, \ldots, x_{l_n(i)+n}]\left(x - x_{l_n(i)}\right)\left(x - x_{l_n(i)+1}\right)\ldots\left(x - x_{l_n(i)+n-1}\right)$$

$$\rightarrow\ u_E(x) = \sum_{j=0}^{n} u\left[x_{l_n(i)}, x_{l_n(i)+1}, \ldots, x_{l_n(i)+j}\right]\prod_{k=0}^{j-1}\left(x - x_{l_n(i)+k}\right), \quad x_{j-1/2} \le x \le x_{j+1/2}$$

Or more hierarchically, we have

$$u_E(x) = u[x_{l_0(i)}] + u[x_{l_1(i)}, x_{l_1(i)+1}]\left(x - x_{l_0(i)}\right) + u\left[x_{l_2(i)}, x_{l_2(i)+1}, x_{l_2(i)+2}\right]\left(x - x_{l_1(i)}\right)\left(x - x_{l_1(i)+1}\right) + \ldots$$

$$+ u[x_{l_n(i)}, x_{l_n(i)+1}, \ldots, x_{l_n(i)+n}]\left(x - x_{l_n(i)}\right)\left(x - x_{l_n(i)+1}\right)\ldots\left(x - x_{l_n(i)+n-1}\right)$$

$$\rightarrow\ u_E(x) = \sum_{j=0}^{n} u\left[x_{l_j(i)}, x_{l_j(i)+1}, \ldots, x_{l_j(i)+j}\right]\prod_{k=0}^{j-1}\left(x - x_{l_{j-1}(i)+k}\right).$$

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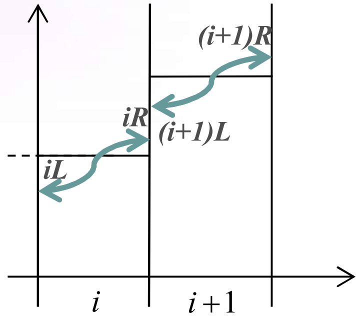
- Flux evaluation using the interpolated values of ENO subcell recfonstruction



• For each cell, we obtain u_{iL} and u_{iR} from $u_E(x)$.
Following the MUSCL approach, we replace the cell-averaged values.

$u_i \to u_{iR} = u_E(x_{i+1/2}),\ x_{i-1/2} \le x \le x_{i+1/2}$

$u_{i+1} \to u_{(i+1)L} = u_E(x_{i+1/2}),\quad x_{i+1/2} \le x \le x_{i+3/2}$

Finally, evaluate $F_{i+1/2} = F_{i+1/2}(u_{iR}, u_{(i+1)L})$ to update u_i^{n+1}

- **Characteristics of ENO scheme**
  - Choose the locally smoothest stencil to avoid stiff gradient or discontinuity
  - Adaptive interpolation(not limiting) to preserve/maximize accuracy across smooth extrema
  - By construction, arbitrary higher-order reconstruction is possible if stencil is available.

• (Stability) With the conservation constraint of $\int_{x_{i-1/2}}^{x_{i+1/2}} u_n(x) = u_i$, it can be shown 'numrically'

that $TV(u^{n+1}) \le TV(u^n) + O(\Delta x^r)$ for $r-th$ order ENO interpolation if there are at least $(r+1)$ smooth points.

$\to$ Strictly speaking, TVB stability(there is a finite M such that $TV(u^n) \le M$ for all n.). Thus, oscillation/monotonicity can be controlled essentially but not strictly.

• (Choice of stencils)

- Behavior near boundaries where proper candidates for ENO stencil is not available.
- Cells between two discontinuities approaching each other

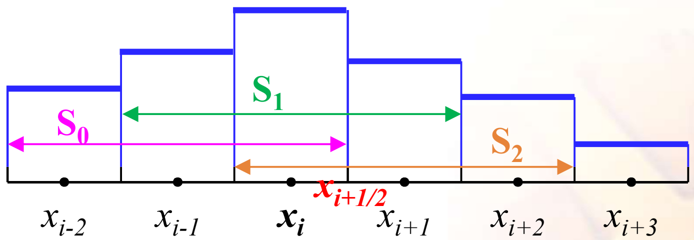
- *WENO schemes*
  - **Adaptive smoothest stencil of ENO scheme susceptible to a small perturbation at round-off level → convergence problem**
    - See the works by Liu, Jiang and Shu(1994, 1996), and others
    - In smooth region, 'free adaptation' of candidate stencils may cause loss of accuracy by choosing unstable stencils.
  - **Single higher-order polynomial → Convex combinations of lower-order (or more local) interpolations → A sophisticated smoothness indicator to determine non-linear weighting for each local polynomial**
  - **WENO polynomial reconstruction using cell-averaged value, u_i**
    - (S1) Interpolation on local and global stencils

• Reconstruct k-th order local polynomial $p_j(x)$ on S_j and $(2k)$-th order global polynomial $Q(x)$ on $T=\sum_j S_j$

$$u_{i+l}=\frac{1}{\Delta x_{i+l}}\int_{x_{i+l-1/2}}^{x_{i+l+1/2}} p_j(x)\,dx,\ \ l=-k+j,\ldots,j \ \ \text{and}\ \ u_{i+l}=\frac{1}{\Delta x_{i+l}}\int_{x_{i+l-1/2}}^{x_{i+l+1/2}} Q(x)\,dx,\ \ l=-k,\ldots,k$$

• Convex combination of local polynomial with linear γ_j weights such that

$$\sum_{j=0}^{k}\gamma_j p_j\left(x_{i\pm1/2}\right)=Q\left(x_{i\pm1/2}\right) \text{ with } \sum_{j=0}^{k}\gamma_j=1$$



- (S2) For each stencil S_j, local smoothness is measured by the indicator. (by Jiang and Shu)

$$\beta_j = \sum_{l=1}^{k} \int_{x_{i-1/2}}^{x_{i+1/2}} \Delta x^{2l-1} (p_j^{(l)}(x))^2 \, dx \left(= \int_{x_{i-1/2}}^{x_{i+1/2}} \left(\Delta x (p_j'(x))^2 + \Delta x^3 (p_j''(x))^2 + \ldots + \Delta x^{2k-1} (p_j^{(k)}(x))^2 \right) dx \right)$$

  - The sum of L_2 norm of all derivatives of $p_j(x)$
- (S3) Design non-linear weights ω_j to satisfy the following ENO property
  - If the stencil S_j is in smooth region : $\omega_j = O(1)$
  - If the stencil S_j is in non-smooth region : $\omega_j \le O(\Delta x^k)$

  $\rightarrow$ Compute the locally adaptive nonlinear weights using the smoothness indicator

$$\bar{\omega}_j = \frac{\gamma_j}{(\varepsilon + \beta_j)^2}, \quad \omega_j = \frac{\bar{\omega}_j}{\sum_l \bar{\omega}_l} \quad \text{with } \varepsilon \cong 10^{-6}$$

  - Huge variants of WENO schemes depending on, among others, the design of ω_j

    $\rightarrow$ WENO, 'WENO-JS', WENO-(M/Z/CU6/PMk/NS/Zn/IM/P/Z+/RM/RIM...)
    Hybrid/Hybrid-compact WENO, etc
- (S4) *(2k+1)-th* order accurate WENO reconstruction via convex combination with the non-linear weights
  - $\sum_{j=0}^{k} \gamma_j p_j(x) \;\rightarrow\; u_W(x) = \sum_{j=0}^{k} \omega_j p_j(x) \;\rightarrow\; u_{iR} = u_W(x_{i+1/2}),\;\; u_{iL} = u_W(x_{i-1/2})$ for all i

$$\left.\begin{array}{l} u_i \rightarrow u_{iR} = u_W(x_{i+1/2}),\; x_{i-1/2} \le x \le x_{i+1/2} \\ u_{i+1} \rightarrow u_{(i+1)L} = u_W(x_{i+1/2}),\quad x_{i+1/2} \le x \le x_{i+3/2} \end{array}\right\} \rightarrow F_{i+1/2} = F_{i+1/2}(u_{iR}, u_{(i+1)L}) \;\rightarrow\; u_i^{n+1}$$

Ex) Construction of 5-th order WENO scheme

- For $k = 2$, local/global polynomials of $p_j(x)$ and $Q(x)$

$$\left.\begin{aligned}
p_0\left(x_{i+1/2}\right) &= \frac{1}{3}u_{i-2} - \frac{7}{6}u_{i-1} + \frac{11}{6}u_i \\
p_1\left(x_{i+1/2}\right) &= -\frac{1}{6}u_{i-1} + \frac{5}{6}u_i + \frac{1}{3}u_{i+1} \\
p_2\left(x_{i+1/2}\right) &= \frac{1}{3}u_i + \frac{5}{6}u_{i+1} - \frac{1}{6}u_{i+2} \\
Q\left(x_{i+1/2}\right) &= \frac{1}{30}u_{i-2} - \frac{13}{60}u_{i-1} + \frac{47}{60}u_i + \frac{9}{20}u_{i+1} - \frac{1}{20}u_{i+2}
\end{aligned}\right\} \rightarrow \left\{\begin{aligned}
&\text{From } Q\left(x_{i\pm1/2}\right) = \sum_{j=0}^{2}\gamma_j p_j\left(x_{i\pm1/2}\right) \text{ with } \sum_{j=0}^{2}\gamma_j = 1, \\
&\text{we have } \gamma_0 = \frac{1}{10},\ \gamma_1 = \frac{6}{10},\ \gamma_2 = \frac{3}{10}.
\end{aligned}\right.$$

- Smoothness indicator

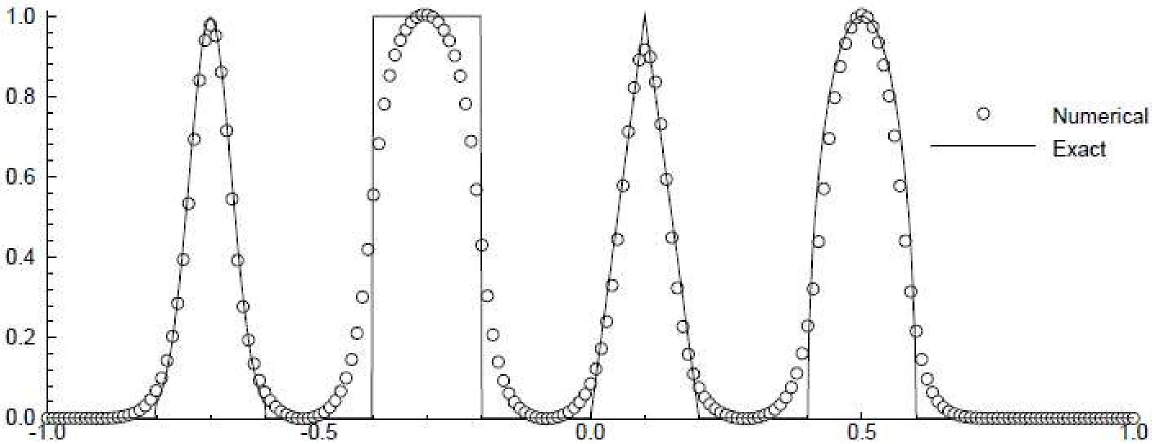
$$\left.\begin{aligned}
\beta_0 &= \frac{13}{12}\left(u_{i-2} - 2u_{i-1} + u_i\right)^2 + \frac{1}{4}\left(u_{i-2} - 4u_{i-1} + 3u_i\right)^2 \\
\beta_1 &= \frac{13}{12}\left(u_{i-1} - 2u_i + u_{i+1}\right)^2 + \frac{1}{4}\left(u_{i-1} - u_{i+1}\right)^2 \\
\beta_2 &= \frac{13}{12}\left(u_i - 2u_{i+1} + u_{i+2}\right)^2 + \frac{1}{4}\left(3u_i - 4u_{i+1} + u_{i+2}\right)^2
\end{aligned}\right\} \rightarrow \bar{\omega}_j = \frac{\gamma_j}{(\varepsilon + \beta_j)^2},\ \omega_j = \frac{\bar{\omega}_j}{\sum_{l=0}^{2}\bar{\omega}_l}$$

- Cell interface values and fluxes

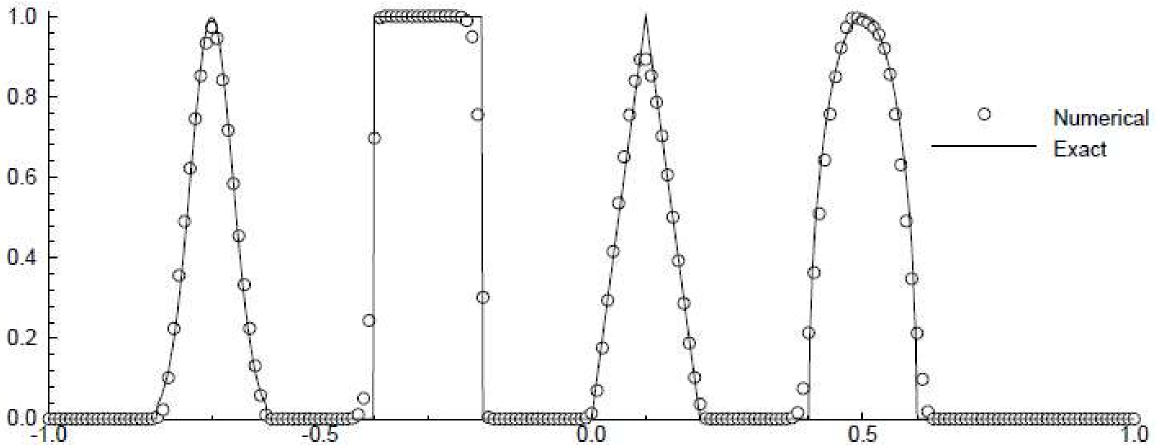
$$u_W(x) = \sum_{j=0}^{2}\omega_j p_j(x) \rightarrow u_{iR} = u_W(x_{i+1/2}),\ \ u_{iL} = u_W(x_{i-1/2}) \rightarrow \text{evaluate } F_{i+1/2} = F_{i+1/2}(u_{iR}, u_{(i+1)L})$$

Chap. 2-3. High Resolution Monotonic Schemes

- **Example**
  - Linear advection problem with smooth and discontinuous profiles



ENO 4th order



WENO 5th order