Chapter 3. Euler Equations

Chap. 3-1. Mathematical and Physical Aspects of Euler Eqns.

- ***One-dimensional Euler Equations***
  - **Conservative vector form of 1-D Euler Equations**

From mass/momentum/energy conservtions neglecting effects of viscosity and heat conductivity, we have

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{F}(\mathbf{U})}{\partial x}=\mathbf{0} \text{ with } \mathbf{U}=\begin{bmatrix}\rho\\ \rho u\\ \rho E\end{bmatrix}=\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}, \mathbf{F}(\mathbf{U})=\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho uH\end{bmatrix}=\begin{bmatrix}f_1\\ f_2\\ f_3\end{bmatrix}, E=e+\frac{u^2}{2}, H=h(\equiv e+\frac{p}{\rho})+\frac{u^2}{2}.$$

Assuming calorically perfect gas/air ($\gamma=c_p/c_v=1.4,\ c_p-c_v=R$), $p=\rho RT=\left(\gamma-1\right)\rho(E-u^2/2)$.

From $\dfrac{\partial \mathbf{U}}{\partial t}+\dfrac{\partial \mathbf{F}}{\partial \mathbf{U}}\dfrac{\partial \mathbf{U}}{\partial x}=\dfrac{\partial \mathbf{U}}{\partial t}+A\dfrac{\partial \mathbf{U}}{\partial x}=\mathbf{0}$,

• A is a quasi-linear function of $\mathbf{U}$.

$$\mathbf{F}(\mathbf{U})=u\mathbf{U}+\begin{bmatrix}0\\ p\\ up\end{bmatrix}=\frac{u_2}{u_1}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}+\begin{bmatrix}0\\ (\gamma-1)\left(u_3-u_2^2/(2u_1)\right)\\ (u_2/u_1)(\gamma-1)\left(u_3-u_2^2/(2u_1)\right)\end{bmatrix} \text{ or } \begin{bmatrix}u_2\\ u_2{}^2/u_1+(\gamma-1)(u_3-u_2{}^2/(2u_1))\\ (u_2/u_1)(\gamma u_3-(\gamma-1)u_2{}^2/(2u_1))\end{bmatrix}$$

$$A(\mathbf{U})=\frac{\partial \mathbf{F}}{\partial \mathbf{U}}=uI+\rho\begin{bmatrix}1\\ u\\ H\end{bmatrix}\frac{\partial u}{\partial \mathbf{U}}+\begin{bmatrix}0\\ 1\\ u\end{bmatrix}\frac{\partial p}{\partial \mathbf{U}} \text{ or } \frac{\partial(f_1,f_2,f_3)}{\partial(u_1,u_2,u_3)}=\begin{bmatrix}0 & 1 & 0\\ (\gamma-3)u^2/2 & (3-\gamma)u & \gamma-1\\ (\gamma-1)u^3-\gamma uE & \gamma E-3(\gamma-1)u^2/2 & \gamma u\end{bmatrix}$$

with $(\gamma-1)u^3-\gamma uE=(\gamma-1)u^3/2-uH$ and $\gamma E-3(\gamma-1)u^2/2=H-(\gamma-1)u^2$

Chap. 3-1. Mathematical and Physical Aspects of Euler Eqns.

- Euler system is hyperbolic since it can be diagonalizable with a set of real eigenvalues and independent right eigenvactors.

  From $\det(A-\lambda I)=0=(\lambda-u)(\lambda^2-2u\lambda+u^2-a^2),\ \lambda_{1,2,3}=u-c,\ u,\ u+c$

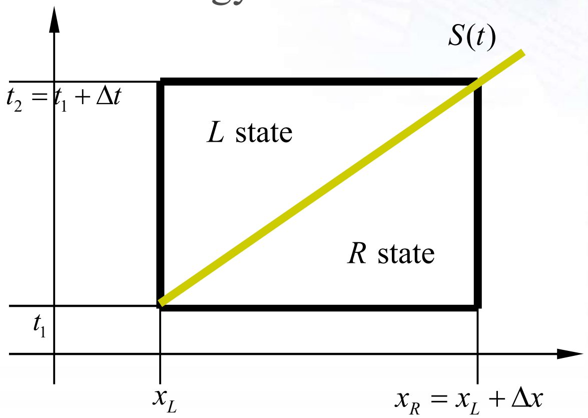
  From $A\mathbf{r}_i=\lambda_i\mathbf{r}_i,\ \mathbf{r}_1=\begin{bmatrix}1\\u-c\\H-uc\end{bmatrix},\mathbf{r}_2=\begin{bmatrix}1\\u\\u^2/2\end{bmatrix},\mathbf{r}_3=\begin{bmatrix}1\\u+c\\H+uc\end{bmatrix}.$

  Using $R=[\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3]$ with $R^{-1}=\dfrac{\gamma-1}{2c^2}\begin{bmatrix}H+\dfrac{c}{\gamma-1}(u-c) & -u-\dfrac{c}{\gamma-1} & 1\\ \dfrac{4}{\gamma-1}c^2-2H & 2u & -2\\ H-\dfrac{c}{\gamma-1}(u-c) & -u+\dfrac{c}{\gamma-1} & 1\end{bmatrix},$

  $$R^{-1}AR=\Lambda=\begin{bmatrix}u-c & 0 & 0\\0 & u & 0\\0 & 0 & u+c\end{bmatrix}$$

- **Conservative form**
  - Differential form obtained by applying integral mass/momentum/energy conservation

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{F}(\mathbf{U})}{\partial x}=\mathbf{0} \quad \text{with } \mathbf{U}=\begin{bmatrix}\rho\\ \rho u\\ \rho E\end{bmatrix},\quad \mathbf{F}(\mathbf{U})=\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho u H\end{bmatrix}$$



  - Rankine-Hugoniot relation

$$\text{From } \iint_\Omega\left(\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{F}}{\partial x}\right)dxdt=\mathbf{0} \quad \text{on } \Omega=[x_L,x_R]\times[t_1,t_2],$$

$$\int_{x_L}^{x_R}\mathbf{U}(x,t_2)dx-\int_{x_L}^{x_R}\mathbf{U}(x,t_1)dx=\int_{t_1}^{t_2}\mathbf{F}(\mathbf{U}(x_L,t))\,dt-\int_{t_1}^{t_2}\mathbf{F}(\mathbf{U}(x_R,t))\,dt$$

$$\Delta x(\mathbf{U}_L-\mathbf{U}_R)=\Delta t(\mathbf{F}_L-\mathbf{F}_R) \text{ or } [\mathbf{F}]=S[\mathbf{U}] \text{ with } S=\Delta x/\Delta t$$

  - Conservative form is meaningful only if it is consistent with physical laws of conservation.

• Isothermal form of 1 – D Euler equations with $\gamma=C_p/C_v=1,\ c^2=\gamma\frac{p}{\rho}=\gamma RT=const$

1-D Euler eqns. become $\begin{bmatrix}\rho\\ \rho u\end{bmatrix}_t+\begin{bmatrix}\rho u\\ \rho u^2+c^2\rho\end{bmatrix}_x=0$

If flow is smooth, momentum equation becomes

$$u_t+uu_x+\frac{c^2}{\rho}\rho_x+\frac{u}{\rho}(\rho_t+u\rho_x+\rho u_x)=0 \to \begin{bmatrix}\rho\\ u\end{bmatrix}_t+\begin{bmatrix}\rho u\\ u^2/2+c^2\ln\rho\end{bmatrix}_x=\mathbf{0} \text{ but, } u \text{ is not a conserved quantity.}$$

Chap. 3-1. Mathematical and Physical Aspects of Euler Eqns.

- **Primitive form, symmetric and symmetrized form**
  - Measurable quantities for **U** → more intuitive and simpler in smooth flow

With $\mathbf{U}=\mathbf{U}_p=\begin{bmatrix}\rho\\u\\p\end{bmatrix}$, conservation laws become $\dfrac{\partial \mathbf{U}_p}{\partial t}+A_p\dfrac{\partial \mathbf{U}_p}{\partial x}=\mathbf{0}$ with $A_p=\dfrac{\partial \mathbf{F}}{\partial \mathbf{U}_p}=\begin{bmatrix}u & \rho & 0\\ 0 & u & 1/\rho\\ 0 & \rho c^2 & u\end{bmatrix}$.

From conservative form with chain rule, we have $\dfrac{\partial \mathbf{U}}{\partial t}+A(\mathbf{U})\dfrac{\partial \mathbf{U}}{\partial x}=\dfrac{\partial \mathbf{U}}{\partial \mathbf{U}_p}\dfrac{\partial \mathbf{U}_p}{\partial t}+A\dfrac{\partial \mathbf{U}}{\partial \mathbf{U}_p}\dfrac{\partial \mathbf{U}_p}{\partial x}=\mathbf{0}.$

Let $\dfrac{\partial \mathbf{U}}{\partial \mathbf{U}_p}\equiv\dfrac{\partial(u_1,u_2,u_3)}{\partial(\rho,u,p)}=P=\begin{bmatrix}1 & 0 & 0\\ u & \rho & 0\\ u^2/2 & \rho u & 1/(\gamma-1)\end{bmatrix}$, $P^{-1}=\dfrac{\partial \mathbf{U}_p}{\partial \mathbf{U}}=\begin{bmatrix}1 & 0 & 0\\ -u/\rho & 1/\rho & 0\\ (\gamma-1)u^2/2 & -(\gamma-1)u & (\gamma-1)\end{bmatrix}$

Thus, $P\dfrac{\partial \mathbf{U}_p}{\partial t}+AP\dfrac{\partial \mathbf{U}_p}{\partial x}=\mathbf{0}=\dfrac{\partial \mathbf{U}_p}{\partial t}+P^{-1}AP\dfrac{\partial \mathbf{U}_p}{\partial x}$ with $P^{-1}AP=A_p$.

Since A is similar to A_p, eigenvalues are the same and eigenvectors are linearly independent.

- From $\det(A_p-\lambda I)=0$, $\lambda_i=u-c,\ u,\ u+c$, and $\mathbf{r}_i=\begin{bmatrix}-\rho/c\\1\\\rho c\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}\rho/c\\1\\\rho c\end{bmatrix}$

- If no shocks (or isentropic flow), energy equation becomes $s_t+us_x=0$

  Thus, $\mathbf{U}_p=\begin{bmatrix}\rho\\u\\s\end{bmatrix}$ is possible to have $\dfrac{\partial \mathbf{U}_p}{\partial t}+A_p\dfrac{\partial \mathbf{U}_p}{\partial x}=0$

with $A_p = \begin{bmatrix} u & \rho & 0 \\ c^2/\rho & u & (1/\rho)\partial p/\partial s \\ 0 & 0 & u \end{bmatrix}$, $\lambda_i = u-c, u, u+c$, $\mathbf{r}_i = \begin{bmatrix} 1 \\ -c/\rho \\ 0 \end{bmatrix}, \begin{bmatrix} -\partial p/\partial s \\ 0 \\ c^2 \end{bmatrix}, \begin{bmatrix} 1 \\ c/\rho \\ 0 \end{bmatrix}$.

- (Symmetric form) Starting from the primitive form, simple and useful symmetric form can be obtained.

$$\frac{\partial \mathbf{U}_p}{\partial t} + A_p \frac{\partial \mathbf{U}_p}{\partial x} = \mathbf{0} \rightarrow \begin{aligned} &\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \\ &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \\ &\frac{\partial p}{\partial t} + \rho c^2\frac{\partial u}{\partial x} + u\frac{\partial p}{\partial x} = 0 \end{aligned} \xrightarrow[ds=(dp-c^2 d\rho)/p\ \ldots]{s=\log(p/\rho^\gamma)=const} \begin{aligned} &\frac{1}{\rho c}\frac{\partial p}{\partial t} + \frac{u}{\rho c}\frac{\partial p}{\partial x} + c\frac{\partial u}{\partial x} = 0 \\ &\frac{\partial u}{\partial t} + \frac{c}{\rho c}\frac{\partial p}{\partial x} + u\frac{\partial u}{\partial x} = 0 \\ &\frac{\partial p}{\partial t} - c^2\frac{\partial \rho}{\partial t} + u\left(\frac{\partial p}{\partial x} - c^2\frac{\partial \rho}{\partial x}\right) = 0 \end{aligned}$$

By introducing $d\mathbf{w}_s = \begin{bmatrix} dp/\rho c \\ du \\ dp - c^2 d\rho \end{bmatrix}$ or $\begin{bmatrix} dp/c^2 \\ (\rho/c)du \\ dp/c^2 - d\rho \end{bmatrix}$, $\dfrac{\partial \mathbf{w}_s}{\partial t} + A_s\dfrac{\partial \mathbf{w}_s}{\partial x} = \mathbf{0}$ with $A_s = \begin{bmatrix} u & c & 0 \\ c & u & 0 \\ 0 & 0 & u \end{bmatrix}$

Now, from $\dfrac{\partial \mathbf{U}}{\partial t} + A\dfrac{\partial \mathbf{U}}{\partial x} = \mathbf{0} = \dfrac{\partial \mathbf{U}}{\partial \mathbf{w}_s}\dfrac{\partial \mathbf{w}_s}{\partial t} + A\dfrac{\partial \mathbf{U}}{\partial \mathbf{w}_s}\dfrac{\partial \mathbf{w}_s}{\partial x}$, we have $\dfrac{\partial \mathbf{w}_s}{\partial t} + P_s^{-1}AP_s\dfrac{\partial \mathbf{w}_s}{\partial x} = \mathbf{0}$

with

$P_s = \dfrac{\partial \mathbf{U}}{\partial \mathbf{w}_s} = \begin{bmatrix} 1 & 0 & -1 \\ u & c & -u \\ H & cu & -u^2/2 \end{bmatrix}$, $P_s^{-1} = \begin{bmatrix} \bar{\gamma}u^2/2 & -\bar{\gamma}u & -\bar{\gamma} \\ -u/c & 1/c & 0 \\ \bar{\gamma}(u^2 - H) & -\bar{\gamma}u & \bar{\gamma} \end{bmatrix}$, $P_s^{-1}AP_s = A_s$, and $\bar{\gamma} = (\gamma - 1)/c^2$.

Chap. 3-1. Mathematical and Physical Aspects of Euler Eqns.

- Symmetric form (cont'd)

Since A_s is symmetric, we have orthonomal eigenvectors with $R_s^{-1} A_s R_s = \Lambda$, where

$$R_s = [\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3] = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}, \; R_s^{-1} = R_s^T$$

From $P_s^{-1} A P_s = A_s = R_s \Lambda R_s^{-1}$, we have $(P_s R_s)^{-1} A (P_s R_s) = \Lambda$.

It is noted that symmetric form can be utilized to obtain symmetrized hyperbolic form with any set of dependent variables.

• (Conservative variables) From $\dfrac{\partial \mathbf{w}_s}{\partial t} + A_s \dfrac{\partial \mathbf{w}_s}{\partial x} = \mathbf{0} = \dfrac{\partial \mathbf{w}_s}{\partial \mathbf{U}} \dfrac{\partial \mathbf{U}}{\partial t} + A_s \dfrac{\partial \mathbf{w}_s}{\partial \mathbf{U}} \dfrac{\partial \mathbf{U}}{\partial x} = P_s^{-1} \dfrac{\partial \mathbf{U}}{\partial t} + A_s P_s^{-1} \dfrac{\partial \mathbf{U}}{\partial x}$,

we have $P_s^{T-1} P_s^{-1} \dfrac{\partial \mathbf{U}}{\partial t} + P_s^{T-1} A_s P_s^{-1} \dfrac{\partial \mathbf{U}}{\partial x} = \mathbf{0} = Q \dfrac{\partial \mathbf{U}}{\partial t} + A_{sc} \dfrac{\partial \mathbf{U}}{\partial x}$, where $Q = (P_s P_s^T)^{-1}$ is symmetric and positive definite, and $A_{sc} = P_s^{T-1} A_s P_s^{-1}$ is symmetric.

• (Primitive variables) From $\dfrac{\partial \mathbf{w}_s}{\partial t} + A_s \dfrac{\partial \mathbf{w}_s}{\partial x} = \mathbf{0} = \dfrac{\partial \mathbf{w}_s}{\partial \mathbf{U}_p} \dfrac{\partial \mathbf{U}_p}{\partial t} + A_s \dfrac{\partial \mathbf{w}_s}{\partial \mathbf{U}_p} \dfrac{\partial \mathbf{U}_p}{\partial x} = M \dfrac{\partial \mathbf{U}_p}{\partial t} + A_s M \dfrac{\partial \mathbf{U}_p}{\partial x}$,

we have $M^T M \dfrac{\partial \mathbf{U}_p}{\partial t} + M^T A_s M \dfrac{\partial \mathbf{U}_p}{\partial x} = N \dfrac{\partial \mathbf{U}_p}{\partial t} + A_{sp} \dfrac{\partial \mathbf{U}_p}{\partial x} = \mathbf{0}$, where $M = \begin{bmatrix} 0 & 0 & 1/c^2 \\ 0 & \rho/c & 0 \\ -1 & 0 & 1/c^2 \end{bmatrix}$,

$N = M^T M$ is symmetric and positive definite, and $A_{sp} = M^T A_s M$ is symmetric.

Chap. 3-1. Mathematical and Physical Aspects of Euler Eqns.

- Symmetric form (cont'd)

- (Entropy variables) By introducing a suitable form of entropy function/flux/variables, symmetrized hyperbolic form can be obtained. Motivated by $\dfrac{\partial(\rho s)}{\partial t}+\dfrac{\partial(\rho u s)}{\partial x}=0$ obtained from energy and continuity equations, consider the generalized entropy function, U_s, and entropy flux, G_s.

(The case of SCL: $u_t + f(u)_x = 0 \rightarrow U(u)_t + F(u)_x = 0$ with $U'f' = F'$ and $U'' \le 0$)

For $\dfrac{\partial \mathbf{U}}{\partial t}+A\dfrac{\partial \mathbf{U}}{\partial x}=\mathbf{0}$, we seek $U_s(\mathbf{U})$ and $G_s(\mathbf{U})$ such that $\dfrac{\partial U_s}{\partial \mathbf{U}}\dfrac{\partial \mathbf{F}}{\partial \mathbf{U}}=\dfrac{\partial G_s}{\partial \mathbf{U}}$ with U_s being a convex function of $\mathbf{U}$.

From $\dfrac{\partial U_s}{\partial \mathbf{U}}\left(\dfrac{\partial \mathbf{U}}{\partial t}+A\dfrac{\partial \mathbf{U}}{\partial x}\right)=0$, we obtain the generalized entropy conservation law of $\dfrac{\partial U_s}{\partial t}+\dfrac{\partial G_s}{\partial x}=0$.

Define entropy variable vector as $\mathbf{S}^T=\dfrac{\partial U_s}{\partial \mathbf{U}}$, and scalar functions $q(\mathbf{S})=\mathbf{S}^{\mathrm{T}}\mathbf{U}-U_s(\mathbf{U})$, $r(\mathbf{S})=\mathbf{S}^{\mathrm{T}}\mathbf{F}-G_s(\mathbf{U})$

i) $\dfrac{\partial q}{\partial \mathbf{S}}=\mathbf{U}^T+\mathbf{S}^{\mathrm{T}}\dfrac{\partial \mathbf{U}}{\partial \mathbf{S}}-\dfrac{\partial U_s}{\partial \mathbf{U}}\dfrac{\partial \mathbf{U}}{\partial \mathbf{S}}=\mathbf{U}^T \rightarrow \dfrac{\partial u_i}{\partial S_j}=\dfrac{\partial}{\partial S_j}\left(\dfrac{\partial q}{\partial S_i}\right)=\dfrac{\partial}{\partial S_i}\left(\dfrac{\partial q}{\partial S_j}\right)=\dfrac{\partial u_j}{\partial S_i} \rightarrow \dfrac{\partial \mathbf{U}}{\partial \mathbf{S}}(=N_s)$ is symmetric.

ii) $\dfrac{\partial r}{\partial \mathbf{S}}=\mathbf{F}^T+\mathbf{S}^{\mathrm{T}}\dfrac{\partial \mathbf{F}}{\partial \mathbf{U}}\dfrac{\partial \mathbf{U}}{\partial \mathbf{S}}-\dfrac{\partial G_s}{\partial \mathbf{U}}\dfrac{\partial \mathbf{U}}{\partial \mathbf{S}}=\mathbf{F}^T \rightarrow \dfrac{\partial f_i}{\partial S_j}=\dfrac{\partial}{\partial S_j}\left(\dfrac{\partial r}{\partial S_i}\right)=\dfrac{\partial}{\partial S_i}\left(\dfrac{\partial r}{\partial S_j}\right)=\dfrac{\partial f_j}{\partial S_i}$

$\rightarrow \dfrac{\partial \mathbf{F}}{\partial \mathbf{S}}=\dfrac{\partial \mathbf{F}}{\partial \mathbf{U}}\dfrac{\partial \mathbf{U}}{\partial \mathbf{S}}(=AN_s)$ is symmetric.

Thus, from $\dfrac{\partial \mathbf{U}}{\partial t}+A\dfrac{\partial \mathbf{U}}{\partial x}=\dfrac{\partial \mathbf{U}}{\partial \mathbf{S}}\dfrac{\partial \mathbf{S}}{\partial t}+A\dfrac{\partial \mathbf{U}}{\partial \mathbf{S}}\dfrac{\partial \mathbf{S}}{\partial x}=\mathbf{0}$, we have a symmetrized form of $N_s\dfrac{\partial \mathbf{S}}{\partial t}+AN_s\dfrac{\partial \mathbf{S}}{\partial x}=\mathbf{0}$.

Chap. 3-1. Mathematical and Physical Aspects of Euler Eqns.

- Symmetric form (cont'd)

  The convexity of $U_s(\mathbf{U})$ can be treated as a generalized energy since U_s is unbounded as $\mathbf{U}$ is unbounded. It can be shown that $-\rho s$ is a convex function of $\mathbf{U}$, and thus we take

  $U_s = \dfrac{-\rho s}{\gamma-1} = \dfrac{\gamma}{\gamma-1}\rho\log\rho - \dfrac{\rho}{\gamma-1}\rho\log p$ as a generalized entropy function.

  From direct calculation, $\mathbf{S}$ and N_s is given by

$$\mathbf{S}^T = \frac{\partial U_s}{\partial \mathbf{U}} = \frac{\rho}{p}\left[\frac{\gamma-s}{\gamma-1}\frac{u^2}{2}, \quad u, \quad -1\right],$$

$$N_s = \frac{\partial \mathbf{U}}{\partial \mathbf{S}} = \begin{bmatrix} \rho & \rho u & \rho E \\ \rho u & \rho u^2+p & \rho uH \\ \rho E & \rho uH & \rho EH + pu^2/2 \end{bmatrix} = \frac{1}{\rho}\mathbf{U}\mathbf{U}^T + p\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & u \\ 0 & u & E+u^2/2 \end{bmatrix}.$$

- **Characteristic form**
  - Wave nature of the Euler equations can be best analyzed by utilizing characteristic form.
  - If $A(\mathbf{U}) =$ const. (locally or globally),

    From $\dfrac{\partial \mathbf{U}}{\partial t} + \dfrac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = \dfrac{\partial \mathbf{U}}{\partial t} + A(\mathbf{U})\dfrac{\partial \mathbf{U}}{\partial x} = \dfrac{\partial \mathbf{U}}{\partial t} + R\Lambda R^{-1}\dfrac{\partial \mathbf{U}}{\partial x} = \mathbf{0},$

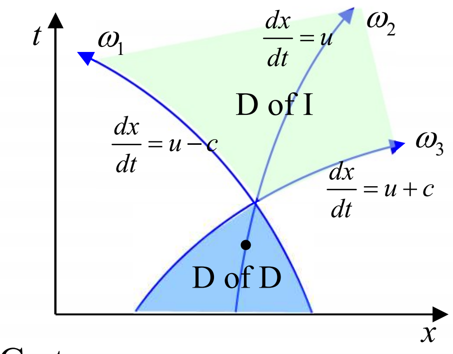
    by introducing characteristic variables $\boldsymbol{\omega} = R^{-1}\mathbf{U}$, we have $\dfrac{\partial \boldsymbol{\omega}}{\partial t} + \Lambda\dfrac{\partial \boldsymbol{\omega}}{\partial x} = \mathbf{0}$ or a family of scalar equations, $\dfrac{\partial \omega_i}{\partial t} + \lambda_i \dfrac{\partial \omega_i}{\partial x} = 0$ with $\lambda_{1,2,3} = u-c,\ u,\ u+c$. Thus, $\mathbf{U} = R\boldsymbol{\omega} = \sum_i \omega_i \mathbf{r}_i$

- If $A(\mathbf{U})$ is not locally constant,

By defining differential characteristic variables as $\partial \boldsymbol{\omega} = R^{-1}\partial \mathbf{U}$, we still have $\frac{\partial \boldsymbol{\omega}}{\partial t} + \Lambda \frac{\partial \boldsymbol{\omega}}{\partial x} = \mathbf{0}$ or a family of scalar equations, $\frac{\partial \omega_i}{\partial t} + \lambda_i \frac{\partial \omega_i}{\partial x} = 0$. Here, λ_i depends on all characteristic variables. With $\frac{d\omega_i}{dt} = 0$, $\omega_i = const.$ along each characteristics $\frac{dx}{dt} = \lambda_i$, yielding limited domain of dependence/influence. The compatibility relation (or $d\omega_i = 0$ along $dx = \lambda_i dt$) is then

$$d\omega_{1,3} = du \mp \frac{dp}{\rho c} = 0 \;\rightarrow\; \omega_{1,3} = u \mp \int \frac{dp}{\rho c} = const. \;\text{ along } dx = (u \mp c)dt,$$

$$d\omega_2 = d\rho - \frac{dp}{c^2} = 0 \;\rightarrow\; \text{from } d\omega_2 = -\frac{\rho}{\gamma} ds,\; s = const. \;\text{ along } dx = udt.$$



Assuming $s = const.$ everywhere (homentropic flow) and calorically perfect gas, all $d\omega_i$ become integrable, and three wave equations are obtained. This formulation is quite useful in implementing far field BCs to minimize wave reflection. With the Riemann invariants of $R^{\mp} = u \mp \frac{2c}{\gamma - 1}$, we have

$$\frac{\partial R^{\mp}}{\partial t} + (u \mp c)\frac{\partial R^{\mp}}{\partial x} = 0 \;\text{ along } dx = (u \mp c)dt, \quad \frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = 0 \;\text{ along } dx = udt.$$

$\rightarrow$ $\omega_1 (= R^-)$ and $\omega_3 (= R^+)$: equations for acoustic waves with the sonic speed of c w.r.t. u

$\omega_2 (= s)$: equation for entropy wave with the local flow speed of u

Chap. 3-1. Mathematical and Physical Aspects of Euler Eqns.

- Simple waves

The flow physics of the 1-D Euler equations is rather complicated by the 'nonlinear interaction' of the three elementary waves. And, overall properties are not the same as those of scalar conservation law. For example,

- Total variation of 1-D Euler equations is not, in general, decreasing.
- From shock/contact interactions, new local maxima/minima can be easily produced.

In order to understand the nature of each wave equation clearly, additional assumption is introduced.

- For each wave with the path of $d\omega_k = 0$ along $dx = \lambda_k dt$, the other two waves are assumed to be constant. $\rightarrow \omega_k = const.$ (actually, all ω_i !) along the straight characteristic line of $x = \lambda_k t + const.$

• Example of simple waves

- Simple acoustic waves: $\dfrac{\partial(V \mp C)}{\partial t} + (V \mp C)\dfrac{\partial(V \mp C)}{\partial x} = 0$ along $x = (V \mp C)t + const.$ $\rightarrow$ Burgers' equation leading to the formation of shock discontinuity/expansion waves
- Simple entropy wave: $\dfrac{\partial s}{\partial t} + V\dfrac{\partial s}{\partial x} = 0$ along $x = Vt + const.$ $\rightarrow$ linear advection equation and contact discontinuity

- Genuinely nonlinear and linearly degenerate field

Linear and nonlinear nature of each wave is often identified in phase space (u_1, u_2, u_3) rather than in physical space (x, t).

• $\omega_i = const.$ along $dx = \lambda_i dt$ in physical space $\rightarrow$ $\omega_i = const.$ along an integral curve $u(\xi)$ in phase

Chap. 3-1. Mathematical and Physical Aspects of Euler Eqns.

- Genuinely nonlinear and linearly degenerate field (cont'd)

space such that $u'(\xi) = (u_1'(\xi), u_2'(\xi), u_3'(\xi)) \propto \mathbf{r}_i$ with some parameter ξ.

Thus, $\dfrac{d\omega_i}{d\xi} = \dfrac{\partial\omega_i}{\partial u_j}\dfrac{\partial u_j}{\partial\xi} = \nabla\omega_i \cdot u'(\xi)^T = 0 = \nabla\omega_i \cdot \mathbf{r}_i^T$.

By checking the behavior of λ_i along the integral curve $u(\xi)$, we define

- Genuinely nonlinear: $\dfrac{d\lambda_i}{d\xi} = \nabla\lambda_i \cdot u'(\xi)^T \neq 0 \neq \nabla\lambda_i \cdot \mathbf{r}_i^T$

- Linearly degenerate: $\dfrac{d\lambda_i}{d\xi} = \nabla\lambda_i \cdot u'(\xi)^T = 0 = \nabla\lambda_i \cdot \mathbf{r}_i^T$

This is an extension of convexity condition into Euler systems since, in SCL, $\nabla\lambda \cdot \mathbf{r}^T = \dfrac{d\lambda}{du} \cdot 1 = \dfrac{d^2 f}{du^2} \neq 0.$

- Example: Euler equations in primitive variables of (ρ, u, p)

From $\lambda_i = u-c,\ u,\ u+c,$ and $\mathbf{r}_i = \begin{bmatrix} -\rho/c \\ 1 \\ \rho c \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \rho/c \\ 1 \\ \rho c \end{bmatrix},$

- linearly degenerate: $\nabla\lambda_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \nabla\lambda_2 \cdot \mathbf{r}_2 = 0$

- Genuinely nonlinear: $\nabla\lambda_{1,3} = \begin{bmatrix} \pm c/2\rho \\ 1 \\ \mp c/2\rho \end{bmatrix} \rightarrow \nabla\lambda_{1,3} \cdot \mathbf{r}_{1,3} = \dfrac{\gamma+1}{2} \neq 0$