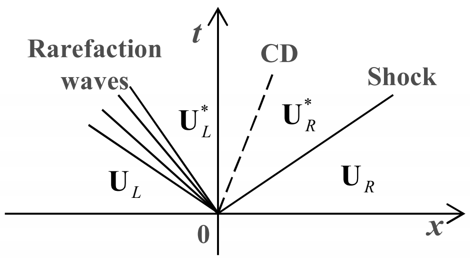
Chap. 3-2. Methods to design $\mathbf{F}_{i+1/2}^{n}$: II. Flux Difference Splitting

- ***Riemann Problem and Its Properties***
  - **An IVP of (1-D) Euler eqns. with discontinuous initial data of two constant state**

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{F}(\mathbf{U})}{\partial x}=\mathbf{0} \text{ with } \mathbf{U}(x,0)=\begin{cases}\mathbf{U}_L & \text{if } x<0\\ \mathbf{U}_R & \text{if } x>0\end{cases},\ \mathbf{U}_L\neq\mathbf{U}_R$$

  - **At $t>0$, solution can be composed of four different states.**
    - Depending on initial condition, four sub-cases can be considered.
      - (RW-CD-SW), (SW-CD-RW), (SW2-CD-SW1), (RW2-CD-RW1)
        SW: shock-wave, CD: contact discontinuity, RW: rarefaction wave
      - Shock-tube problem is a special case of Riemann problem.
    - Solution of Riemann problem is self-similar, or, $U(x,t)=U(x/t)$.
      - Solution is constant along $x/t=const.$ in (x,t) plane.



  - **Solution of Riemann problem for $\mathbf{F}_{i+1/2}$ → Godunov Scheme**
    - Firstly introduced by Godunov to obtain a cell-interface flux at $x_{i+1/2}$ by $(0,0)=(x_{i+1/2},t^n)$
    - For the case of (RW-CD-SW),
      - Region $(\mathbf{U}_R^*,\ \mathbf{U}_R)$: from conservation relations across a right moving shock,

$$\frac{T_R^*}{T_R}=\frac{c_R^{*2}}{c_R^2}=\frac{p_R^*}{p_R}\frac{(\gamma+1)/(\gamma-1)+p_R^*/p_R}{1+((\gamma+1)/(\gamma-1))(p_R^*/p_R)},\quad u_R^*=u_R+\frac{c_R}{\gamma}\frac{p_R^*/p_R-1}{\sqrt{(p_R^*/p_R-1)((\gamma+1)/(2\gamma))+1}},$$

$$S=u_R+c_R\sqrt{(p_R^*/p_R-1)((\gamma+1)/(2\gamma))+1}$$

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- For the case of (SW-CD-RW), (cont'd)
  - Region $(\mathbf{U}^*_L,\ \mathbf{U}^*_R)$: conditions across contact discontinuity

    $u^*_L = u^*_R,\ p^*_L = p^*_R$
  - Region $(\mathbf{U}_L,\ \mathbf{U}^*_L)$: from the MOC of $R^\pm$ across left moving centered expansion waves,

$$u_L + \frac{2c_L}{\gamma-1} = u^*_L + \frac{c^*_L}{\gamma-1},\quad \frac{p^*_L}{p_L} = \left(\frac{\rho^*_L}{\rho_L}\right)^\gamma = \left(\frac{T^*_L}{T_L}\right)^{\gamma/\gamma-1} = \left(\frac{c^*_L}{c_L}\right)^{2\gamma/\gamma-1}$$

From the Region $(\mathbf{U}_L,\ \mathbf{U}^*_L)$ and Region $(\mathbf{U}^*_L,\ \mathbf{U}^*_R)$,

$$u^*_L = u_L + \frac{2c_L}{\gamma-1}\left[1-\left(\frac{p^*_L}{p_L}\right)^{(\gamma-1)/2\gamma}\right] = u_L + \frac{2c_L}{\gamma-1}\left[1-\left(\frac{p_R}{p_L}\frac{p^*_R}{p_R}\right)^{(\gamma-1)/2\gamma}\right] \rightarrow \frac{p_R}{p_L} = \frac{p_R}{p^*_R}\left[1+\frac{\gamma-1}{2c_L}(u_L-u^*_R)\right]^{2\gamma/(\gamma-1)}$$

Thus, from the Region $(\mathbf{U}^*_R,\ \mathbf{U}_R)$, we have

$$\frac{p_R}{p_L} = \left(\frac{p^*_R}{p_R}\right)^{-1}\left[1+\frac{\gamma-1}{2c_L}\left(u_L-u_R-\frac{c_R}{\gamma}\frac{p^*_R/p_R-1}{\sqrt{(p^*_R/p_R-1)((\gamma+1)/2\gamma)+1}}\right)\right]^{2\gamma/(\gamma-1)}.$$

This is a nonlinear algebraic equation in terms of (p^*_R/p_R), which can be obtained by a iterative root finder, such as Newton's method.

p^*_R or $p^*_R/p_R \ \rightarrow$ all flow variables in Region $(\mathbf{U}^*_L,\ \mathbf{U}^*_R) \ \rightarrow\ \mathbf{F}_{i+1/2}(\mathbf{U}^*_{L/R}(0,t))$

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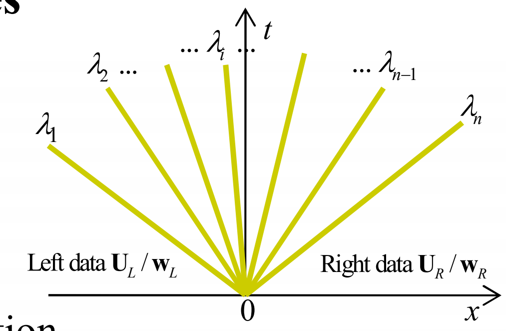
- ***Linear Riemann Problem for Small Fluctuation***
  - **If $A = \partial \mathbf{F}/\partial \mathbf{U}$ is (globally or locally) constant,**

$$\frac{\partial \mathbf{U}}{\partial t} + A\frac{\partial \mathbf{U}}{\partial x} = \mathbf{0}, \quad \text{with} \quad \mathbf{U}(x,0) = \begin{cases} \mathbf{U}_L & \text{if } x<0, \\ \mathbf{U}_R & \text{if } x>0 \end{cases} \quad \text{but } \mathbf{U}_R = \mathbf{U}_L + \delta\mathbf{U} \;\&\; \|\delta\mathbf{U}\| \ll \|\mathbf{U}_L\| \quad \text{Eq.(*)}$$

  - **From hyperbolicity, eigenvalues are real and distinct with λ_i , and linearly independent right/left eigenvectors with $\mathbf{r}_i \cdot \mathbf{l}_j = \delta_{ij}$.**
  - **From $R^{-1}\mathbf{U} = [\mathbf{l}_1, \mathbf{l}_2, \ldots, \mathbf{l}_n]^T\,\mathbf{U} = \boldsymbol{\omega} = [\omega_i]_{1\le i\le n}$, Eq.(*) becomes**

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \Lambda \frac{\partial \boldsymbol{\omega}}{\partial x} = \mathbf{0} \text{ with } \boldsymbol{\omega}(x.0) = \begin{cases} \boldsymbol{\omega}_L = R^{-1}\mathbf{U}_L & \text{if } x<0, \\ \boldsymbol{\omega}_R = R^{-1}\mathbf{U}_R & \text{if } x>0 \end{cases}$$

$$\rightarrow \frac{\partial \omega_i}{\partial t} + \lambda_i \frac{\partial \omega_i}{\partial x} = 0 \text{ with } \omega_i^0(x,0) = \begin{cases} \omega_i^l = \mathbf{l}_i^T \mathbf{U}_L & \text{if } x<0, \\ \omega_i^r = \mathbf{l}_i^T \mathbf{U}_R & \text{if } x>0 \end{cases}$$



Thus, the exact solution is simply a superposition of n linear advection equations for ω_i with a wave speed of λ_i. And, the exact soln. of ω_i is given by

$$\omega_i(x,t) = \omega_i^0(x-\lambda_i t) = \begin{cases} \omega_i^l & \text{if } x-\lambda_i t<0 \; (x/t \le \lambda_i), \\ \omega_i^r & \text{if } x-\lambda_i t>0 \; (x/t \ge \lambda_i). \end{cases}$$

- If x/t is located at $[\lambda_I, \lambda_{I+1}]$, $\mathbf{U}(x,t) = R\boldsymbol{\omega} = \sum_{i=1}^{I} \omega_i^r \mathbf{r}_i + \sum_{i=I+1}^{n} \omega_i^l \mathbf{r}_i$.
- If $(x/t) < \lambda_1$, $\mathbf{U}(x/t) = \sum_{i=1}^{n} \omega_i^l \mathbf{r}_i = R\boldsymbol{\omega}_L = \mathbf{U}_L$ and if $x/t > \lambda_n$, $\mathbf{U}(x/t) = \sum_{i=1}^{n} \omega_i^r \mathbf{r}_i = R\boldsymbol{\omega}_R = \mathbf{U}_R$

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- If $(x/t)_a < \lambda_k < (x/t)_b$, $\mathbf{U}(x,t)\big|_a = \sum_{i=1}^{k-1}\omega_i^r \mathbf{r}_i + \sum_{i=k}^{n}\omega_i^l \mathbf{r}_i$ and $\mathbf{U}(x,t)\big|_b = \sum_{i=1}^{k}\omega_i^r \mathbf{r}_i + \sum_{i=k+1}^{n}\omega_i^l \mathbf{r}_i$

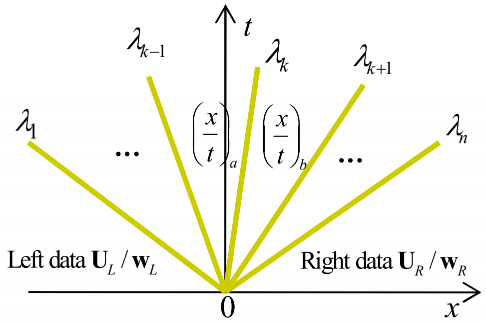
  $\rightarrow$ a jump across λ_k characteristic wave: $\mathbf{U}_b - \mathbf{U}_a = \Delta\mathbf{U}_k = \left(\omega_k^r - \omega_k^l\right)\mathbf{r}_k = \Delta\omega_k \mathbf{r}_k$,

  and from $\mathbf{F} = A\mathbf{U}$, $\Delta\mathbf{F}_k = \mathbf{F}_b - \mathbf{F}_a = A\Delta\mathbf{U}_k = \lambda_k \Delta\omega_k \mathbf{r}_k$

- From $A = R\Lambda R^{-1} = A^+ + A^-$ with $A^\pm = R\Lambda^\pm R^{-1}$, $\lambda_i^- = \min(0,\lambda_i)$ and $\lambda_i^+ = \max(0,\lambda_i)$,

  the cell-interface flux, $\mathbf{F}_{i+1/2} = A\mathbf{U}(0)$, can be expressed as

$$\mathbf{F}_{i+1/2} = \begin{cases} A\mathbf{U}_L + \sum_{i=1}^{n}\lambda_i^- \Delta\omega_i \mathbf{r}_i = A\mathbf{U}_L + A^-(\mathbf{U}_R - \mathbf{U}_L) \\ A\mathbf{U}_R - \sum_{i=1}^{n}\lambda_i^+ \Delta\omega_i \mathbf{r}_i = A\mathbf{U}_R - A^+(\mathbf{U}_R - \mathbf{U}_L) \end{cases} \quad \text{or}$$



$$\mathbf{F}_{i+1/2} = \frac{1}{2}A(\mathbf{U}_L + \mathbf{U}_R) - \mathbf{D}_{i+1/2} == \frac{1}{2}(\mathbf{F}_L + \mathbf{F}_R) - \mathbf{D}_{i+1/2}$$

with $\mathbf{D}_{i+1/2} = \frac{1}{2}(A^+ - A^-)(\mathbf{U}_R - \mathbf{U}_L) = \frac{1}{2}\left|A\right|(\mathbf{U}_R - \mathbf{U}_L) = \frac{1}{2}R\left|\Lambda\right|R^{-1}\Delta\mathbf{U}_{R/L} = \sum_{i=1}^{n}\left|\lambda_i\right|\Delta\omega_i \mathbf{r}_i$.

- ***Approximate Riemann Solver***
  - **From the original work by Godunov, a flux at a cell-interface can be obtained by exactly solving the local Riemann problem defined at** $x_{i+1/2}$, $x_i \le x \le x_{i+1}$.
    - Computationally expensive due to iterative computation of the non-linear algebraic eqn.
    - In many cases, local fluctuation across a cell-interface is small.

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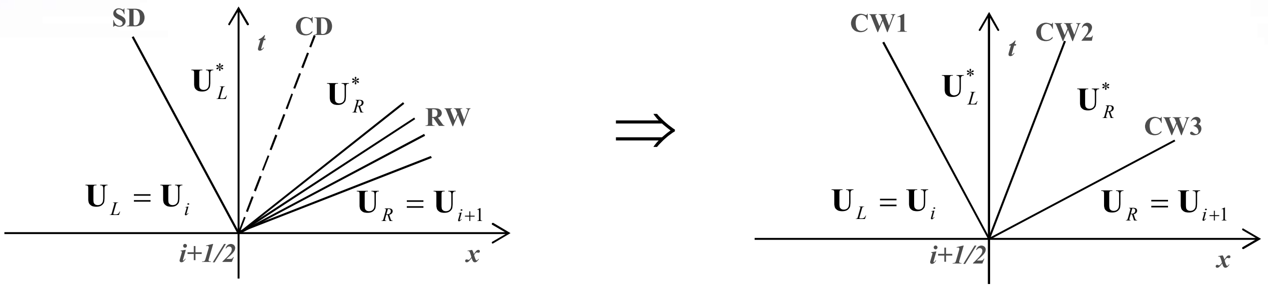
- **Local linearization by approximating the flux Jacobian matrix at $x_{i+1/2}$**
  - See the works by P. Roe(1981)



$$\mathbf{U}_t + 'A(\mathbf{U})'\mathbf{U}_x = \mathbf{0} \xrightarrow[\|\delta\mathbf{U}\| \ll \|\mathbf{U}\|]{\text{small fluctuation}} \mathbf{U}_t + 'A_{i+1/2}'\mathbf{U}_x = \mathbf{0}$$

Design $A_{i+1/2} = A(\mathbf{U}_i, \mathbf{U}_{i+1})$ satisfying

i) consistency: $A_{i+1/2} = \left(\dfrac{\partial \mathbf{F}}{\partial \mathbf{U}}\right)_i = \left(\dfrac{\partial \mathbf{F}}{\partial \mathbf{U}}\right)_{i+1} = A(\mathbf{U})$, if $\mathbf{U}_i = \mathbf{U}_{i+1} = \mathbf{U}$

ii) conservation: $A_{i+1/2}(\mathbf{U}_{i+1} - \mathbf{U}_i) = \mathbf{F}_{i+1} - \mathbf{F}_i$ or $A_{i+1/2}\Delta\mathbf{U}_{i+1/2} = \Delta\mathbf{F}_{i+1/2}$, if $\mathbf{U}_i \neq \mathbf{U}_{i+1}$

* Note that this is exactly the extension of upwinding of SCL $(u_t + f(u)_x = 0)$ into Euler system.

- **Upwinding for SCL**
  - For $u_t + f(u)_x = u_t + a(u)u_x = 0$ with $a(u) = a$ or u, upwind differencing changes the direction of discretization depending on the sign of the local wave speed.

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{1}{\Delta x}\left[a^+\left(u_i^n - u_{i-1}^n\right) + a^-\left(u_{i+1}^n - u_i^n\right)\right] = 0 \quad \text{with} \quad a^\pm = \frac{a \pm |a|}{2}$$

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- Linear case with *a(u)=a*

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+\frac{F_{i+1/2}^n-F_{i-1/2}^n}{\Delta x}=0 \ \text{ with } \ F_{i+1/2}=\frac{a}{2}\left(u_i^n+u_{i+1}^n\right)-\frac{|a|}{2}\Delta u_{i+1/2}^n=\frac{1}{2}\left(f_i^n+f_{i+1}^n\right)-\frac{|a|}{2}\Delta u_{i+1/2}^n$$

- Nonlinear case with $a(u)=f'(u)$

$$F_{i+1/2}=\frac{1}{2}\left(f_i^n+f_{i+1}^n\right)-\frac{\left|a_{i+1/2}^n\right|}{2}\Delta u_{i+1/2}^n \ \text{ with } \ a_{i+1/2}=\begin{cases}\left.\dfrac{\partial f}{\partial u}\right)_{u=u_i \text{ or } u_{i+1}} & \text{if } u_{i+1}=u_i \\ \dfrac{\Delta f_{i+1/2}}{\Delta u_{i+1/2}}=\dfrac{f_{i+1}-f_i}{u_{i+1}-u_i} & \text{if } u_{i+1}\neq u_i\end{cases}$$

- From the mean value theorem, $a_{i+1/2}=a(\xi)$ for $\xi\in[u_i,\, u_{i+1}]$.
- Note that this is a secant approximation of the true flux function.

$$f(u)\cong f_i + a_{i+1/2}(u-u_i) \text{ or } f_{i+1} + a_{i+1/2}(u-u_{i+1}) \ \rightarrow \ u_t+a_{i+1/2}u_x=0$$

- Can we realize the above observation for Euler system? If so, which one is most desirable?

$$\Delta f_{i+1/2}=a_{i+1/2}\Delta u_{i+1/2} \ \rightarrow \ \Delta\mathbf{F}_{i+1/2}=A_{i+1/2}\Delta\mathbf{U}_{i+1/2}$$

Construction of $A_{i+1/2}=A(\mathbf{U}_{i+1/2})$

- Define a parameter vector as $\mathbf{q}=\sqrt{\rho}\,[1,\ u,\ H]^T=[q_1,\ q_2,\ q_3]^T$

$$\mathbf{U}=\begin{bmatrix} q_1^2 \\ q_1q_2 \\ \dfrac{q_1q_3}{\gamma}+\dfrac{\gamma-1}{2\gamma}\left(\dfrac{q_2^2}{2}\right)\end{bmatrix} \ \text{ and } \ \mathbf{F}=\begin{bmatrix} q_1q_2 \\ q_2^2+\dfrac{q_1q_3}{\gamma}+\dfrac{\gamma-1}{2\gamma}\left(\dfrac{q_2^2}{2}\right) \\ q_2q_3\end{bmatrix} \rightarrow u_i \text{ and } f_i \text{ are quadratic function of } q_i.$$

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- For any quadratic product uv

  $$\Delta(uv) \equiv (u+\Delta u)(v+\Delta v) - uv = \hat{u}\Delta v + \hat{v}\Delta u \quad \text{with} \quad \begin{cases} \hat{u} = u + \Delta u/2 = \left[u + (u+\Delta u)\right]/2 \\ \hat{v} = v + \Delta v/2 = \left[v + (v+\Delta v)\right]/2 \end{cases}$$

  Thus, if $Q(q_i, q_j)$ is quadratic in q's,

  $$\Delta Q = Q(q_i + \Delta q_i, q_j + \Delta q_j) - Q(q_i, q_j) = \frac{\partial Q}{\partial q_i}\Delta q_i + \frac{\partial Q}{\partial q_j}\Delta q_j,$$

  where $\dfrac{\partial Q}{\partial q_i}, \dfrac{\partial Q}{\partial q_j}$ are evaluated by the mean values of $(q_i, q_i + \Delta q_i)$ and $(q_j, q_j + \Delta q_j)$.

- If $\mathbf{q} \to \mathbf{q} + \Delta\mathbf{q}$,

  $\mathbf{U}(\mathbf{q}) \to \mathbf{U}(\mathbf{q}) + \Delta\mathbf{U}(\mathbf{q})$ and $\mathbf{F}(\mathbf{q}) \to \mathbf{F}(\mathbf{q}) + \Delta\mathbf{F}(\mathbf{q})$ with quadratic components of q_i

  Thus, $\Delta\mathbf{U}$ and $\Delta\mathbf{F}$ can be exactly computed.

  $\Delta\mathbf{U}(\mathbf{q}) = \left(\dfrac{\partial \mathbf{U}}{\partial \mathbf{q}}\right)\Delta\mathbf{q}$ and $\Delta\mathbf{F}(\mathbf{q}) = \left(\dfrac{\partial \mathbf{F}}{\partial \mathbf{q}}\right)\Delta\mathbf{q}$ by evaluating $\dfrac{\partial \mathbf{U}}{\partial \mathbf{q}}, \dfrac{\partial \mathbf{F}}{\partial \mathbf{q}}$ with the mean values of

  $\mathbf{q}$ and $\mathbf{q} + \Delta\mathbf{q}$. $\to$ $\Delta\mathbf{q} = \left(\dfrac{\partial \mathbf{U}}{\partial \mathbf{q}}\right)^{-1}\Delta\mathbf{U}$ and $\Delta\mathbf{F} = \left(\dfrac{\partial \mathbf{F}}{\partial \mathbf{q}}\right)\left(\dfrac{\partial \mathbf{U}}{\partial \mathbf{q}}\right)^{-1}\Delta\mathbf{U} = \left(\dfrac{\partial \mathbf{F}}{\partial \mathbf{U}}\right)\Delta\mathbf{U}$

  Thus, conservation requirement can be satisfied by evaluating $\dfrac{\partial \mathbf{F}}{\partial \mathbf{U}}$ by the average values of

  the parameter vector $\mathbf{q}$ and $\mathbf{q} + \Delta\mathbf{q}$.

  If $\mathbf{q} = \mathbf{q}_i$, $\mathbf{q} + \Delta\mathbf{q} = \mathbf{q}_i + (\mathbf{q}_{i+1} - \mathbf{q}_i) = \mathbf{q}_{i+1}$, we have $\mathbf{F}_{i+1} - \mathbf{F}_i = \Delta\mathbf{F}_{i+1/2} = \tilde{A}_{i+1/2}\Delta\mathbf{U}_{i+1/2}$ with

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$$\tilde{A}_{i+1/2} = \left(\frac{\partial \mathbf{F}}{\partial \mathbf{U}}\right)_{i+1/2} = A(\tilde{\mathbf{U}}) = \begin{bmatrix} 0 & 1 & 0 \\ (\gamma-3)\dfrac{\tilde{u}^2}{2} & (3-\gamma)\tilde{u} & \gamma-1 \\ \dfrac{(\gamma-1)}{2}\tilde{u}^3 - \tilde{u}\tilde{H} & \tilde{H}-(\gamma-1)\tilde{u}^2 & \gamma\tilde{u} \end{bmatrix}_{i+1/2} \rightarrow \Delta\mathbf{F}_{i+1/2} = \tilde{A}_{i+1/2}\Delta\mathbf{U}_{i+1/2}.$$

From direct computations of $\Delta\mathbf{F}_{i+1/2} = \tilde{A}_{i+1/2}\Delta\mathbf{U}_{i+1/2}$, we have two relations to identify $\tilde{u}$ and $\tilde{H}$ as

$$\tilde{u} = \frac{\sqrt{\rho_i}u_i + \sqrt{\rho_{i+1}}u_{i+1}}{\sqrt{\rho_i}+\sqrt{\rho_{i+1}}}, \quad \tilde{H} = \frac{\sqrt{\rho_i}H_i + \sqrt{\rho_{i+1}}H_{i+1}}{\sqrt{\rho_i}+\sqrt{\rho_{i+1}}}$$

- $\tilde{q}\big|_{\tilde{u}\text{ or }\tilde{H}}$ are convex combination of q_i and q_{i+1}, $\tilde{q} = \theta q_i + (1-\theta)q_{i+1}$, with $\theta = \dfrac{\sqrt{\rho_i}}{\sqrt{\rho_i}+\sqrt{\rho_{i+1}}}$
- With $\tilde{\rho} = \sqrt{\rho_i\rho_{i+1}}$, $\tilde{c}^2 = (\gamma-1)\left[\tilde{H} - \dfrac{\tilde{u}^2}{2}\right] = \gamma\dfrac{\tilde{p}}{\tilde{\rho}}$, but $\tilde{c}^2 = \dfrac{\sqrt{\rho_i}a_i^2 + \sqrt{\rho_{i+1}}a_{i+1}^2}{\sqrt{\rho_i}+\sqrt{\rho_{i+1}}} + \dfrac{\gamma-1}{2}\dfrac{\tilde{\rho}}{\sqrt{\rho_i}+\sqrt{\rho_{i+1}}}(u_i - u_{i+1})^2$

  $\rightarrow \tilde{c}^2 \neq \theta a_i^2 + (1-\theta)a_{i+1}^2$

- **Ex) Construction of $\tilde{A}_{i+1/2}$ for isothermal 1-D Euler eqns.**

  $\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0}$ with $\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \end{bmatrix}$ and $\mathbf{F}(\mathbf{U}) = \begin{bmatrix} \rho u \\ \rho u^2 + \rho c^2 \end{bmatrix}$

  $A(\mathbf{U}) = \begin{bmatrix} 0 & 1 \\ c^2 - u^2 & 2u \end{bmatrix}$, $\lambda_{1,2} = u-c, u+c$ and $\mathbf{r}_{1,2} = \begin{bmatrix} 1 \\ u-c \end{bmatrix}$, $\begin{bmatrix} 1 \\ u+c \end{bmatrix}$

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Define the parameter vector $\mathbf{q}=\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}=\sqrt{\rho}\begin{bmatrix} 1 \\ u \end{bmatrix} \rightarrow \mathbf{U}=\begin{bmatrix} q_1^2 \\ q_1 q_2 \end{bmatrix}$ and $\mathbf{F}(\mathbf{U})=\begin{bmatrix} q_1 q_2 \\ q_2^2+c^2 q_1^2 \end{bmatrix}$

And arithmatic average of $\mathbf{q}$, $\bar{\mathbf{q}}=\begin{bmatrix} \bar{q}_1 \\ \bar{q}_2 \end{bmatrix}=\frac{1}{2}(\mathbf{q}_i+\mathbf{q}_{i+1})=\frac{1}{2}\begin{bmatrix} \sqrt{\rho_i}+\sqrt{\rho_{i+1}} \\ \sqrt{\rho_i}u_i+\sqrt{\rho_{i+1}}u_{i+1} \end{bmatrix}$

Then, $\Delta\mathbf{U}=\left(\frac{\partial\mathbf{U}}{\partial\mathbf{q}}\right)\Delta\mathbf{q}$, $\Delta\mathbf{F}=\left(\frac{\partial\mathbf{F}}{\partial\mathbf{q}}\right)\Delta\mathbf{q}$ with $\frac{\partial\mathbf{U}}{\partial\mathbf{q}}=\begin{bmatrix} 2\bar{q}_1 & 0 \\ \bar{q}_2 & \bar{q}_1 \end{bmatrix}$, $\frac{\partial\mathbf{F}}{\partial\mathbf{q}}=\begin{bmatrix} \bar{q}_2 & \bar{q}_1 \\ 2c^2\bar{q}_1 & 2\bar{q}_2 \end{bmatrix}$

$$\left(\frac{\partial\mathbf{F}}{\partial\mathbf{U}}\right)=\left(\frac{\partial\mathbf{F}}{\partial\mathbf{q}}\right)\left(\frac{\partial\mathbf{U}}{\partial\mathbf{q}}\right)^{-1}=\frac{1}{2q_1^2}\begin{bmatrix} \bar{q}_2 & \bar{q}_1 \\ 2c^2\bar{q}_1 & 2\bar{q}_2 \end{bmatrix}\begin{bmatrix} \bar{q}_1 & 0 \\ -\bar{q}_2 & 2\bar{q}_1 \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ c^2-\left(\frac{\bar{q}_2}{\bar{q}_1}\right)^2 & 2\frac{\bar{q}_2}{\bar{q}_1} \end{bmatrix}$$

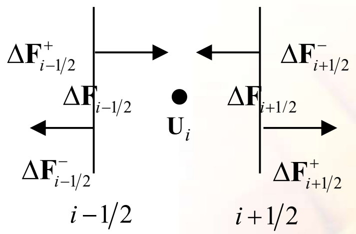
Thus, we have $\tilde{A}_{i+1/2}=\begin{bmatrix} 0 & 1 \\ c^2-\tilde{u}^2 & 2\tilde{u} \end{bmatrix}$ with $\tilde{u}=\frac{\bar{q}_2}{\bar{q}_1}\equiv\frac{\sqrt{\rho_i}u_i+\sqrt{\rho_{i+1}}u_{i+1}}{\sqrt{\rho_i}+\sqrt{\rho_{i+1}}}$.

$\tilde{A}_{i+1/2}$ is identical to $A(\mathbf{U})$ except that each component is evaluated with the arithmatic average of q_i (or Roe-averaged variables: $\tilde{\rho}$, $\tilde{u}$, $\tilde{H}$)

- **Flux at a cell interface**
  - Split the flux difference using eigenvalues of $\tilde{A}_{i+1/2}$

$$\Delta\mathbf{F}_{i+1/2}=\tilde{A}_{i+1/2}\Delta\mathbf{U}_{i+1/2}=\tilde{A}^+_{i+1/2}\Delta\mathbf{U}_{i+1/2}+\tilde{A}^-_{i+1/2}\Delta\mathbf{U}_{i+1/2}=\Delta\mathbf{F}^+_{i+1/2}+\Delta\mathbf{F}^-_{i+1/2}$$

with $\tilde{A}^{\pm}_{i+1/2}=(\tilde{R}\tilde{\Lambda}^{\pm}\tilde{R}^{-1})_{i+1/2} \rightarrow \mathbf{U}^{n+1}_i=\mathbf{U}^n_i-\frac{\Delta t}{\Delta x}\left(\Delta\mathbf{F}^+_{i-1/2}+\Delta\mathbf{F}^-_{i+1/2}\right)$



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- Numerical dissipation at a cell-interface

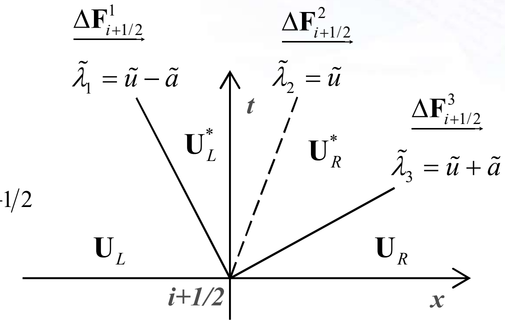
$$\mathbf{F}_{i+1/2} = \frac{1}{2}\left(\mathbf{F}_i + \mathbf{F}_{i+1}\right) - \mathbf{D}_{i+1/2} \text{ with } \mathbf{D}_{i+1/2} = \frac{1}{2}\left|\tilde{A}_{i+1/2}\right|\Delta\mathbf{U}_{i+1/2} \text{ (matrix dissipation)}$$

- Characteristic wave splitting (three-wave approximation)

$$\Delta\mathbf{F}_{i+1/2} = \tilde{A}_{i+1/2}\Delta\mathbf{U}_{i+1/2} = (\tilde{R}\tilde{\Lambda}\tilde{R}^{-1})_{i+1/2}\Delta\mathbf{U}_{i+1/2} = (\tilde{R}\tilde{\Lambda})_{i+1/2}(\tilde{R}^{-1}\Delta\mathbf{U})_{i+1/2}$$

$$= \tilde{\lambda}_1\Delta\tilde{w}^1_{i+1/2}\tilde{\mathbf{r}}_1 + \tilde{\lambda}_2\Delta\tilde{w}^2_{i+1/2}\tilde{\mathbf{r}}_2 + \tilde{\lambda}_3\Delta\tilde{w}^3_{i+1/2}\tilde{\mathbf{r}}_3 = \Delta\mathbf{F}^1_{i+1/2} + \Delta\mathbf{F}^2_{i+1/2} + \Delta\mathbf{F}^3_{i+1/2}$$

$$\rightarrow \mathbf{F}_{i+1/2} = \mathbf{F}_L + \sum_{\lambda_k<0}\tilde{\lambda}_k\Delta\tilde{w}^k_{i+1/2}\tilde{\mathbf{r}}_k \text{ or } \mathbf{F}_R - \sum_{\lambda_k>0}\tilde{\lambda}_k\Delta\tilde{w}^k_{i+1/2}\tilde{\mathbf{r}}_k$$



Characteristics of FDS (vs. FVS)

- **Split the flux difference by defining a cell-interface state followed by upwinding → wave interaction between left and right states through** $\tilde{A}_{i+1/2} = A(\tilde{\mathbf{U}})$

$\left|\tilde{\lambda}_i\right| = \tilde{\lambda}_i$ and $\left|\tilde{A}_{i+1/2}\right| = \tilde{A}_{i+1/2} \rightarrow \mathbf{F}_{i+1/2} = 0.5\left(\mathbf{F}_i + \mathbf{F}_{i+1}\right) - 0.5\tilde{A}_{i+1/2}\Delta\mathbf{U}_{i+1/2} = 0.5\left(\mathbf{F}_i + \mathbf{F}_{i+1}\right) - 0.5\left(\mathbf{F}_{i+1} - \mathbf{F}_i\right) = \mathbf{F}_i$

- **Capturing discontinuities**
  - $\mathbf{F}_{i+1} = \mathbf{F}_i \rightarrow \Delta\mathbf{F}_{i+1/2} = \tilde{A}_{i+1/2}\Delta\mathbf{U}_{i+1/2} = \mathbf{0} = \sum_k \tilde{\lambda}_k\Delta\tilde{w}^k_{i+1/2}\tilde{\mathbf{r}}_k = \mathbf{0} \rightarrow \tilde{\lambda}_k = 0 \text{ or } \Delta\tilde{w}^k_{i+1/2} = 0$

Thus, $\mathbf{D}_{i+1/2} = 0.5\left|\tilde{A}\right|_{i+1/2}\Delta\mathbf{U}_{i+1/2} = 0.5\sum_k\left|\tilde{\lambda}_k\right|\Delta\tilde{w}^k_{i+1/2}\tilde{\mathbf{r}}_k = \mathbf{0}$, or more generally,

$\Delta\mathbf{F}_{i+1/2} = \tilde{A}_{i+1/2}\Delta\mathbf{U}_{i+1/2} = S\Delta\mathbf{U}_{i+1/2} \rightarrow S = \tilde{\lambda}_j$ and $\Delta\mathbf{U}_{i+1/2} = \tilde{\mathbf{r}}_j$. From $\Delta\boldsymbol{\omega} = \tilde{R}^{-1}\Delta\mathbf{U} = \tilde{\mathbf{l}}_i \cdot \tilde{\mathbf{r}}_j = \delta_{ij}$,

only non-zero wave strength is $\Delta\omega_j$, indicating that $\tilde{\lambda}_j$ is an only non-zero characteristic wave allowing transition from $\mathbf{U}_i$ to $\mathbf{U}_{i+1}$. →Exact capturing of shock/CD and Good for N-S computations