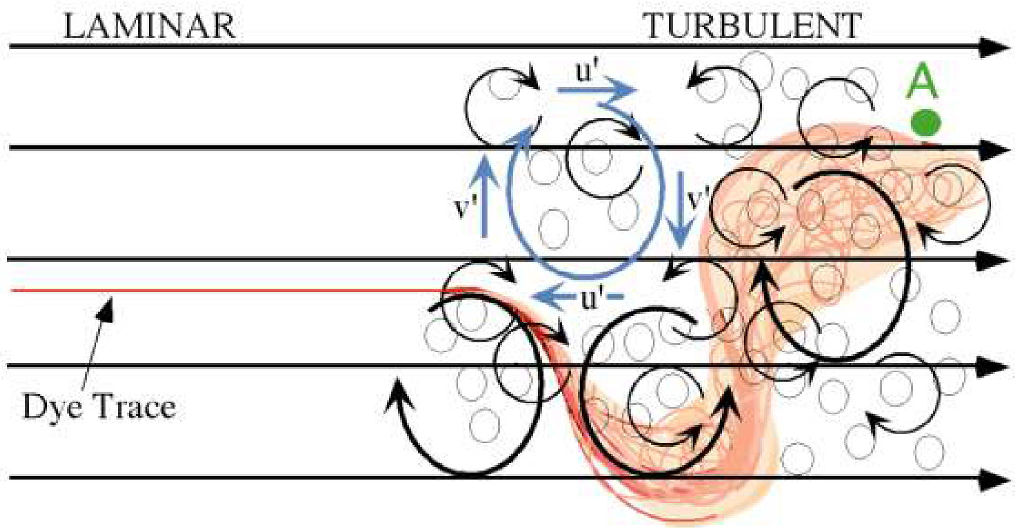
Chapter 3

Turbulent Diffusion



Kawahara (2016)

Chapter 3 Turbulent Diffusion

Contents

Chapter 3 Turbulent Diffusion

Objectives

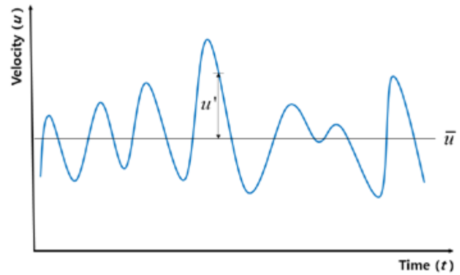
- Introduce kinematics of turbulence which affects the mixing of pollutants
- Visualize spreading of particles due to turbulent motion
- Introduce the statistical concepts for the analysis of the turbulent mixing
- Derive turbulent diffusion equation using Taylor' analysis (1921)

Taylor, Geoffrey – English fluid mechanician

3.1 Introduction

- Turbulent mixing

\- Compared to molecular diffusion, turbulent diffusion is very efficient in rapidly decreasing the concentrations of contaminants that are released into the natural environment.



- Water quality analysis

\- Water quality standards are usually written in terms of <u>time-averaged values</u>.

\- However, the contaminant signal consists of a <u>small mean value</u> with <u>intermittent fluctuations</u> that range from zero to levels that are <u>orders of magnitude higher than the mean</u>.

3.1 Introduction

- Most mathematical models of turbulent diffusion predict only time-averaged concentrations.
- Mathematical models usually do not predict higher order measures of these signals such as their intermittency, peak values, probability density functions, and spatial correlations.
- Thus, the statistical variations may sometimes be needed.

• In this chapter, we consider only the case of the turbulent diffusion, that is, the spreading of a scalar quantity due to irregular turbulent velocity excluding mixing (shear-flow dispersion) due to the combined effect of diffusion plus shear in the mean velocity.

3.1 Introduction

3.1.1 Basic characteristics of turbulent flows

· Turbulent flow: Irregularity, randomness, non-linearity

↔ coherent structure – large scale eddies

High diffusivity and mixing

High Reynolds number

3-D fluctuations and rotational eddies

↔ tend to be isotropic– small scale eddies

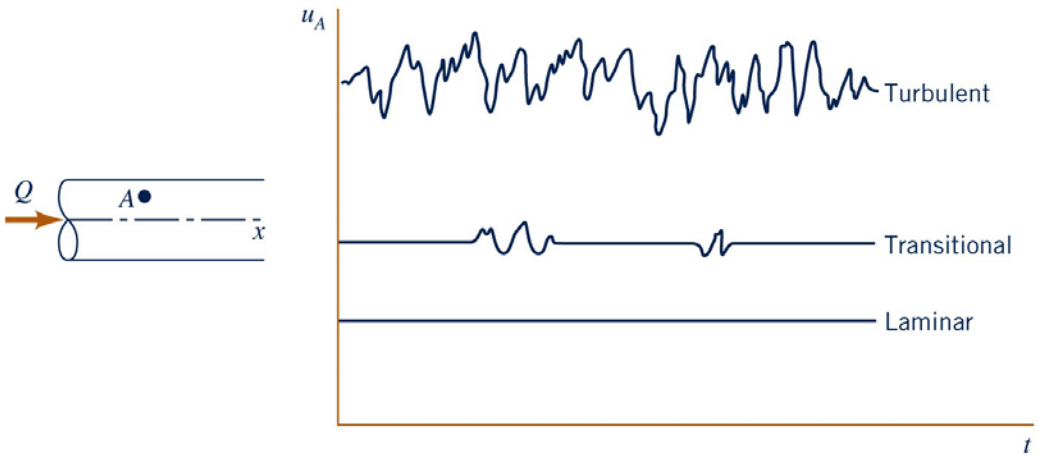
Transfer and dissipation of kinetic energy

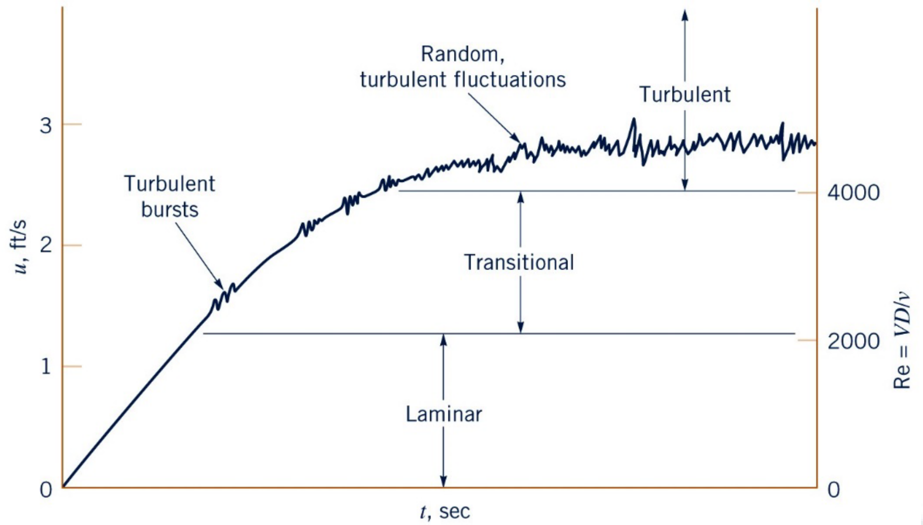
Continuum phenomenon

Feature of flow ↔ property of fluid $(\rho, \mu,)$

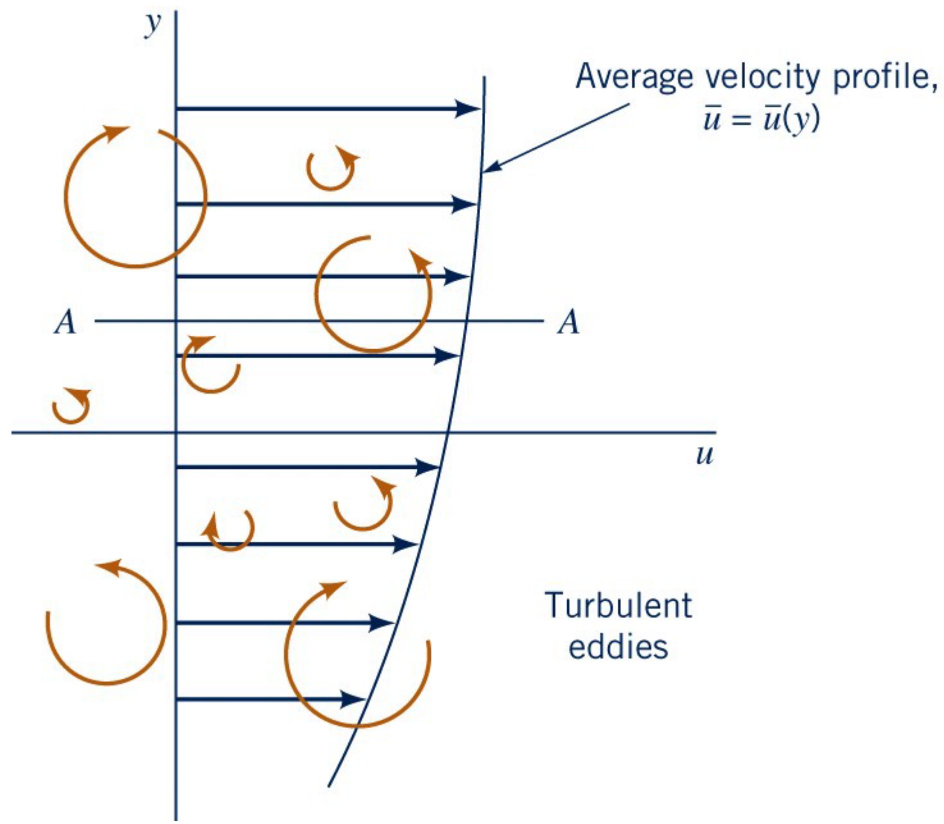
· Navier-Stokes and scalar transport equations can be used to describe turbulent flows and mixing.

3.1 Introduction



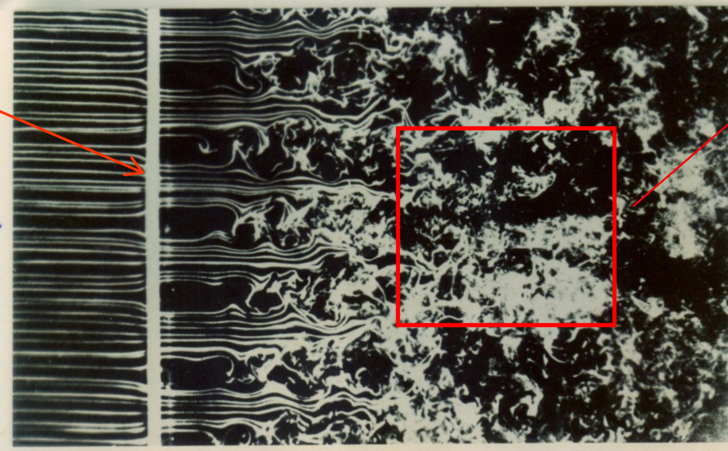


3.1 Introduction



3.1 Introduction

Coarse grid

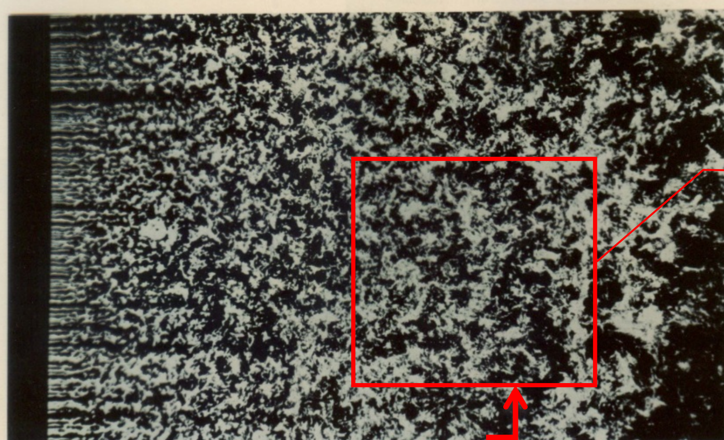


152. Generation of turbulence by a grid. Smoke wires show a uniform laminar stream passing through a $1/16$-inch plate with $3/4$-inch square perforations. The Reynolds number is 1500 based on the 1-inch mesh size. Instability of the shear layers leads to turbulent flow downstream. *Photograph by Thomas Corke and Hassan Nagib*

grid turbulence

Non-isotropic turbulence

Fine grid



153. Homogeneous turbulence behind a grid. Behind a finer grid than above, the merging unstable wakes quickly form a homogeneous field. As it decays downstream, it provides a useful approximation to the idealization of isotropic turbulence. *Photograph by Thomas Corke and Hassan Nagib*

Isotropic turbulence

3.1 Introduction

3.1.2 Description of turbulent flows

The velocity at a point in a turbulent flow will appear to an observer to be "random" or "chaotic."

→ The velocity is unpredictable in the sense that knowing the instantaneous velocity at some instant of time is insufficient to predict the velocity a short time later.

→ Thus, we describe the motion through statistical measures.

3.1 Introduction

For a stationary (steady) velocity record, the instantaneous velocity can be decomposed into the sum of time-averaged and fluctuating contributions as shown in Fig. 3.1.

$$u = \overline{u} + u' \tag{3.1}$$

where $\overline{u}$ is a time-averaged value which is given as

$$\overline{u} = \frac{1}{T}\int_0^T u\,dt \tag{3.2}$$

where T is a time much longer than the longest turbulent fluctuations in the flow, and u' is the fluctuating component, i.e. the deviation from the mean values.

3.1 Introduction

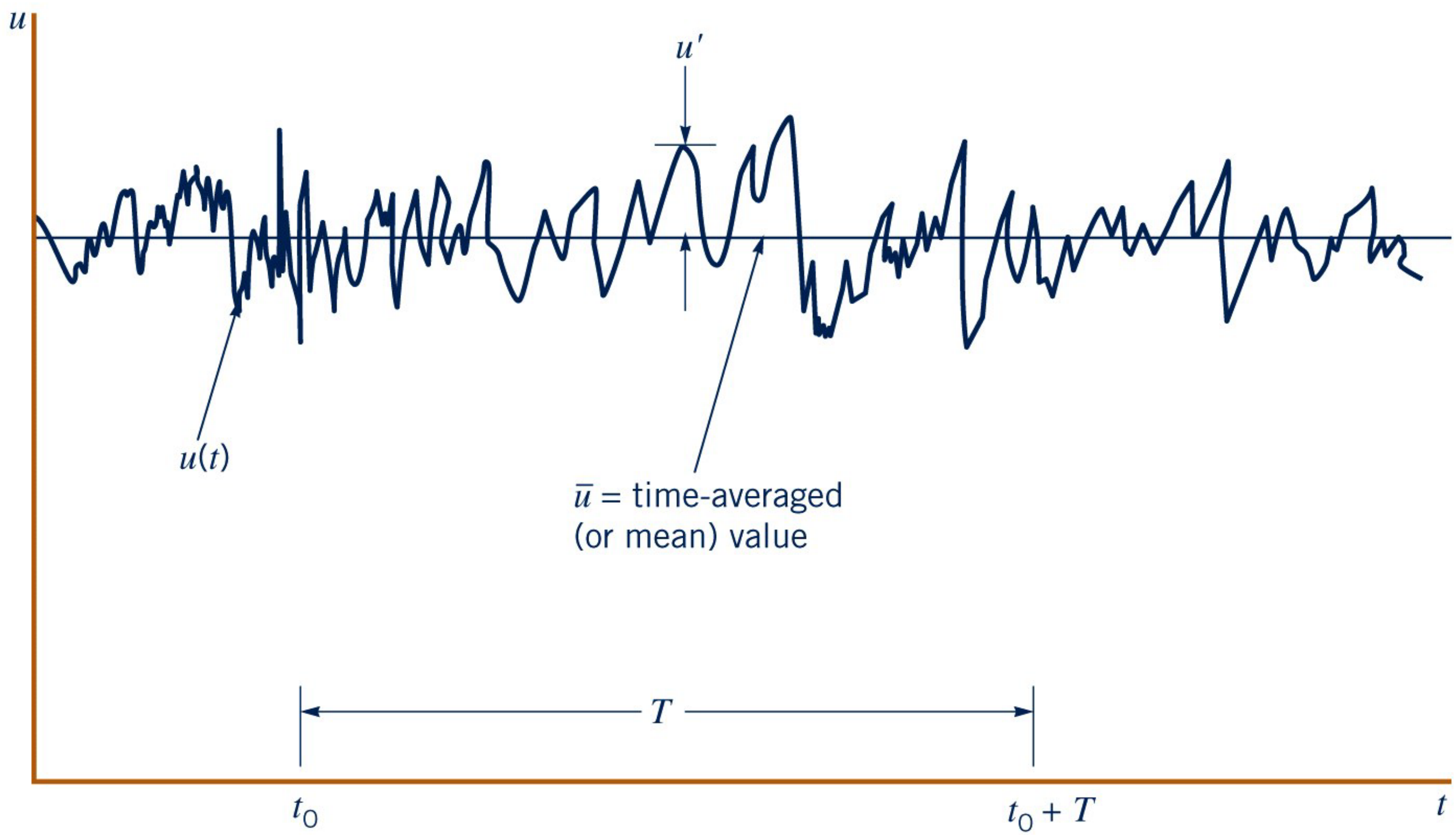


Fig. 3.1

3.1 Introduction

Higher order statistical quantities, such as the variance, are used to describe the magnitude (intensity) of the fluctuations:

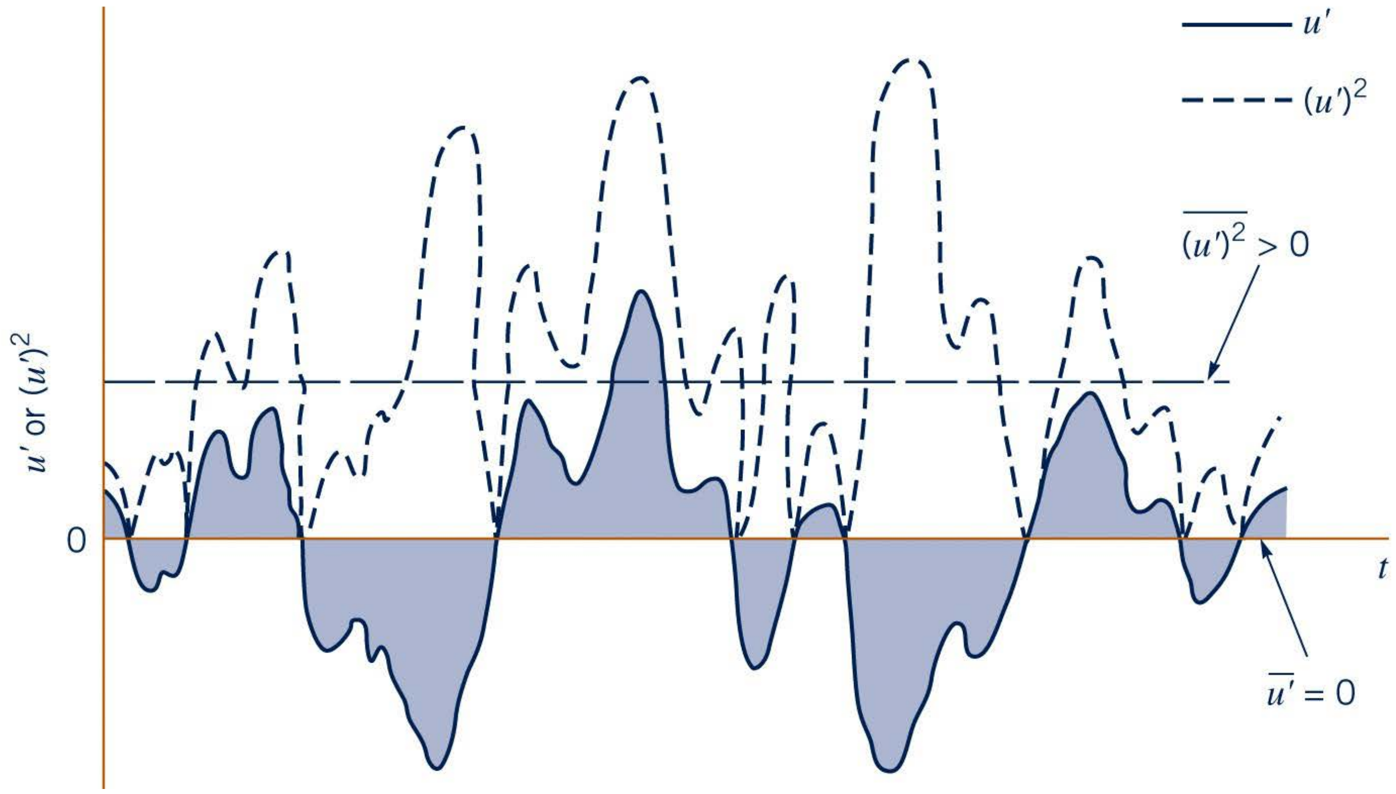
$$\tilde{u}^2 = \overline{u'^2} = \frac{1}{T}\int_0^T (u - \bar{u})^2 \, dt \qquad (3.3)$$

where $\tilde{u}$ is the standard deviation and is defined as the turbulent intensity.

Actual velocity records obtained at two depths in the open channel flow photographed in Fig. 3.2 are shown in Fig. 3.3.

→ The time-averaged velocity is greater farther from the wall, but the turbulence intensity is significantly larger near the wall.

3.1 Introduction



3.1 Introduction

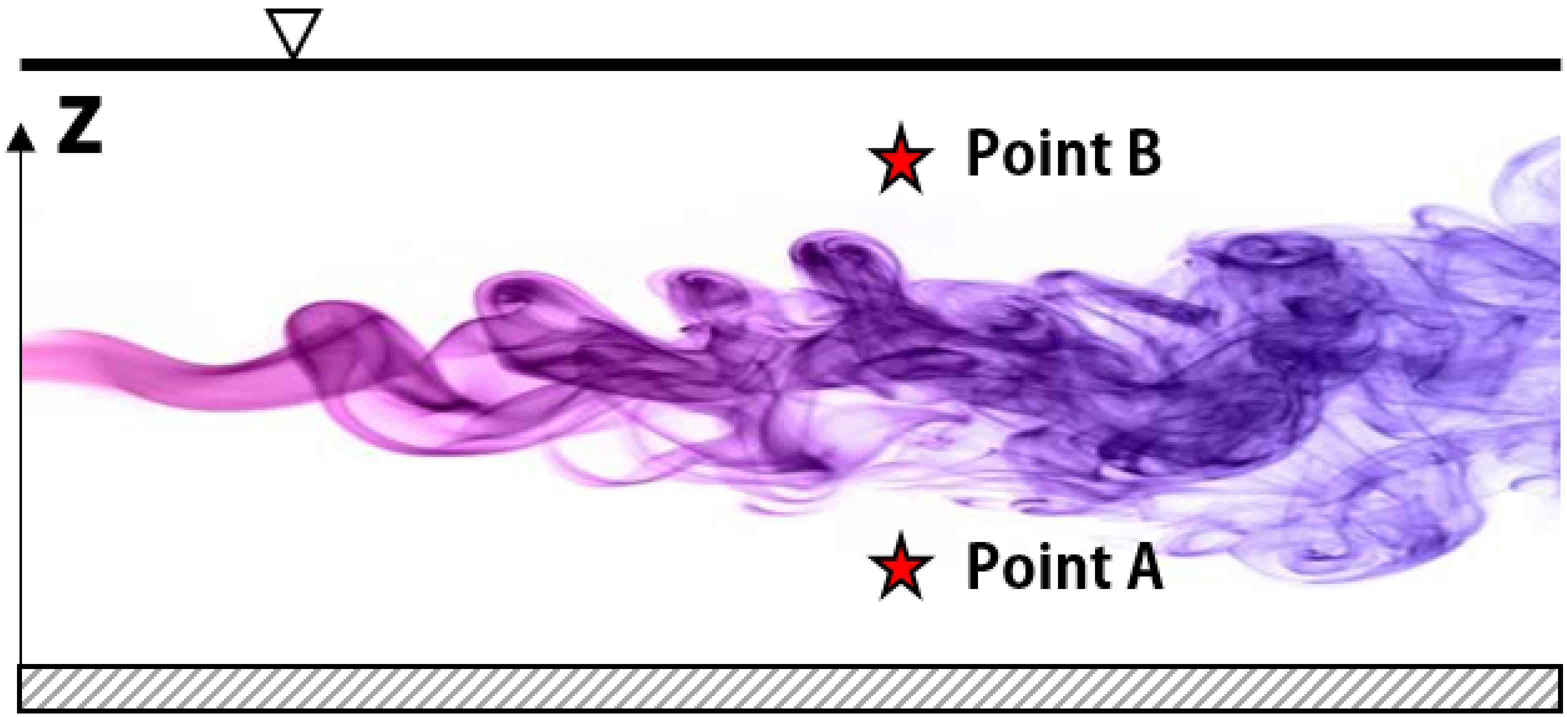


Fig. 3.2

3.1 Introduction

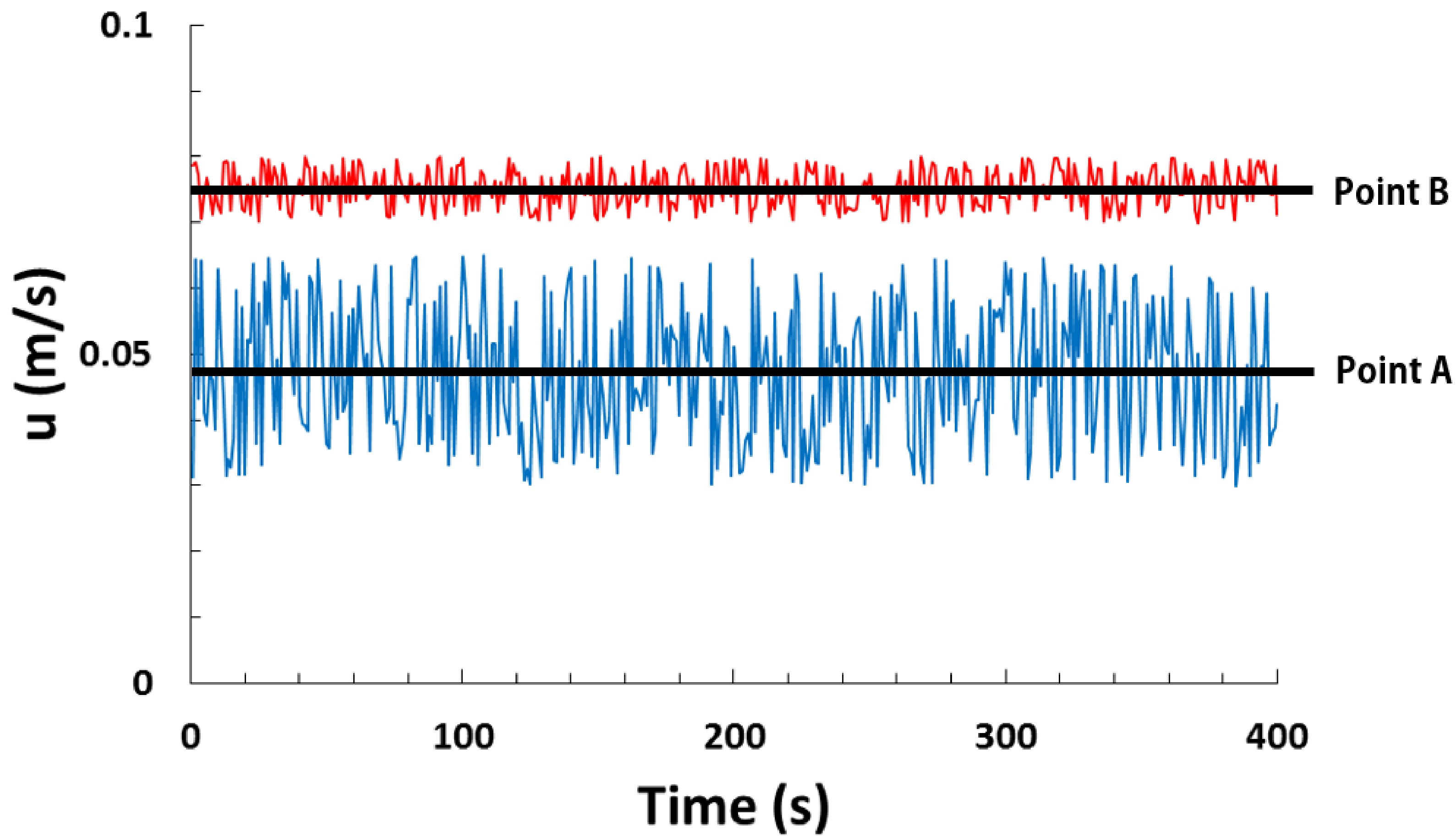


Fig. 3.3

3.1 Introduction

The variation of the time-averaged velocity and the turbulence intensity with distance, z, are shown in Fig. 3.4.

→ The time-averaged velocity increases monotonically from zero at the wall to be approximated into the logarithmic profile.

→ The turbulence intensity increases rapidly from zero at the wall to a local maximum near the wall and then monotonically decreases.

The turbulent fluctuations act to efficiently transport momentum, heat, and tracer concentration.

→ It is common to model the transport due to the fluctuations by defining an effective diffusion coefficient called the eddy diffusivity.

3.1 Introduction

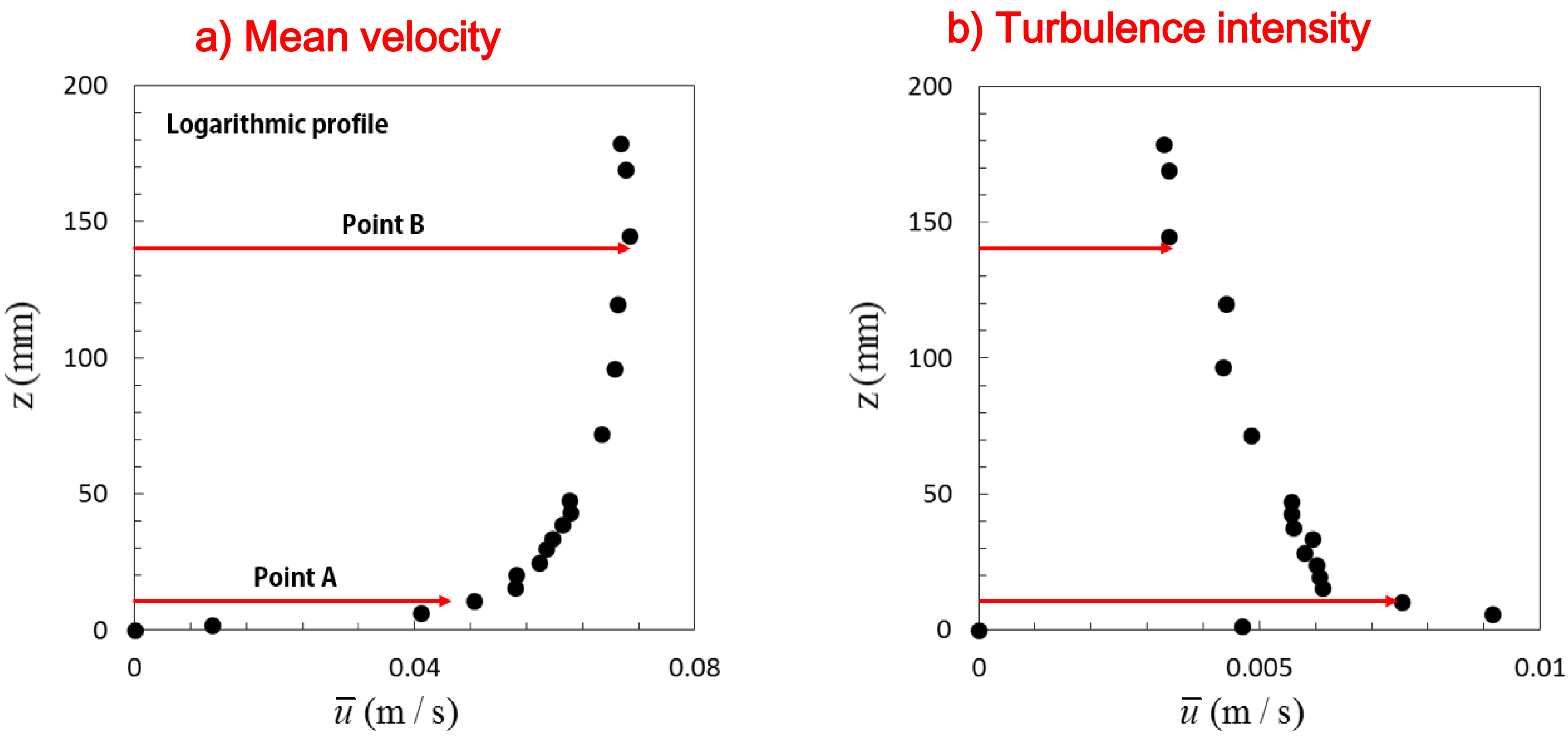


Fig. 3.4

3.1 Introduction

3.1.3 Length Scales of turbulent flows

Motions in a turbulent flow exist over a broad range of length and time scales.

(1) Integral length scale

→ The largest scales are bounded by the geometric dimensions of the flow, for instance the diameter of a pipe or the depth of an open channel.

Eddies lose the most of their energy after one or two overturns.

→ Thus, because the rate of energy transferred from the largest eddies is proportional to their energy times their rotational frequency.

The rate of energy dissipation, ε, is of the order:

$$\varepsilon \propto \tilde{u}^2 \cdot \frac{\tilde{u}}{l} \propto \frac{\tilde{u}^3}{l} \qquad (3.4)$$

3.1 Introduction

The rate of dissipation is independent of the fluid viscosity and only depends on large-scale motions.

→ The scale at which the dissipation occurs is strongly dependent on the fluid viscosity.

(2) Kolmogorov microscale

~ turbulent velocity field

Dissipation length scale (에너지를 소멸시키는 와의 크기) is

$$\eta \propto \left(\frac{\nu^3}{\varepsilon} \right)^{\frac{1}{4}} \qquad (3.5)$$

ν = kinematic viscosity

3.1 Introduction

Time scale of the smallest eddies is

$$\tau \propto \left(\frac{\nu}{\varepsilon}\right)^{\frac{1}{2}} \tag{3.6}$$

Velocity scale is

$$u \propto \left(\nu\varepsilon\right)^{\frac{1}{4}} \tag{3.7}$$

(3) Batchelor scale (molecular scale)

~ turbulent concentration field

$$L_B \propto \left(\frac{D}{\gamma}\right)^{\frac{1}{2}} \tag{3.8}$$

3.1 Introduction

where D = molecular diffusivity; γ = the strain rate of the smallest velocity scale which is given as

$$\gamma = \frac{u}{\eta} = \frac{(\nu\varepsilon)^{1/4}}{\left(\frac{\nu^3}{\varepsilon}\right)^{1/4}} = \left(\frac{\varepsilon}{\nu}\right)^{1/2} \tag{3.9}$$

Therefore the Batchlor's length scale can be recast into a form that include both the molecular diffusivity and the kinematic viscosity.

$$L_B \propto \left(\frac{\nu D^2}{\varepsilon}\right)^{1/4} \tag{3.10}$$

3.1 Introduction

- **Schmidt number**

~ is defined as the ratio of the Kolmogorov and Batchelor length scales

$$S_c \approx \left(\frac{\eta}{L_B}\right)^2 \approx \left(\frac{\nu}{D}\right) \qquad (3.11)$$

[Ex] For the open channel flow, $\bar{u} = 50\,mm/s$, $\tilde{u} \approx 5\,mm/s$

· Integral length scale, $l \approx \frac{1}{2} h \approx 100mm$

Water @ 20℃:; $\nu = 1\times10^{-6}\,m^2/s = 1\times10^{0}\,mm^2/s$; $D = 1\times10^{-10} m^2 / s$

3.1 Introduction

Solution:

· Kolmogorov scales

$$\varepsilon \approx \frac{(5)^3 (mm/s)^3}{100(mm)} = 1.25 \frac{mm^2}{s^3}$$

$$\eta = \left[\frac{\left(1 \times 10^0 \, mm^2/s\right)^3}{1.25} \right]^{1/4} = \ 0.7 \ mm$$

$$\tau = \left[\frac{1\left(mm^2/s\right)}{1.25\left(mm^2/s^3\right)} \right]^{\frac{1}{2}} = \ 0.5 \ s$$

3.1 Introduction

· Batchelor scale

$$L_B = 0.02mm$$

~ Batchelor scale is 35 times smaller than the Kolmogorov scale.

→ We would expect a much finer structure of the concentration field than the velocity field.

3.1 Introduction

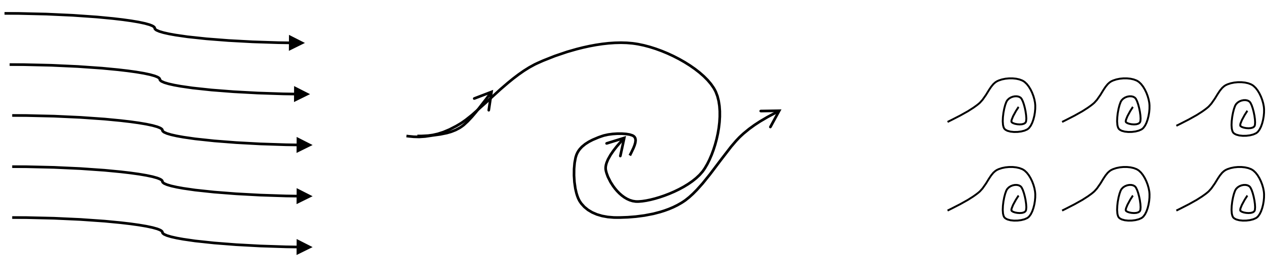
3.1.4 Energy cascade

- Spreading of kinetic energy of the fluid motion over a range of eddy sizes through the non-linear interaction of the large and smaller scales of motion

- 난류는 크고 작은 와로 구성되어 있으며 이들은 서로 비선형적으로 연계되어 있음
- 큰 규모의 와는 에너지 공급원, 에너지의 대부분은 포함하고 있음
- 아주 작은 와는 에너지 소멸의 주원인

3.1 Introduction

· Energy transfer



mean flow → large eddy → small eddy → heat

generation of turbulence — energy cascade — dissipation by viscosity

mean flow	→	large eddy	→	small eddy	→	heat
	generation of turbulence		energy cascade		dissipation by viscosity	

Kolmogorov (1941): In equilibrium, transfer rate = dissipation rate

3.1 Introduction

- Turbulent kinetic energy is passed down from the largest eddies to the smallest through the process called the energy cascade.

  ~ To maintain turbulence, a constant supply of energy must be fed to the turbulent fluctuations at the largest scales from the mean motion.

  ~ At the smallest scales, the energy is dissipated into heat by viscous effects.

- The small scale motions tend to have small time scales.

  → These motions are statistically independent of the relatively slow, large-scale turbulence and of the mean flow. → isotropic turbulence

  → The small scale motion should depend only on the rate of energy transfer from the larger scales and on the viscosity of the fluid.

3.1 Introduction

3.1.5 Kolmogorov's universal equilibrium theory of turbulence

The kinetic energy of the small and intermediate scale motions varies only at the rate at which the mean flow varies.

→ The behavior of the intermediate scales (inertial surange) is governed by the transfer of energy which, in turn, is exactly balanced by dissipation at the smallest scales.

~ In equilibrium, the transfer process must be in equilibrium with the energy dissipation rate.

- The energy spectrum characterizes the turbulent kinetic energy distribution as a function of length scale.

  → The spectrum indicates the amount of turbulent kinetic energy contained at a specific length scale.

3.1 Introduction

- The large turbulent length scales in the flow dictate the rate of dissipation.

  ~ These large scales draw energy from the mean flow, then transfer the energy to successively smaller scales until it is dissipated at the Kolmogorov microscales.

  ~ The energy distribution at the largest length scales is generally dictated by the flow geometry and mean flow velocity.

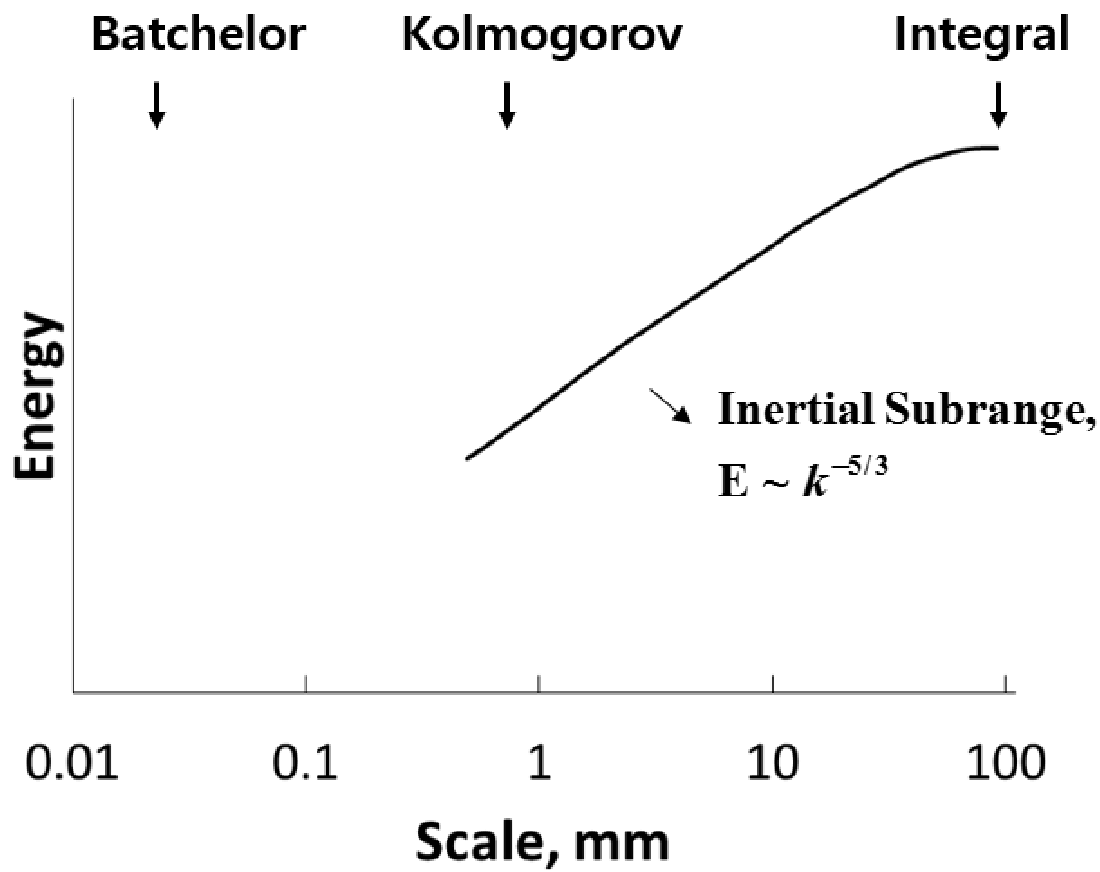
  ~ The smallest length scales are isotropic in nature.

- For inertial subrange $(\eta \leq L \leq \ell)$, the energy spectrum will only be a function of the length scale and the dissipation rate.

$$E = \alpha \varepsilon^{2/3} k^{-5/3} \qquad (3.12)$$

k = wave number ~ inverse of length

3.1 Introduction



3.1 Introduction

3.1.6 Evolution equations

Turbulent flows must instantaneously satisfy conservation of mass and momentum.

→ The incompressible continuity and Navier-Stokes equations can be solved for the instantaneous flow field.

→ However, to accurately simulate the turbulent field, the calculation must span from the largest geometric scales down to the Kolmogorov and Batchelor length scales.

3.1 Introduction

In many situations, engineers are satisfied with an accurate assessment of the time-averaged flow quantities.

→ The time-averaged flow equations, known as Reynolds-Averaged Navier-Stokes (RANS) equations, read as

$$\frac{\partial \overline{u_i}}{\partial x_i} = 0 \tag{3.13}$$

$$\rho\left(\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j}\right) = -\frac{\partial \overline{p}}{\partial x_i} + \mu\frac{\partial^2 \overline{u_i}}{\partial {x_j}^2} - \rho\frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right) \tag{3.14}$$

where $\rho\frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right)$ = Reynolds stress tensor; it physically corresponds to the transport of momentum due to the turbulent fluctuations.

3.2 Mixing by Turbulent Motion

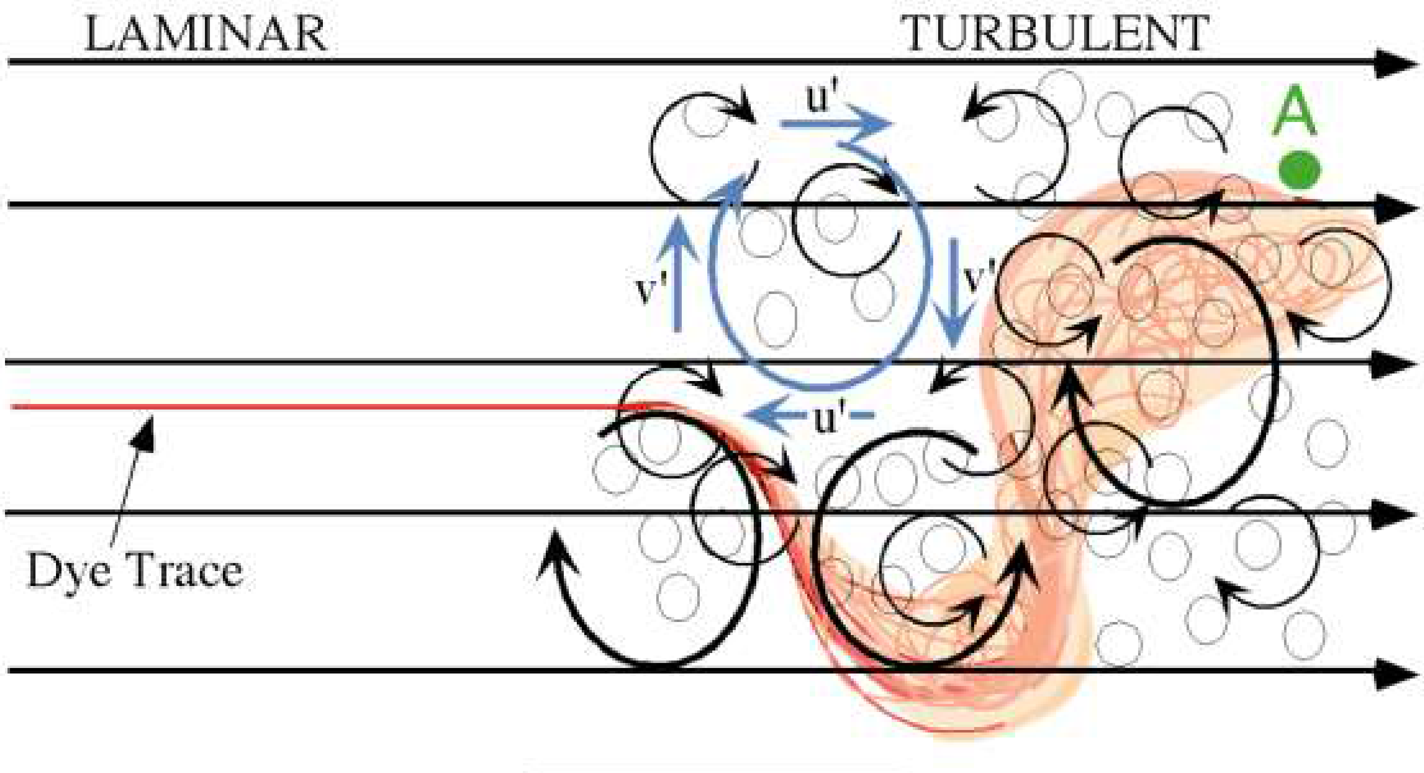
3.2.1 Mixing by turbulent motion

- Eddies range in size from the largest geometric scales of the flow down to small scale where molecular diffusion dominates.

- These eddies are continuously evolving in time, and the superposition of their induced motions leads to the fluctuating time records as shown in Fig. 3.2.

- Large eddies transport the whole cloud of tracer while small eddies distort the shape of the cloud.

3.2 Mixing by Turbulent Motion



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3.2 Mixing by Turbulent Motion

*3.2.2 Two approaches for analysis of turbulent **mixing***

1) In Environmental Fluid Mechanics, they try to model the transport due to the fluctuations by defining an effective diffusion coefficient called the eddy diffusivity.

$$-\overline{u_i c} = \Gamma_t \frac{\partial C}{\partial x_i}$$

1) In Turbulence Modeling, they model the turbulent motion first, and then relate the turbulent diffusion to the eddy viscosity.

→ use turbulent Prandtl (heat) or Schmidt number (mass), σ_t

$$\Gamma_t = \frac{\nu_t}{\sigma_t}$$

3.2 Mixing by Turbulent Motion

3.2.3 Mixing mechanism in turbulent flows

(1) Experiments of slugs of tracer

- Consider the spreading of a slug of tracer or a group of marked particles in a steady turbulent flow.

- Suppose a mass M of tracer (or a total number M of marked particles) is released at a fixed point in the stationary, homogeneous turbulent flow.

- Subsequent spread of the tracer is to be viewed by an observer moving with the mean velocity of the fluid.

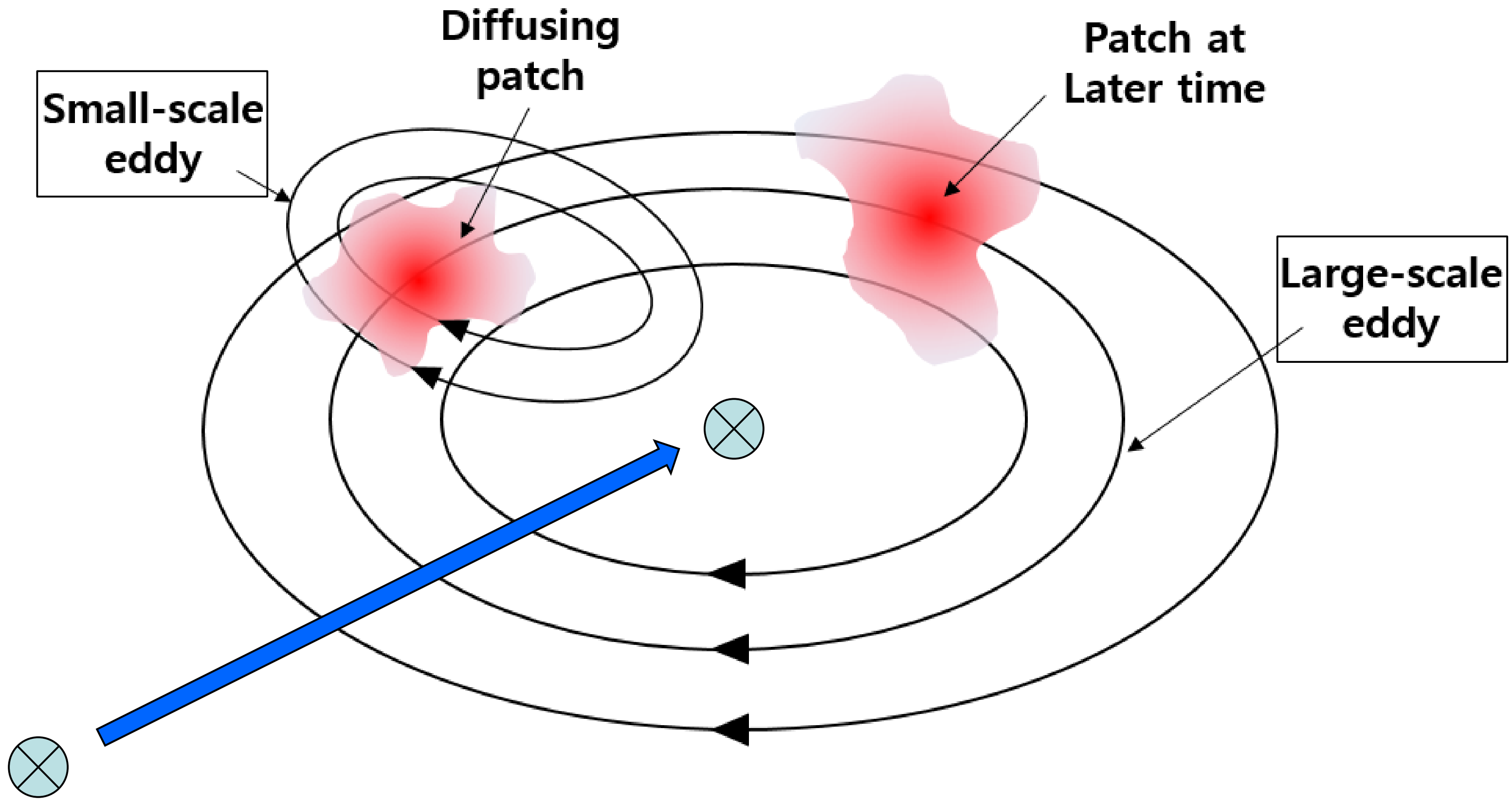
[Re] Stationary, homogeneous turbulence

~the variance of the velocity is steady and does not change with time and position.

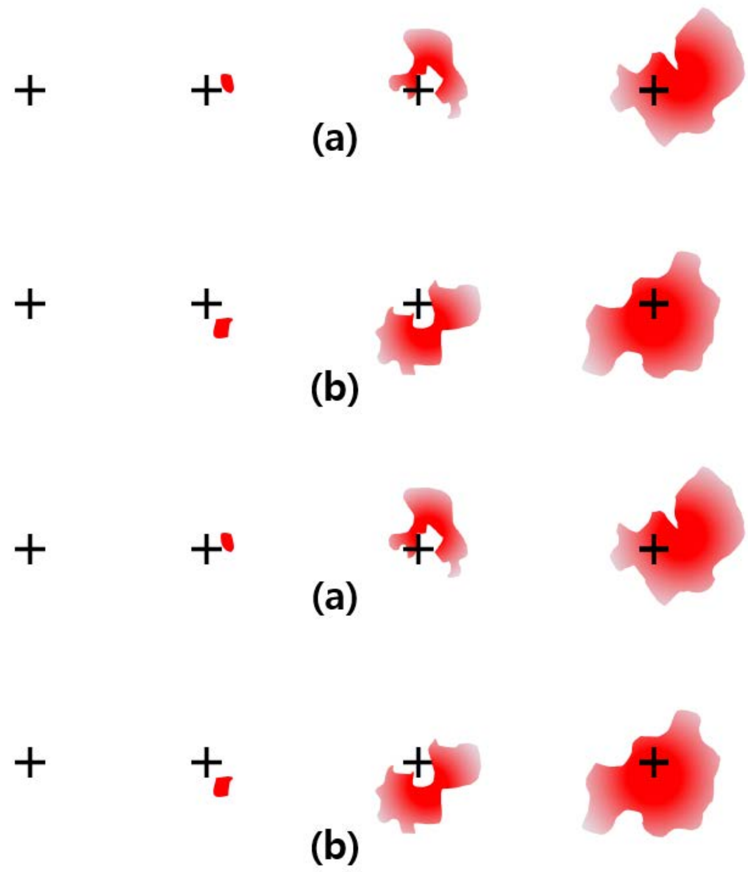
3.2 Mixing by Turbulent Motion

- We can take a series of photographs using a camera traveling with the mean velocity of the flow.
- The photos were taken at equal times after the release for two identical experiments
- The experiment could be repeated many times
- The results of the two sets of experiments are quite different, and the differences are two sorts: first, the small scale fluctuations, which are different for each cloud; second, the large scale fluctuations.

3.2 Mixing by Turbulent Motion



3.2 Mixing by Turbulent Motion



3.2 Mixing by Turbulent Motion

① The small scale fluctuations

~ distort the shape of the cloud and produce steep concentration gradient over short distances.

~ These local differences will eventually be smoothed out by molecular diffusion.

② The large scale fluctuations

~ some of them are larger than the cloud itself.

~ transport the entire cloud

~ Each cloud of particles encounters a different set of large scale motions, so the motion of the center of mass of each cloud is different.

3.2 Mixing by Turbulent Motion

(2) Ensemble averages

- Now, suppose we release a large number of clouds of particles, one after another, and watch the spread of each cloud over a long period of time.
- Since we subtracted out the mean motion, the average position of the center of mass will be at the origin, but the center of mass of each cloud may diverge from the origin because of the large scale eddies.
- If we wait long enough, each separate cloud will grow to be bigger than the largest eddies and will average out their effects.
- The center of mass of each cloud will tend to return to the origin through the process of averaging the random motions. We can use "ensemble average" which is the average over all of the release.

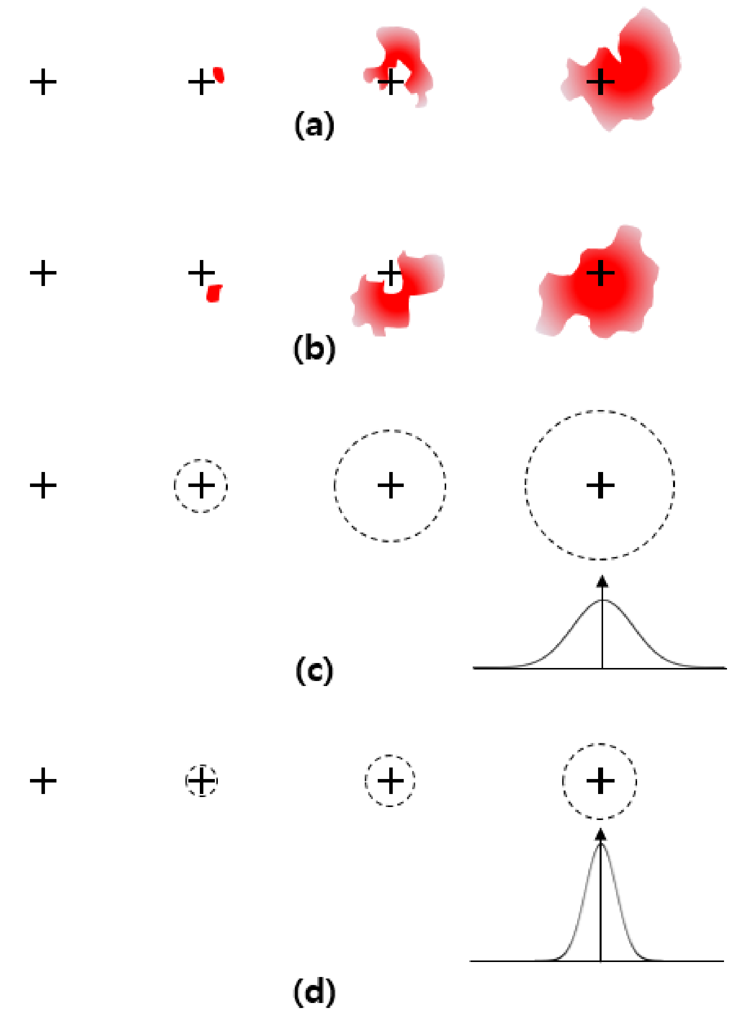
3.2 Mixing by Turbulent Motion

1st release

2nd release

Ensemble mean-LT (EALT)

Ensemble mean-ST (EAST)



3.2 Mixing by Turbulent Motion

- However, we also want to solve the problems of the turbulent diffusion during short times.

→ We try to follow each release separately.

- Two kinds of average

1) Ensemble average for long times (EALT)

- average over all of the releases after long times
- However, the ensemble average concentration at a point in space and time is likely to be an average of a large number of zeros (times when the tracer cloud does not cover the point) plus a few large values.
- This type of average may be meaningless when high concentration is of concern.

3.2 Mixing by Turbulent Motion

2) Ensemble average for short times (EAST)

- follow each release separately during short times
- superpose the centers of mass of each of the individual clouds and then average over the ensemble of releases.
- The average extent of each individual cloud is smaller than the extent of the ensemble average (EALT) because the EALT includes the distribution of the centers of mass. $\rightarrow C_{II} > C_{I}$
- However, in applying this method, not enough is known about the turbulent flow to permit the computation of the spread of each individual cloud or the peak concentration at any instant.
- use a statistical estimate of the size of an individual cloud

3.3 Statistic Concepts

- Study various ways of computing averages, variances, and correlation coefficients of a random time series

3.3.1 Statistics of particle position

- Suppose that a particle in a turbulent fluid is, at time t_o, located at the point, $\vec{\xi}$, in Cartesian coordinates (ξ, η, ζ).
- At a subsequent time *t*, the particle moves to a new point, $\vec{X}\,(X, Y, Z)$
- The trajectory of the particular particle is given as

$$\vec{X} = \vec{X}(\vec{\xi}, t, t_0) \quad (3.15)$$

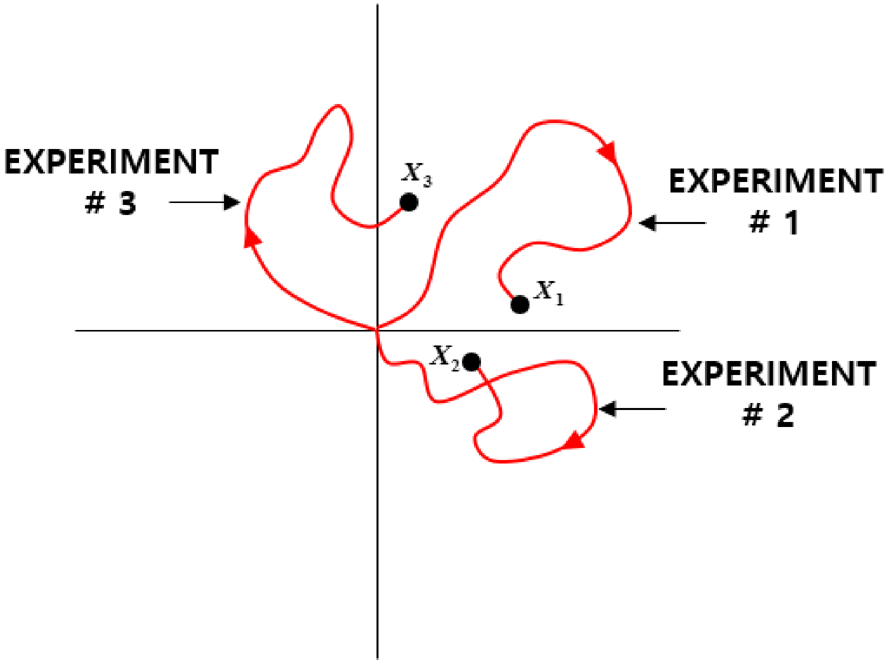
where

$$X = X(\xi, \eta, \zeta, t, t_0)$$

$$Y = Y(\xi, \eta, \zeta, t, t_0)$$

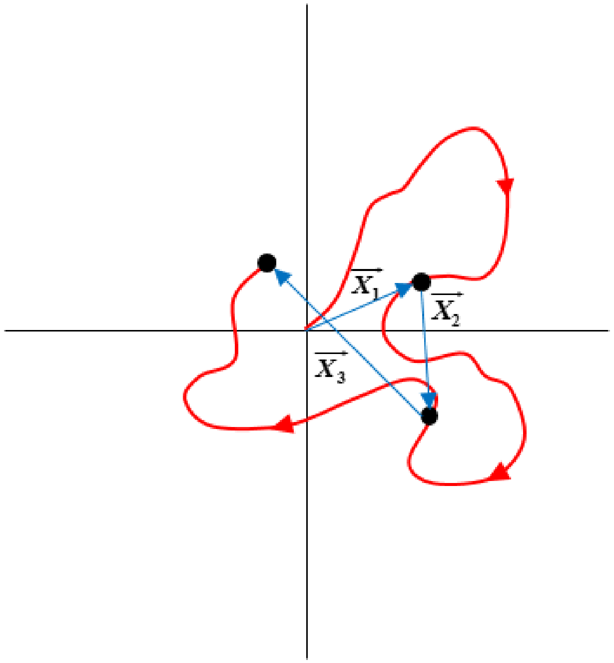
$$Z = Z(\xi, \eta, \zeta, t, t_0) \quad (3.16)$$

3.3 Statistic Concepts



(a)

Trajectories of three particles
for a time T



(b)

Trajectory of a single particle
for a time 3 T

3.3 Statistic Concepts

- Two interpretations of Eq. (3.15)

i) For each release of a particle, there will be a different functional form of $\vec{X}$ reflecting the random nature of turbulent flows. → Fig. 3.8(a)

ii) For any one release, $\vec{X}$ may be interpreted as a random variable in time since the position that a particle takes at some later time $t+\tau$ may bear little correlation to the position at time t. → Fig. 3.8(b)

3.3 Statistic Concepts

• Two averages

i) Ensemble average

~ mean over many trials (releases)

~ average across the totality of experiments performed

→ As shown in Fig. 3.8(a), a number of particles are released at different times, and the displacement $\vec{X}$ of each particle a time T after its release is observed.

ii) Time series average

As shown in Fig. 3.8(b), a single particle is released and followed through a large number of time increments each of duration T.

3.3 Statistic Concepts

- The displacement $\vec{X}$ during each time increment is a random variable.
- The statistical properties may differ from those obtained by an ensemble of many releases.
- If the time series average and the ensemble average are the same, this process satisfies the ergodic property.

3.3 Statistic Concepts

• Correlation

Because of the correlation of turbulent motion, the probability that the value of x coordinate of the random variable $\vec{X}$ is between X and $X + dX$ at time t will not be independent of where, the particle was at some previous instant, $t - \Delta t$.

• Joint probability

= Probability that the value will be between X_1 and $X_1 + dX_1$ at t_1 , between and X_2 at $X_2 + dX_2$, t_2 between X_3 and $X_3 + dX_3$ at t_3 , and so on.

→ Two turbulent mixing processes are identical if all the joint probabilities are the same.

→ This does not mean that any two trials will produce the same position time history.

3.3 Statistic Concepts

① Ensemble mean of particle displacement

~ average taken over a large number of trials

$$\langle \mathrm{X} \rangle = \int_{-\infty}^{\infty} X p(X \,|\, \xi, t, t_0) dX \qquad (3.17)$$

where $p(X \,|\, \xi, t, t_0) dX$ is the probability density function of the process X

= probability that the random variable X has a value between X and $X + dX$ at time t given that it was ξ at time t_0 .

3.3 Statistic Concepts

② *k*th moment

$$\langle X^k \rangle = \int_{-\infty}^{\infty} X^k p(X | \xi, t, t_0) dX \tag{3.18}$$

③ Auto-covariance

$$B_{XX}(\xi, t_1, t_2, t_0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X_1 X_2 p(X_1 X_2 | \xi, t_1, t_2, t_0) dX_1 dX_2 \tag{3.19}$$

3.3 Statistic Concepts

④ Cross-covariance

$$B_{XY}(\xi,t,t_0)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} XYp(XY\,|\,\xi,t,t_0)\,dX_1 dX_2 \qquad (3.20)$$

where $p(XY\,|\,\xi,t,t_0)$ is the joint probability density function for the X and Y components of particle displacement.

The particle displacement X is stationary if all the moments of the distribution of the displacement are independent of the time origin and depend only on the time difference $t-t_0$, and in addition the covariances depend only on the time difference.

The X is homogeneous if the moments of the distribution of the displacement depend only on the relative displacement $|X-\xi|$, and not the initial position ξ.

3.3 Statistic Concepts

3.3.2 Statistical distribution of concentration

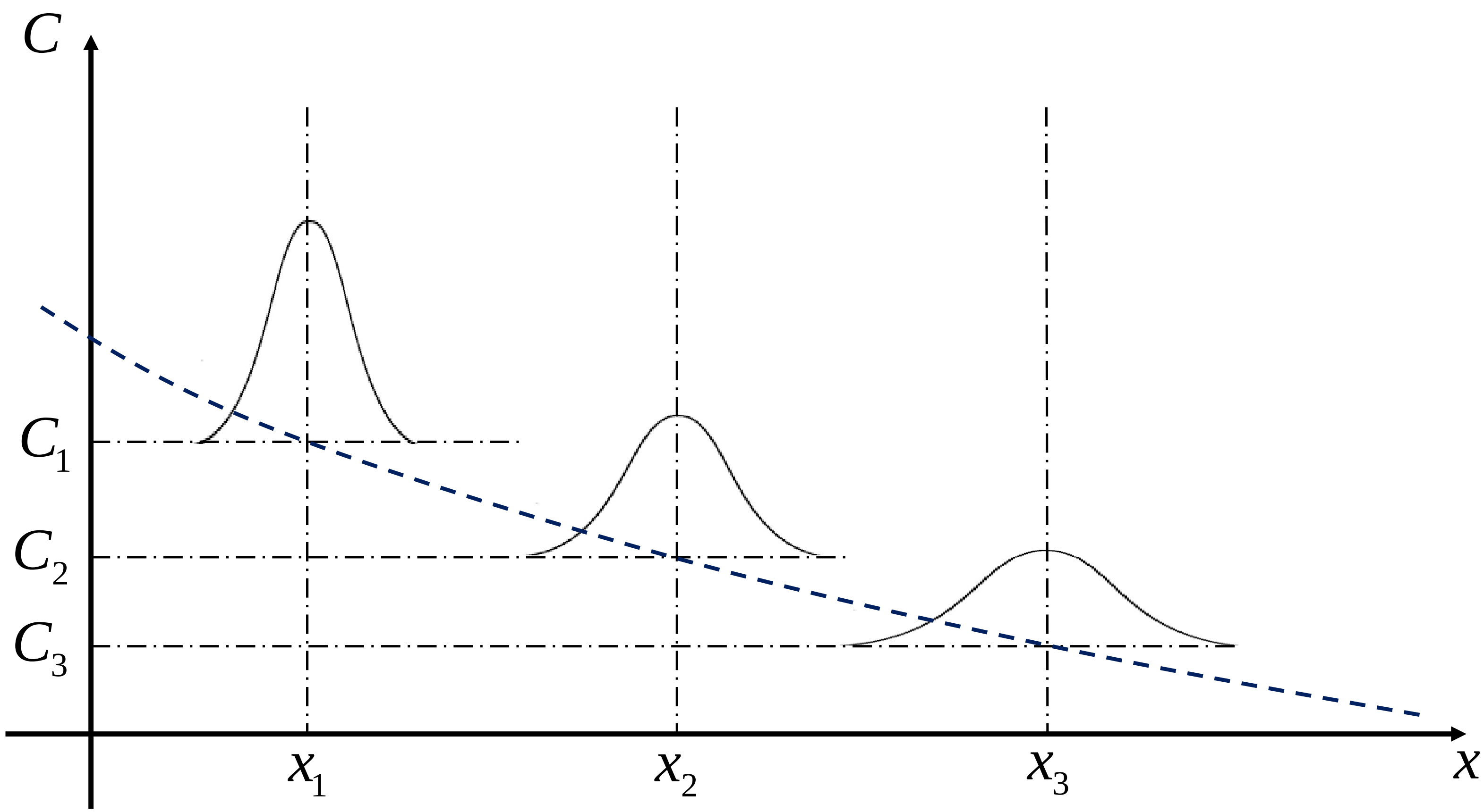
Ensemble average of the concentration $n(x,t)$ measured at point x at time t after many repeated trials in which identical cloud of particles are released under the same statistical conditions is given as:

$$C(x,t) = \langle n(x,t) \rangle = \int_0^\infty np(n|x,t)dn \qquad (3.21)$$

where $n(x,t) =$ concentration observed at point x and at time t

$p(n|x,t)dn$ = probability that the concentration of tracer material has a value between n and $n+dn$ at the point x at time t .

3.3 Statistic Concepts



3.3 Statistic Concepts

- ***Statistics of a single trial (cloud)***

① Centroid of cloud

$$\overline{X} = \frac{1}{M}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xn(x, y, z, t)dxdydz \qquad (3.22)$$

where M is the total mass (= number of particles) in the cloud

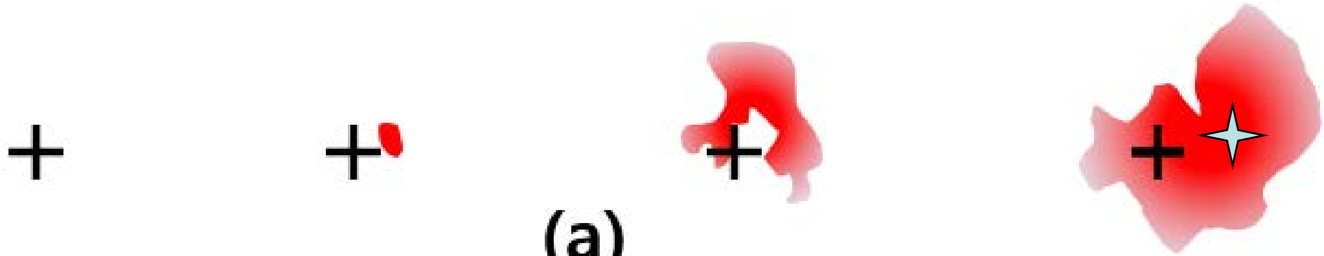
$$M = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} n(x, y, z, t)dxdydz \qquad (3.23)$$

② Variance of the cloud

~ the mean square x displacement about the center of mass (centroid) of particles in a single cloud.

$$\sigma_x^2 = \frac{1}{M}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - \overline{X})^2 n(x, y, z, t)dxdydz \qquad (3.24)$$

3.3 Statistic Concepts



(a)

(b)

3.3 Statistic Concepts

- ***Statistics of an ensemble clouds of many trials***

① <u>Center of mass</u> for <u>the ensemble</u>

$$\langle \overline{X} \rangle = \frac{1}{M} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x C(x, y, z, t) dx dy dz \qquad (3.25)$$

② Overall variance of the concentration in the ensemble of clouds

$$\Sigma_x^2 = \frac{1}{M} \iiint (x - \langle \overline{X} \rangle)^2 C(x, y, z, t) dx dy dz \qquad (3.26)$$

3.3 Statistic Concepts

Expand Eq. (3.24) and then take the ensemble average (EALT)

$$\boxed{\Sigma_x^2 = \langle \sigma_x^2 \rangle + \langle (\overline{X} - \langle \overline{X} \rangle)^2 \rangle} \qquad (3.27)$$

The variance of the ensemble distribution about its expected position is equal to the ensemble average of the variance of each cloud about its center of mass plus the ensemble mean square displacement of an individual cloud's center of mass from its expected position.

We can define the size of an individual cloud as

$$\ell(t) = \left[\frac{1}{3} \left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2 \right) \right]^{1/2} \qquad (3.28)$$

3.3 Statistic Concepts

The size of the ensemble-average cloud is

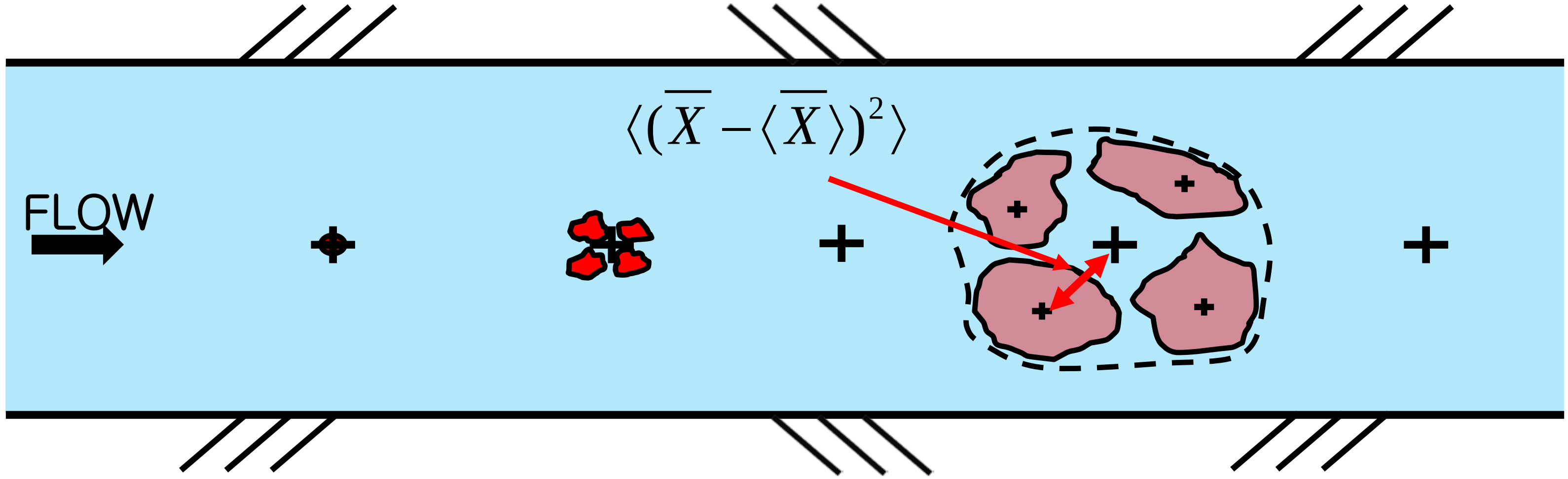
$$L(t) = \left[\frac{1}{3}\left(\Sigma_x^2 + \Sigma_y^2 + \Sigma_z^2\right)\right]^{\frac{1}{2}} \quad (3.29)$$

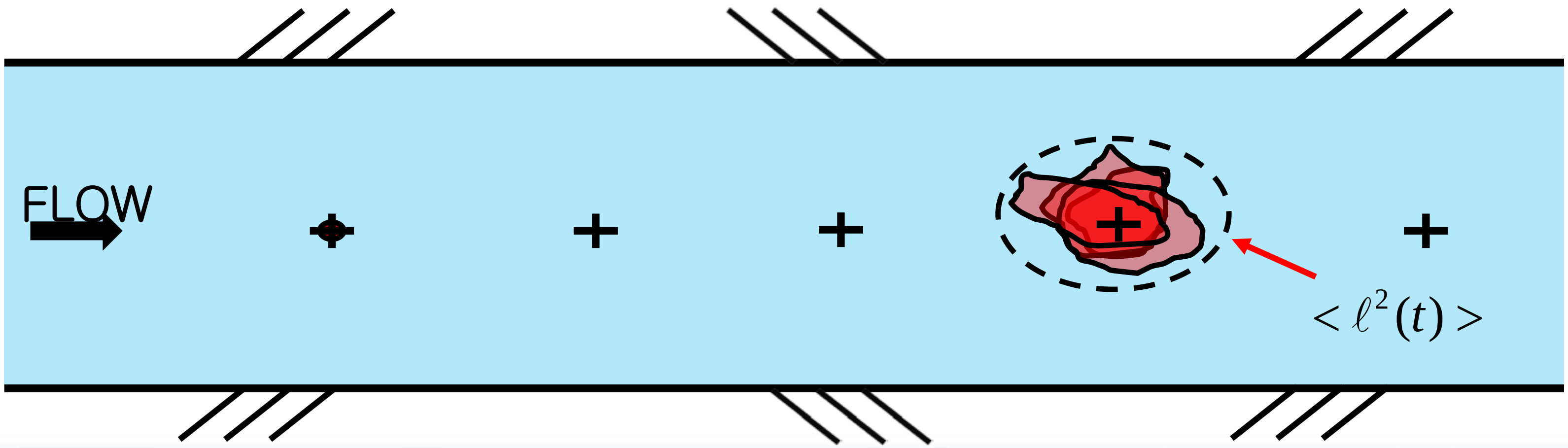
Combine Eqs. (3.27) ~ (3.29)

$$\boxed{L^2(t) = < \ell^2(t) > + \frac{1}{3}\left[\langle(\overline{X} - \langle\overline{X}\rangle)^2\rangle + \langle(\overline{Y} - \langle\overline{Y}\rangle)^2\rangle + \langle(\overline{Z} - \langle\overline{Z}\rangle)^2\rangle\right]} \quad (3.30)$$

→ The width of the ensemble mean concentration profile is larger than the average width of a cloud as indicated in Fig. 3.7 (c) and (d).

3.3 Statistic Concepts





3.4 Diffusion of the Ensemble Mean Concentration

3.4.1 Taylor's Theory (1921) of Turbulent Mixing

Assume a field of stationary homogeneous turbulence.

Consider a series of experiments with zero mean velocity where, in each experiment, mass *m* of matter is added to a single fluid particle at time to at point ξ.

→ $\overline{X}$ and $\langle \overline{X} \rangle$ are zero.

→ $$L^2(t) = \Sigma_x^2 = \frac{1}{M}\iiint x^2 C(x,y,z,t)\,dxdydz \qquad (3.31)$$

Overall variance

where number of particles $N = \dfrac{M}{m}$

3.4 Diffusion of the Ensemble Mean Concentration

• The effects of <u>molecular diffusion are neglected</u>.

→ The mass m allocated to each particle remains with that particle at all times.

→ The average concentration $C(x,t)$ measured at a fixed point in space will be proportional to the probability that the particle is at x .

$$C(x,t) = Mp(x|\xi,t,t_0) \tag{3.32}$$

Substituting Eq. (3.32) into Eq. (3.31) and using Eq. (3.18) gives

$$L^2(t) = \Sigma_x^2 = \iiint x^2 p(x|\xi,t,t_0)dxdydz$$

$$= \int_{-\infty}^{\infty} x^2 p(x|\xi,t,t_0)dxdydz = \langle x^2 \rangle \tag{3.33}$$

3.4 Diffusion of the Ensemble Mean Concentration

Eq. (3.33) holds if all particles begin their motion at the same point ξ, but it can also be generalized to the case in which we begin with a cloud of particles having a spatial distribution $C(\xi, t)$ at time t_0.

By superposition of the point source result Eq. (3.32), as discussed in Problem 1-3 in Chapter 2, we have

$$C(x,t) = \int_V C(\xi, t_0) p(x \mid \xi, t, t_0) d\xi \qquad (3.32a)$$

By substituting Eq. (3.32a) into (3.31), the more general result follows:

$$\boxed{L^2(t) = \langle x^2 \rangle + L^2(t_0)} \qquad (3.33a)$$

3.4 Diffusion of the Ensemble Mean Concentration

where $L^2(t_0)$ is the ensemble mean square size of the cloud at the initial time t_0.

→ Thus, the problem of finding the ensemble mean size of the cloud is equivalent to finding the ensemble mean square displacement of the fluid particles.

→ This problem was solved by Taylor in 1921.

Let U be the velocity of the particle, with zero mean.

Take $\xi = 0$ and $t_0 = 0$, then the location of the particle, *X(t)* is

$$X(t) = \int_0^t U dt$$

$$X^2(t) = \left(\int_0^t U d\tau_1 \right) \left(\int_0^t U d\tau_2 \right) = \int_0^t \int_0^t U(\tau_1) U(\tau_2) d\tau_1 d\tau_2 \qquad (3.34)$$

3.4 Diffusion of the Ensemble Mean Concentration

The ensemble mean is

$$\langle X^2 \rangle = \int_0^t \int_0^t \langle U(\tau_1) U(\tau_2) \rangle d\tau_1 d\tau_2 \tag{3.35}$$

The ensemble average $\langle U(\tau_1)U(\tau_2)\rangle$ means the average over a large number of trials of the product of the velocity of a single particle at time τ_1 multiplied by the velocity of the same particle at time τ_2.

→ Since the turbulence is stationary this can only be a function of the difference between τ_1 and τ_2 .

3.4 Diffusion of the Ensemble Mean Concentration

Define a correlation coefficient as

$$\boxed{R_x(\tau_2 - \tau_1) = \frac{\langle U(\tau_1)U(\tau_2)\rangle}{\langle U^2\rangle}} \tag{3.36}$$

where $\langle U^2\rangle = \langle U(0)U(0)\rangle =$ square of the <u>turbulence intensity</u>.

R_x is called the <u>Lagrangian autocorrelation function</u>.

Substitute Eq. (3.36) into Eq. (3.35)

$$\langle X^2(t)\rangle = \langle U^2\rangle \int_0^t \int_0^t R_x(\tau_2 - \tau_1)\, d\tau_2\, d\tau_1 \tag{3.37}$$

3.4 Diffusion of the Ensemble Mean Concentration

Changing the variables of integration to

$$s = \tau_2 - \tau_1$$

$$\tau = (\tau_1 + \tau_2)/2$$

Performing the integration with respect to τ gives

$$\boxed{\langle X^2(t)\rangle = 2\langle U^2\rangle \int_0^t (t-s) R_x(s)\,ds} \qquad (3.38)$$

For the case of diffusion in three dimensional flow

$$\langle Y^2(t)\rangle = 2\langle V^2\rangle \int_0^t (t-s) R_y(s)\,ds$$

$$\langle Z^2(t)\rangle = 2\langle W^2\rangle \int_0^t (t-s) R_z(s)\,ds$$

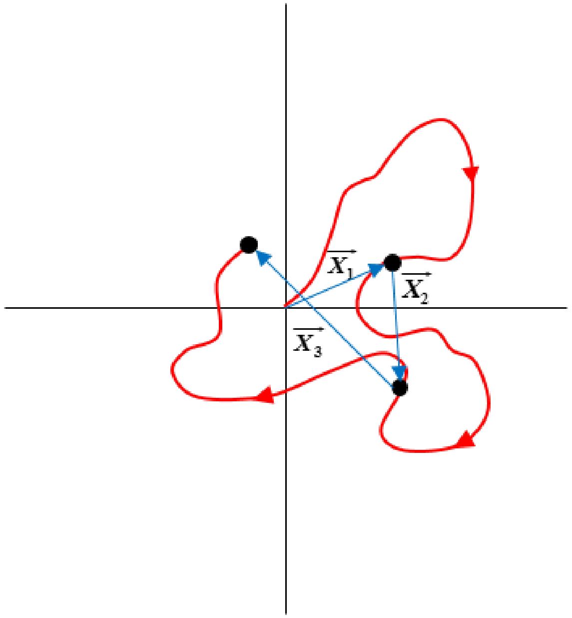
3.4 Diffusion of the Ensemble Mean Concentration

① For very short times after release of the particles, $(t << T_x)$

→ The particle velocity is nearby constant and $R_x \approx 1$.

Then, Eq. (3.38) is simplified as

$$\langle X^2 \rangle = \langle U^2 \rangle t^2 \qquad (3.39)$$



② For very long times $(t >> T_x)$

$\rightarrow R_x \rightarrow 0$ because motions become less and less correlated at longer and longer times.

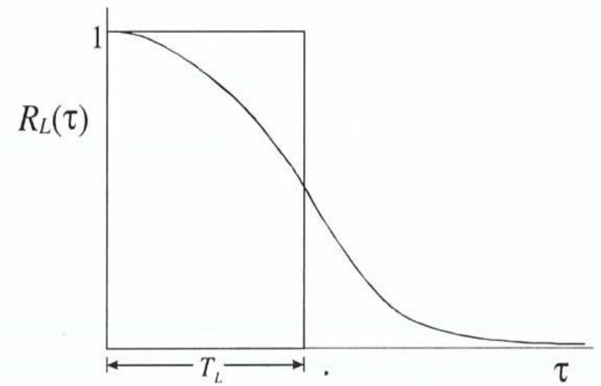
$$\boxed{\langle X^2 \rangle \rightarrow 2\langle U^2 \rangle T_x t + const} \qquad (3.40)$$

3.4 Diffusion of the Ensemble Mean Concentration

where

$$\boxed{T_x = \int_0^\infty R_x(s)ds} \quad (3.41)$$

T_x is known as the Lagrangian time scale which is a measure of how long the particle takes to lose memory of its initial velocity



3.4 Diffusion of the Ensemble Mean Concentration

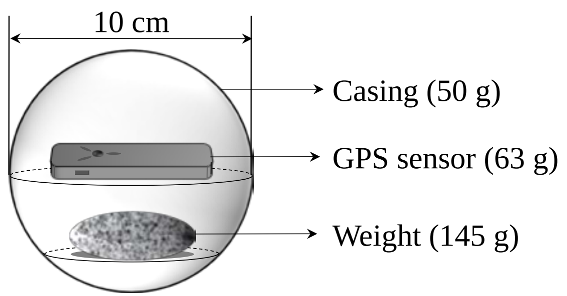
Differentiating Eq. (3.40) gives

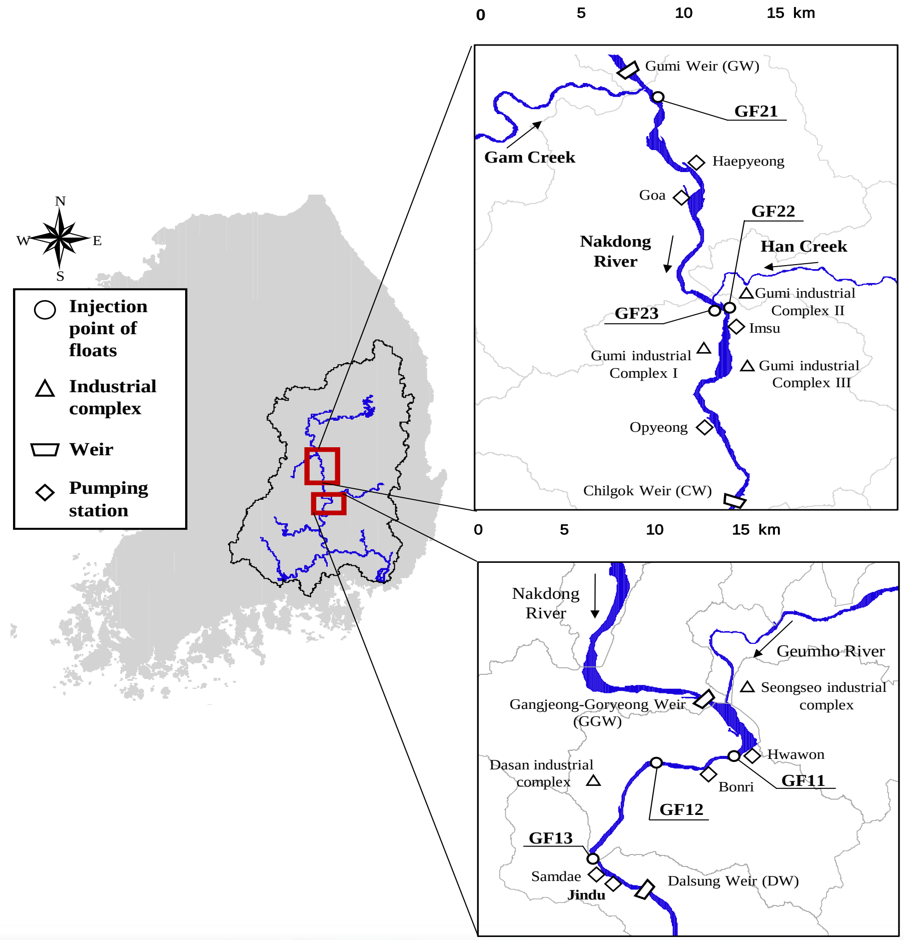
$$\boxed{\frac{d}{dt}\langle X^2\rangle = 2\langle U^2\rangle T_x} \tag{3.42}$$

→ After some long enough time, the variance of the ensemble averaged concentration distribution for clouds dispersing in a stationary homogeneous field of turbulence grows linearly with time. → Fickian diffusion

3.4 Diffusion of the Ensemble Mean Concentration

[Case study] Field study using GPS floaters in Nakdong River

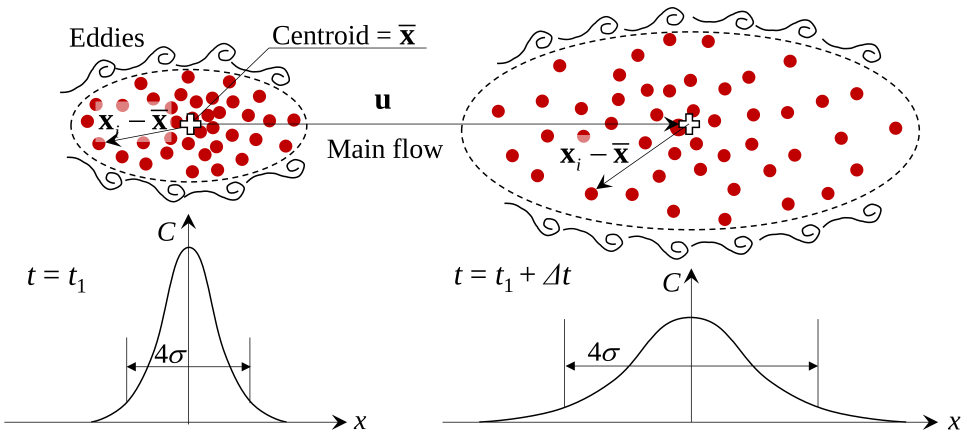


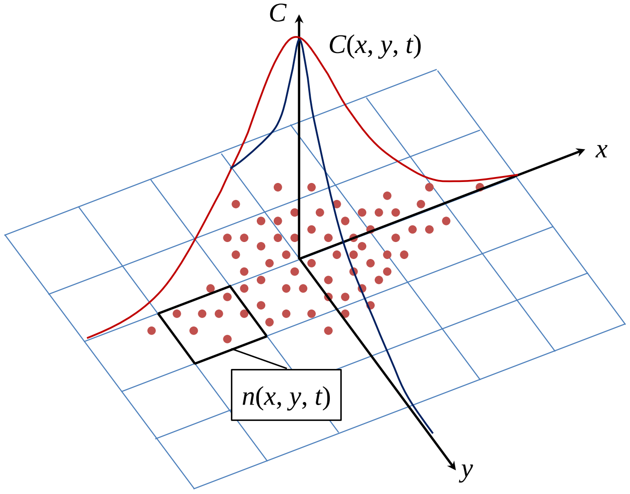


3.4 Diffusion of the Ensemble Mean Concentration

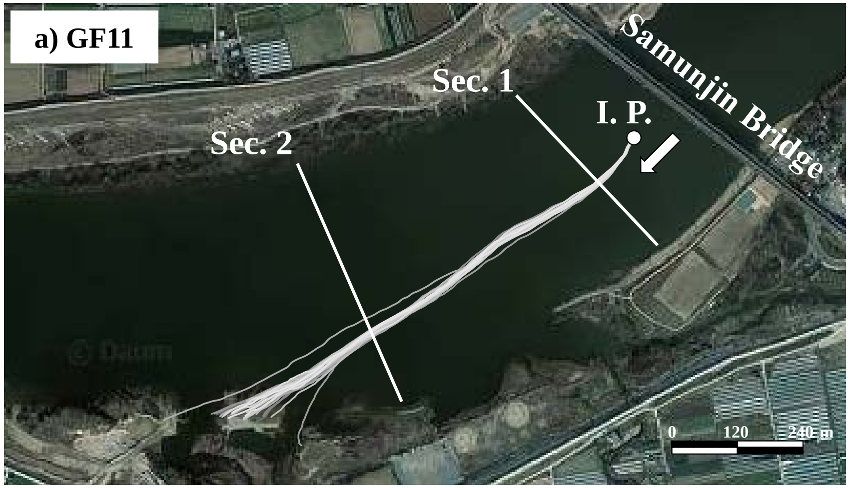
Case	Date	Q (m^3/s)	U (m/s)	H (m)	W (m)	Wind speed (m/s) (Direction)
GF11	Sep. 12, 2012	547	0.19	6.0	383	0.15 ~ 0.22 (SW)
GF12	Sep. 15, 2013	681	0.21	7.1	308	0.50 ~ 2.00 (NE)
GF13	Sep. 15, 2013	697	0.22	8.5	360	0.50 ~ 2.00 (NW)
GF21	Oct. 11, 2013	169	0.29	3.4	263	0.16 ~ 1.25 (NW)
GF22	Oct. 12, 2013	352	0.32	5.6	480	0.82 ~ 1.90 (NW)
GF23	Oct. 12, 2013	352	0.32	5.6	480	0.82 ~ 1.90 (NW)

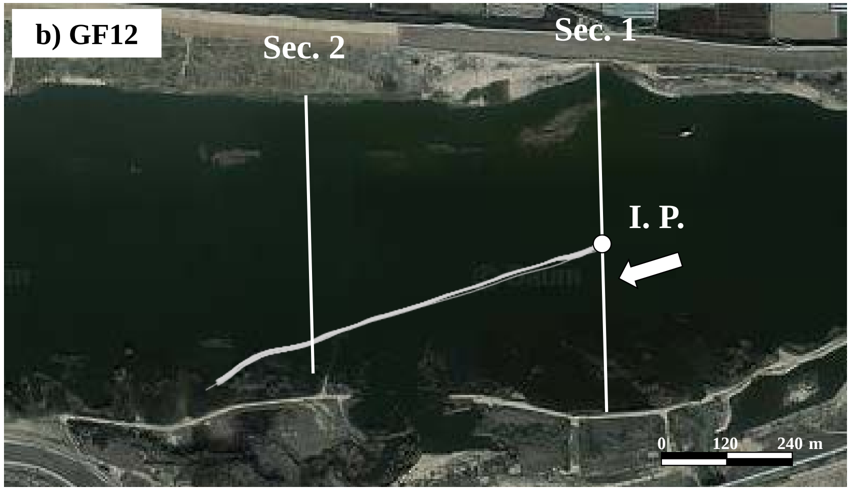
3.4 Diffusion of the Ensemble Mean Concentration

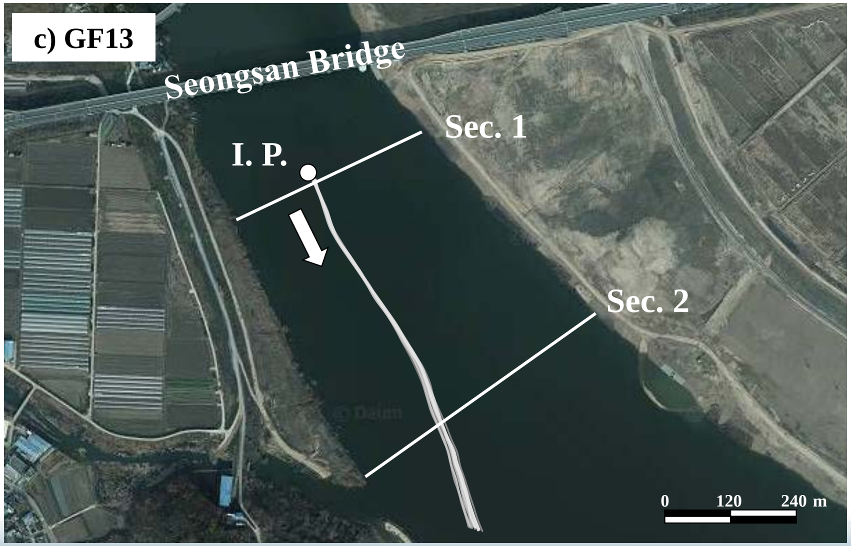




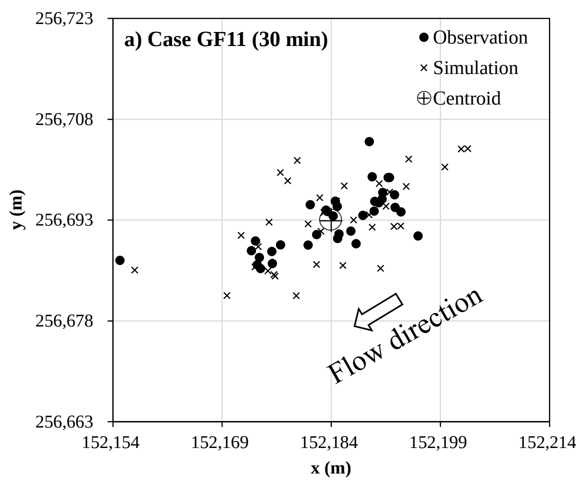
3.4 Diffusion of the Ensemble Mean Concentration

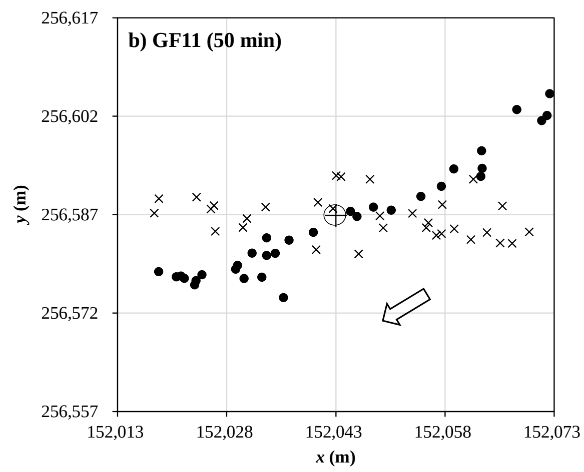


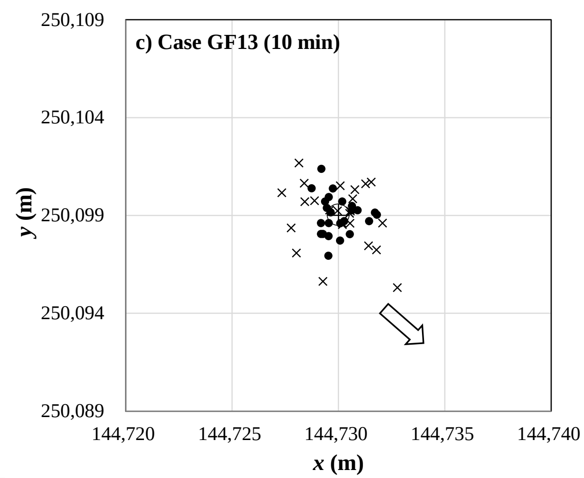


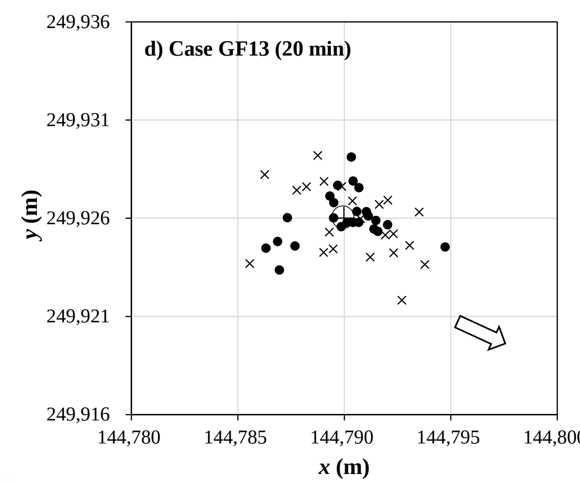


3.4 Diffusion of the Ensemble Mean Concentration

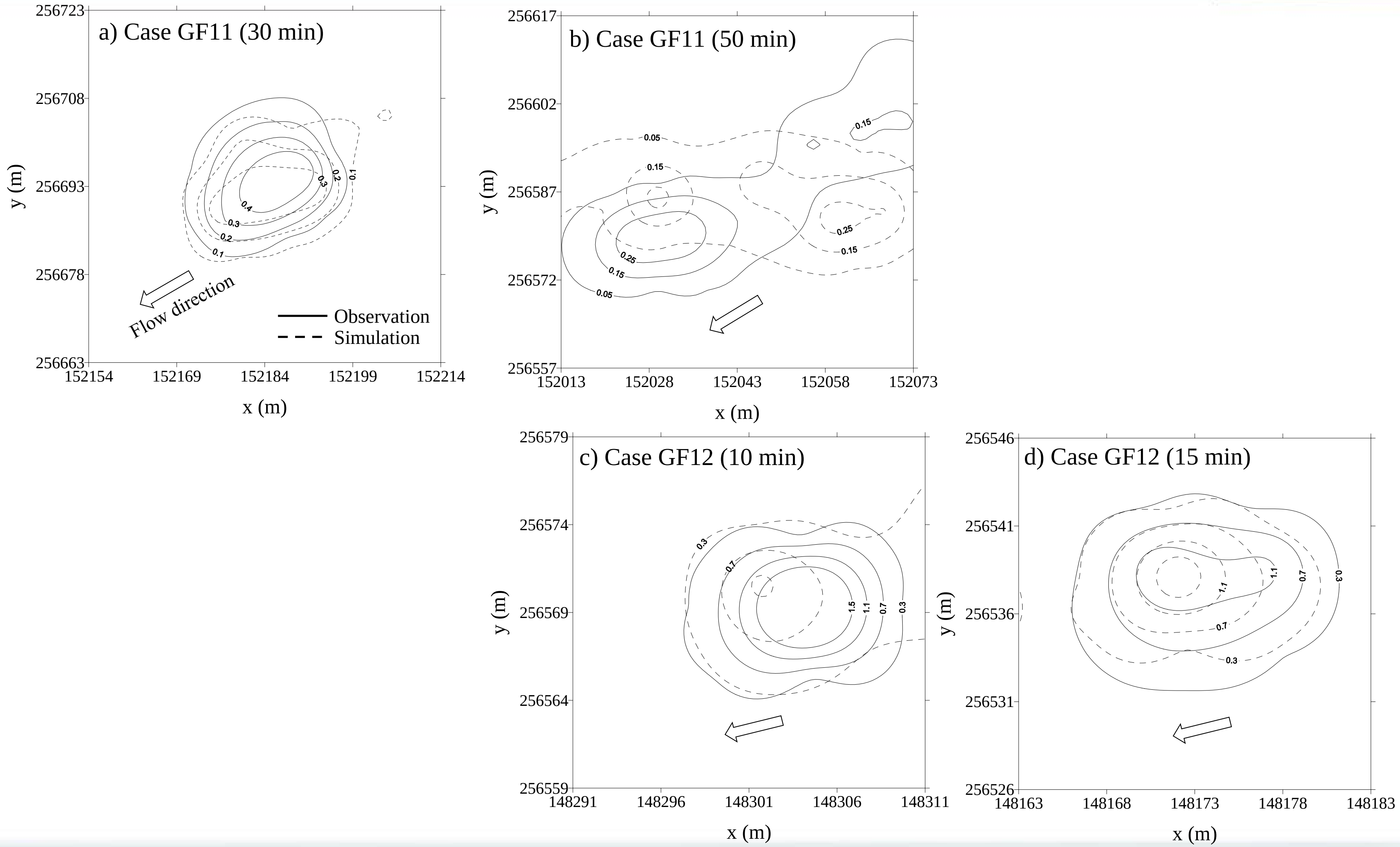




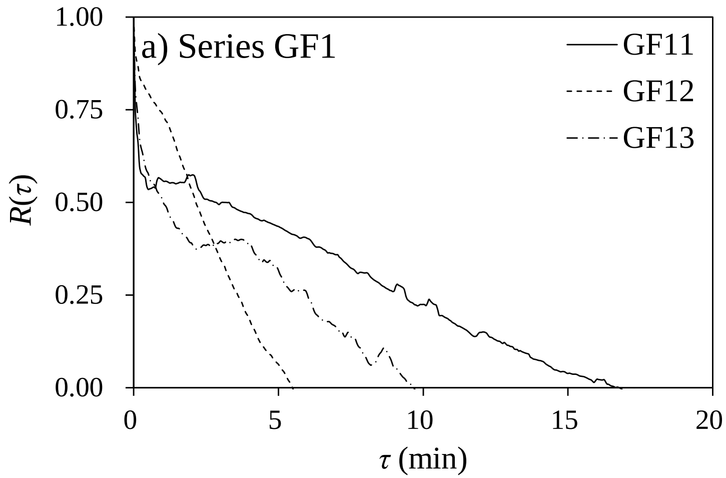


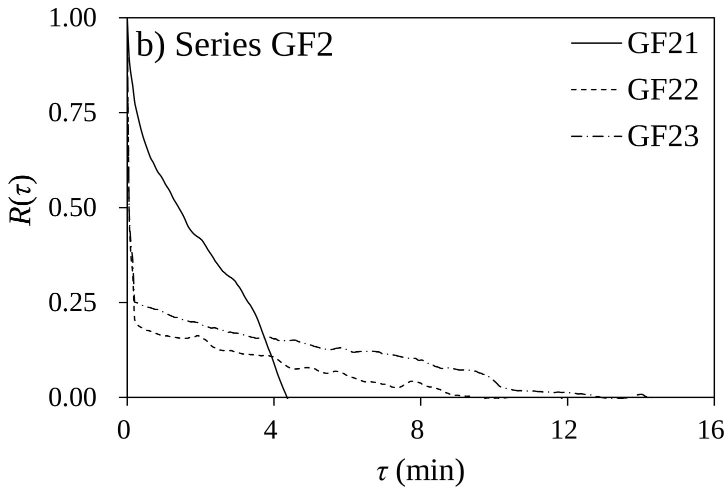


3.4 Diffusion of the Ensemble Mean Concentration



3.4 Diffusion of the Ensemble Mean Concentration





3.4 Diffusion of the Ensemble Mean Concentration

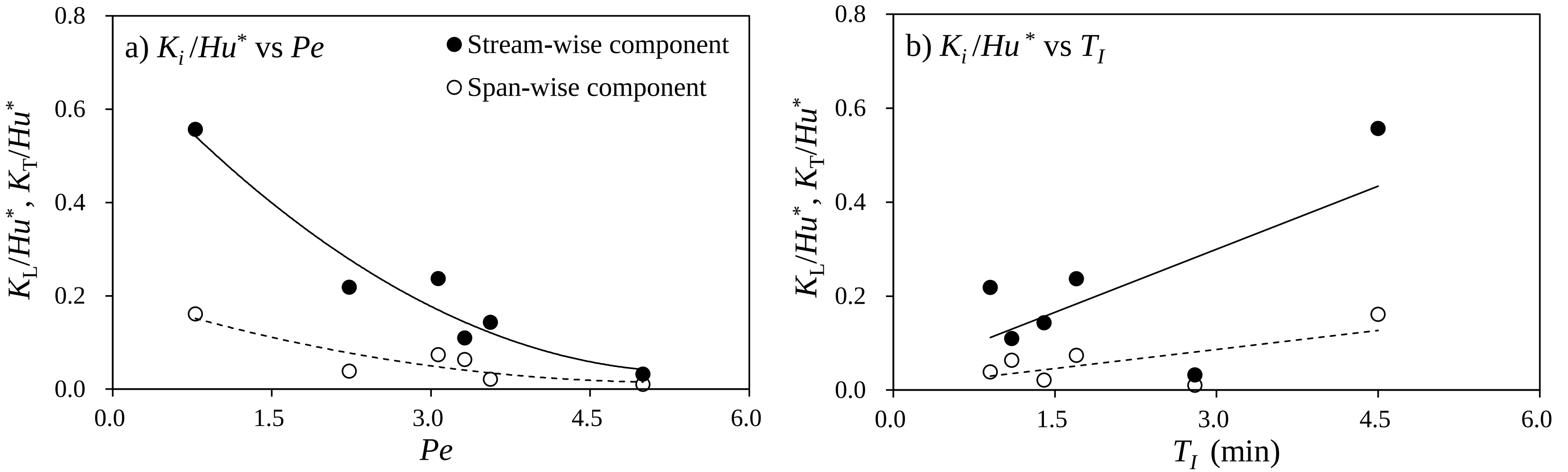
$$K_i = \left\langle u_i'^2 (t) \right\rangle T_I = \frac{1}{2}\frac{d\sigma_i^2}{dt}$$

$$\sigma_i^2 = \frac{\sum_{i=1}^{N} \left(\mathbf{x}_i - \overline{\mathbf{x}}\right)^2}{N}$$

$$\overline{\mathbf{x}} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{x}_i$$

Case	T_I (min)	K_L (m²/s)	K_T (m²/s)	K_L / Hu^*	K_T / Hu^*	K_L/K_T	Pèclet number
GF11	4.5	0.041	0.012	0.56	0.16	3.46	0.78
GF12	1.4	0.012	0.002	0.14	0.02	6.75	3.56
GF13	2.8	0.003	0.001	0.03	0.01	3.14	5.00
GF21	1.7	0.016	0.005	0.24	0.07	3.20	3.07
GF22	0.9	0.021	0.004	0.22	0.04	5.68	2.23
GF23	1.1	0.011	0.006	0.11	0.06	1.73	3.32

3.4 Diffusion of the Ensemble Mean Concentration



3.4 Diffusion of the Ensemble Mean Concentration

3.4.2 Derivation of diffusion equation for turbulent flows

Recall molecular diffusion

→ Fick's law led to the derivation of the diffusion equation.

→ The fundamental solution to the diffusion equation is the Gaussian distribution.

→ A property of the diffusion equation is that the variance of a concentration distribution always grows linearly with time.

$$\boxed{\frac{d\sigma^2}{dt} = 2D} \qquad (2.22)$$

3.4 Diffusion of the Ensemble Mean Concentration

After some start-up time, the random walk generates the Gaussian distribution.

→ The Gaussian distribution implies that the diffusion equation describes the process, and that the variance of a spreading cloud of molecules undergoing a random walk grows linearly with time.

• Taylor's analysis, Eq. (3.42), has shown that after some start-up time the variance of a spreading cloud of particles in stationary homogeneous turbulent motion grows linearly with time.

→ We can define a turbulent mixing coefficient, analogous to the molecular diffusion coefficient, by the relationship

3.4 Diffusion of the Ensemble Mean Concentration

$$\boxed{\varepsilon_x = \frac{1}{2}\frac{d}{dt}\langle X^2 \rangle = \langle U^2 \rangle T_x} \qquad (3.43)$$

However, linear growth of the variance is a necessary condition for the diffusion equation to apply, but it is not a sufficient one.

The velocity $U(t)$ is a random variable for any fixed time *t*, so that $X(t) = \int_0^t U(t)dt$ is the sum or integral of random variables.

3.4 Diffusion of the Ensemble Mean Concentration

The central limit theorem of probability theory tells us that such sums approach normality as $t \to \infty$ provided that the variable $U(t)$ satisfies certain weak independence requirements.

→ Since a stationary homogeneous turbulent velocity field satisfies these requirements we may expect that $X(t)$ becomes a normal or Gaussian random variable for large time.

→ If so, as provided in Section 2.3, the spread of the ensemble mean concentration may be described by a diffusion equation.

→ The diffusion equation for zero mean flow velocity is

$$\frac{\partial C}{\partial t} = \varepsilon_x \frac{\partial^2 C}{\partial x^2} + \varepsilon_y \frac{\partial^2 C}{\partial y^2} + \varepsilon_z \frac{\partial^2 C}{\partial z^2} \qquad (3.44)$$

3.4 Diffusion of the Ensemble Mean Concentration

→ If the fluid has a mean velocity,

$$\boxed{\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=\varepsilon_x\frac{\partial^2 C}{\partial x^2}+\varepsilon_y\frac{\partial^2 C}{\partial y^2}+\varepsilon_z\frac{\partial^2 C}{\partial z^2}} \quad (3.44a)$$

☞ **Derivation by time-averaging the advection-diffusion equation:**

For turbulent flow, the advection-diffusion equation which was derived from conservation of matter is given as

$$\frac{\partial c}{\partial t}+U\frac{\partial c}{\partial x}+V\frac{\partial c}{\partial y}+W\frac{\partial c}{\partial z}=D\frac{\partial^2 c}{\partial x^2}+D\frac{\partial^2 c}{\partial y^2}+D\frac{\partial^2 c}{\partial z^2} \quad (3.45)$$

3.4 Diffusion of the Ensemble Mean Concentration

Now, neglecting molecular diffusion gives

$$\frac{\partial c}{\partial t}+U\frac{\partial c}{\partial x}+V\frac{\partial c}{\partial y}+W\frac{\partial c}{\partial z}=0 \quad (3.45a)$$

where U , V, and W are randomly varying velocities; c is the time-varying point concentration in turbulent flow.

If this equation is averaged over a time long enough to average the turbulent velocity and concentration fluctuations, we obtain

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=-\frac{\partial}{\partial x}(\overline{U'C'})-\frac{\partial}{\partial y}(\overline{V'C'})-\frac{\partial}{\partial z}(\overline{W'C'}) \quad (3.46)$$

3.4 Diffusion of the Ensemble Mean Concentration

where $U = u + U'$

$$V = v + V'$$

$$W = w + W'$$

$$c = C + C'$$

Comparing Eq. (3.46) with Eq. (3.44) gives

$$\overline{U'C'} = -\varepsilon_x \frac{\partial \overline{C}}{\partial x}$$

$$\overline{V'C'} = -\varepsilon_y \frac{\partial \overline{C}}{\partial y}$$

$$\overline{W'C'} = -\varepsilon_z \frac{\partial \overline{C}}{\partial z} \qquad (3.47)$$

3.4 Diffusion of the Ensemble Mean Concentration

where $\varepsilon_x, \varepsilon_y, \varepsilon_z$ are referred to as "Fickian turbulent diffusion coefficient"; since they are result from a process involving larger scale random motions they are often called "eddy diffusivities." Then Eq. (3.46) becomes

$$\boxed{\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=\frac{\partial}{\partial x}\left(\varepsilon_x\frac{\partial C}{\partial x}\right)+\frac{\partial}{\partial y}\left(\varepsilon_y\frac{\partial C}{\partial y}\right)+\frac{\partial}{\partial z}\left(\varepsilon_z\frac{\partial C}{\partial z}\right)} \quad (3.48)$$

→ This is the same as Eq. (3.44a).

3.4 Diffusion of the Ensemble Mean Concentration

3.4.3 Requirements of Taylor's analysis

We must now resolve the question of under what conditions a Fickian turbulent diffusion equation like Eq. (3.48) can be used.

① Requirement in terms of time

If we are dealing with a cloud of particles originating at a point, requirement given by Taylor's analysis is that more time has elapsed than the Lagrangian time scale.

$$t > T_L = \frac{1}{3}(T_x + T_y + T_z)$$

3.4 Diffusion of the Ensemble Mean Concentration

② Requirement in terms of the size of the dispersing cloud

In many cases, the time of origin of the cloud as a point source may not be known. Then, use a Lagrangian length scale, ℓ_L , defined by the relation

$$\ell_L^{\,2} = \langle U^2 \rangle T_L^2 \qquad (3.49)$$

The scale ℓ_L gives the order of magnitude of the distance a fluid particle will travel before losing memory of its initial velocity.

The size of the dispersing cloud, L, should substantially exceed the distance over which turbulent motions are correlated, which is given using Eq. (3.40)

$$L^2 \cong 2\langle U^2 \rangle T_L t > 2\ell_L^2 \qquad (3.50)$$

3.4 Diffusion of the Ensemble Mean Concentration

■ Summary

- Taylor's analysis has shown that there is such a thing as a turbulent mixing coefficient that is analogous to the molecular diffusion coefficient.

- Taylor's analysis shows, by combining Eqs. (3.43) and (3.49) that

$$\boxed{\varepsilon_x = \ell_L [\langle U^2 \rangle]^{1/2}} \qquad (3.54)$$

where $[\langle U^2 \rangle]^{1/2} =$ the intensity of the turbulence; $\ell_L =$ Lagrangian length scale which is a measure of how for a particle travels before it forgets its initial velocity.

3.5 Relative Diffusion of Clouds

3.5.1 Growth of clouds of particles for $t < T_L$

• Two ways of forming ensemble average concentrations

① Ensemble average of random clouds

~ average the concentration at each point in space at identical times

$$C(x,t) = \langle n(x,t) \rangle \qquad (3.55)$$

where $n(x,t)$ is the concentration in one trial at point and time x .

The measure of the spread of the concentration distribution is given as t

$$\Sigma_x^2 = \frac{1}{M} \iiint \left(x - \langle \overline{X} \rangle \right)^2 C(x,t) dx \qquad (3.56)$$

3.5 Relative Diffusion of Clouds

where

$$\langle \overline{X} \rangle = \frac{1}{M} \iiint xC(x,t)dx \tag{3.57}$$

From the Taylor's analysis, the size of the cloud changes with time.

$$L^2(t) = \frac{1}{3}(\Sigma_x^2 + \Sigma_y^2 + \Sigma_z^2) \tag{3.58}$$

② Ensemble average of superposed clouds

The ensemble mean concentration is formed by averaging the concentration at points equidistant from the center of mass of each cloud in the trial.

The rate of growth of the ensemble of clouds with superposed centroid is

3.5 Relative Diffusion of Clouds

$$\langle \sigma^2 \rangle = \left\langle \frac{1}{M} \iiint (x - \overline{X})^2 n(x,t) dxdydz \right\rangle \quad (3.59)$$

where

$$\overline{X} = \frac{1}{M} \iiint x n(x,t) dxdydz \quad (3.60a)$$

$$\langle \overline{X} \rangle = \left\langle \frac{1}{M} \iiint x n(x,t) dxdydz \right\rangle \quad (3.60)$$

The size of an ensemble cloud is defined as

$$\langle \ell^2(t) \rangle = \frac{1}{3} \langle \sigma_x^2 + \sigma_y^2 + \sigma_z^2 \rangle \quad (3.61)$$

3.5 Relative Diffusion of Clouds

The ensemble concentration formed by aligning the centers of mass is

$$\psi(x,t) = \langle n(x-\overline{X},t)\rangle \tag{3.62}$$

where

$$x = x - \overline{X} \tag{3.63}$$

By translating the origin $x=0$ Eq. (3.63) becomes

$$\langle \sigma^2 \rangle = \frac{1}{M}\iiint x_x^2 \psi(x,t)\,dx_x dydz \tag{3.64}$$

The mean concentrations by Eq. (3.62) are actually higher than those described by Eq. (3.55).

$$L^2(t) = \langle \ell^2(t)\rangle + \frac{1}{3}\langle[(\overline{X}-\langle\overline{X}\rangle)^2 + (\overline{Y}-\langle\overline{Y}\rangle)^2 + (\overline{Z}-\langle\overline{Z}\rangle)^2]\rangle \tag{3.65}$$

3.5 Relative Diffusion of Clouds

3.5.2 Batchelor's analysis

Consider the statistics of the separation of a particular pair of particles in a cloud. We define the probability density function

$Q(s,t;s_0,t_0)ds =$ probability that a pair of particles separated by a distance between s_0 and $s_0 + ds_0$ at time t_0 will be separated by a distance s and $s + ds$ at time t .

• Batchelor's analysis (1952)

If the pair of particles are not widely separated compared to the scale of the turbulent eddying motion, then only two length scales are important to the statistics of their relative motion.

3.5 Relative Diffusion of Clouds

Two scales are their <u>initial separation s_0</u> and <u>Kolmogorov scale</u> $\left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$.

Two time scales are important to the motion: scale $t-t_0$ and Kolmogorov scale $\left(\frac{\nu}{\varepsilon}\right)^{1/2}$.

Dimensional analysis implied that

$$\frac{d\langle s^2\rangle}{dt} = \varepsilon\tau^2 f\left(\frac{s_0}{\varepsilon^{1/2}\tau^{3/2}}, \frac{\tau\varepsilon^{1/2}}{\nu^{1/2}}\right) \qquad (3.66)$$

where $\tau = t - t_0$

$$\langle s^2\rangle = \int_{-\infty}^{\infty} s^2 Q(s,t;s_{0,}t_0)ds \qquad (3.67)$$

3.5 Relative Diffusion of Clouds

① For very long times from release

~ <u>two particles will wander independently</u>

$$\langle s^2 \rangle \to 2\langle X^2 \rangle \tag{3.68}$$

$$\psi(x,t) \to C(x,t) \tag{3.69}$$

② For very short times

~ the rate of increase is a function of viscosity and the initial separation.

If s is much larger than the Kolomogorov scale $\nu^{3/4}\varepsilon^{-1/4}$, the viscosity is not important.

If enough time has passed that the initial separation of the particles s_0 has been forgotten, Eq. (3.66) becomes

3.5 Relative Diffusion of Clouds

$$\frac{d}{dt}\langle s^2 \rangle \sim C_2 \varepsilon (\tau - t_1)^2 \tag{3.70}$$

where t_1 is proportional to $s_0^{2/3}\varepsilon^{-1/3}$ and C_2 is a universal constant.

Integrating Eq. (3.70) gives $\langle s^2 \rangle \sim t^3$ so that we have

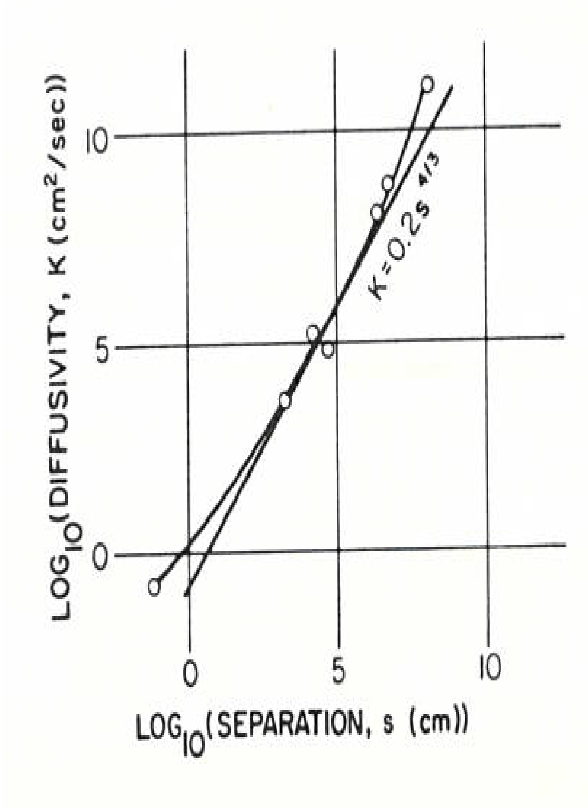
$$\frac{d}{dt}\langle s^2 \rangle \sim \varepsilon^{1/3}[\langle s^2 \rangle]^{2/3} \tag{3.71}$$

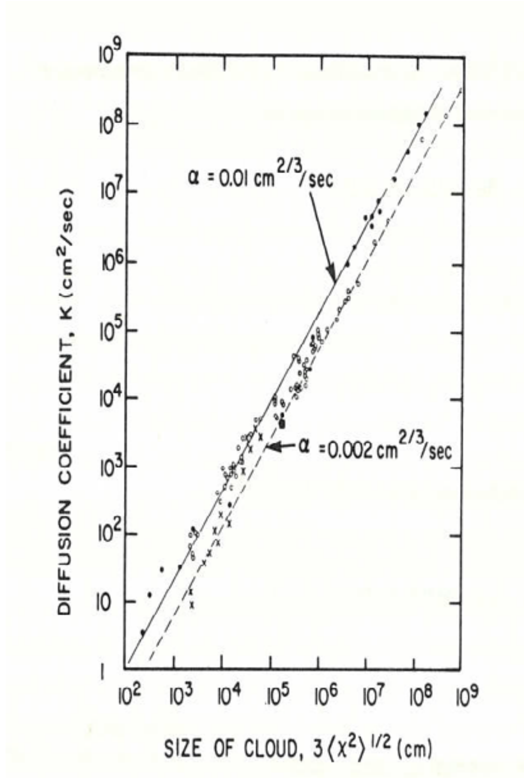
→ The rate of increase of the mean square separation of particles is proportional to the <u>mean square separation to the power 2/3</u>.

→ Experimental results collected by Richardson (1926) showed that

$$\boxed{\frac{d\langle s^2 \rangle}{dt} \propto s^{4/3}} \tag{3.72}$$

3.5 Relative Diffusion of Clouds





3.5 Relative Diffusion of Clouds

3.5.3 The 4/3 law of Diffusion

If the initial separation of pairs of particles in a cloud has been forgotten, and the turbulence is homogeneous, the description of the mean square separating of all pairs of point in an ensemble will be exactly the same as the description of the mean square displacement from the center of mass. If we let $\chi = |\chi|$ then we have that

$$\langle \chi^2 \rangle = \frac{1}{M} \iiint \chi^2 \psi(x,t) d\chi_x d\chi_y d\chi_z = 3(\ell^2(t)) \qquad (3.73)$$

$$\frac{d\langle \ell^2(t) \rangle}{dt} = k\varepsilon^{1/3} \langle \ell^2(t) \rangle^{2/3} \qquad (3.74)$$

3.5 Relative Diffusion of Clouds

From this result we postulate the existence of a differential equation describing $\psi(x,t)$ of the form

$$\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial \chi_x}\left(K\frac{\partial \psi}{\chi_x}\right) + \frac{\partial}{\partial \chi_y}\left(K\frac{\partial \psi}{\chi_y}\right) + \frac{\partial}{\partial \chi_z}\left(K\frac{\partial \psi}{\chi_z}\right) \qquad (3.75)$$

where

$$K = \alpha \langle \chi^2 \rangle^{2/3} \qquad (3.76)$$

$$K = \frac{1}{2}\frac{d\langle \chi^2 \rangle}{dt} = \alpha \langle \chi^2 \rangle^{2/3} \qquad (3.77)$$

3.5 Relative Diffusion of Clouds

The solution of Eq. (3.75) corresponding to a point source at the origin $|\chi| = 0$ is

$$\psi(\chi,t) = \frac{1}{\left[2\pi\langle\chi^2\rangle\right]^{n/2}} \exp\left(-\frac{\chi^2}{2\langle\chi^2\rangle}\right) \qquad (3.78)$$

With $n = 3$ and

$$\langle\chi^2\rangle = \left(\frac{2}{3}\alpha t\right)^3 \qquad (3.79)$$

This solution should only apply where

$$t >> \langle\chi_0^2\rangle^{2/3}\varepsilon^{-1/3} \qquad (3.80)$$

3.5 Relative Diffusion of Clouds

which implied that the original cloud size has been forgotten.

Experimental results of 2D field diffusion studies collected by Okubo (1974) showed that the 4/3 laws of Eq. (3.77) is valid over a much larger scale than the Batchelor-Kolmogorov theory would indicate as appropriate.

The constant α decreases with an increase in the scale of the diffusion.

→ However, unless the scale of the problem is very large a reasonable estimate for α for engineering purpose is given by taking the universal constant to have a value in the range of $0.002 \sim 0.01 cm^{2/3}/\sec$, as shown by Fig. 3.11.

3.6 Summary

- This chapter introduces the kinematics of turbulence which affects the mixing of pollutants

- It discusses the ranges of scales of motion in turbulent flow, and how they interact with the size of a dispersing cloud of tracer

• **Three concentrations**

① $n(x,t)$

~ concentration observed at point x at time t after the release of a single cloud of particles from point ξ

3.6 Summary

② $\psi(x,t)$

~ concentration computed at point x at time t by releasing a large number of clouds of particles at point ξ, superposing the centers of mass of each cloud, and then averaging the values of n over all clouds. x is measured relative to the center of mass of the cloud and t is measured relative to the time of release.

③ $C(x,t)$

~ the ensemble average obtained by releasing a large number of clouds particles at point ξ at various times and averaging the values of n observed at point x for all clouds at time t after their release.

3.6 Summary

- **What to use?**

From the point of view of pollution control, $n(x,t)$ is the most relevant concentration because it is what is actually seen by an organism in water.

→ Unfortunately we have not been able to give a general method for predicting values of $n(x,t)$.

→ We have seen, however, that spreading of both $\psi(x,t)$ and $C(x,t)$ can be modeled by Fickian diffusion equations.

3.6 Summary

- **Their uses and limitations**

① $\psi(x,t)$

The average concentration we called ψ is <u>not an observable concentration</u> because it requires an averaging by superposition of center of mass that never actually occurs in the environment. It leads to the "4/3 law" which says that the <u>diffusion coefficient is proportional to the 4/3 power of the size of the cloud</u>. The 4/3 law is useful for studies in the ocean where turbulence is homogeneous.

3.6 Summary

② $C(x,t)$

The ensemble average concentration C is obtained by averaging at a fixed point over a large number of releases of tracer. It is likely that C is what will be observed after even just one release of tracer if the tracer has been in the flow longer than the Lagrangian time scale of turbulence. After the tracer has been in the flow longer than the Lagrangian time scale, further changes in C are governed by the Fickian diffusion equation with constant coefficients.

3.6 Summary

- **Stages of diffusion**

(i) Individual clouds grow at a rate which increases with their size, and which is different for each cloud.

(ii) In the intermediate stage, the average growth of the cloud can be described by the diffusion equation with a diffusion coefficient proportional to the 4/3 power of the size of the cloud, if the turbulence is homogeneous.

3.6 Summary

(iii) After the clouds reach a size larger than the largest scales of turbulence motion, the further spread is described by the Fickian diffusion equation with constant coefficients.

→ This theory can be used for practical problems of rivers and estuaries because the size of pollutant cloud is usually greater than the largest eddy sizes.