3. Graphical methods: Stereographic projection

1) Types of projection

- Projection: Mapping 3D images into 2D ones on a planar surface

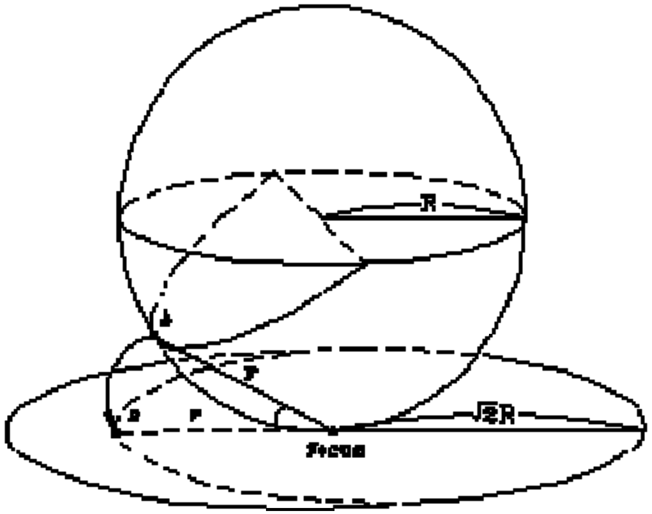
- Parallel projection: parallel rays are projected to a planar surface from an object. It is useful to convey measurements of distances or angles.

  ex.) Orthographic projection (Fig.3.1, Fig 3.4),
  oblique projection (Fig.3.2)

- Perspective projection: nonparallel rays connecting one or more foci and a object are projected to a surface.
  It is useful to convey perspective views of objects.

  ex.) Equal area projection,
  equal angle (stereographic) projection

1) Equal-Area projection (등면적 투영법)



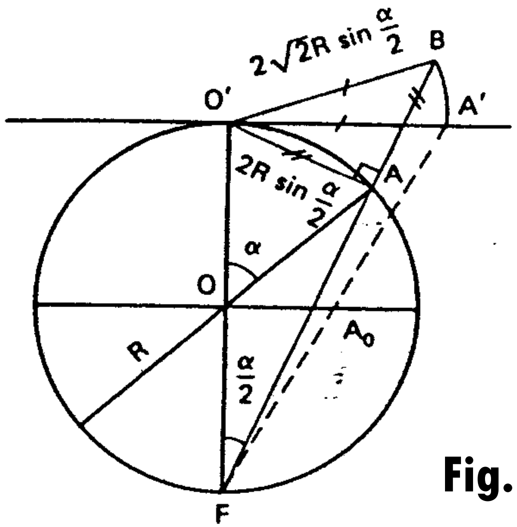
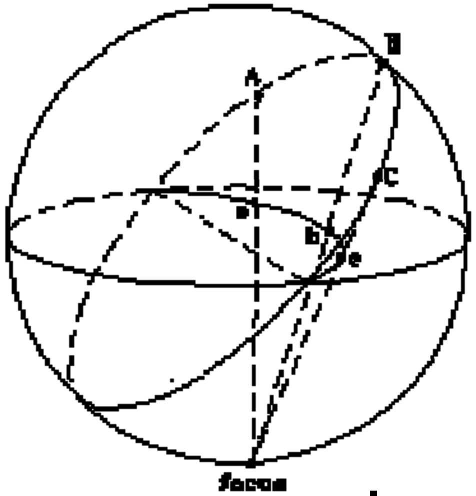


Fig. 3.5

Advantage: Area of a small circle is preserved.

Disadvantage: Shape of a small circle is distorted according to its location on the sphere.

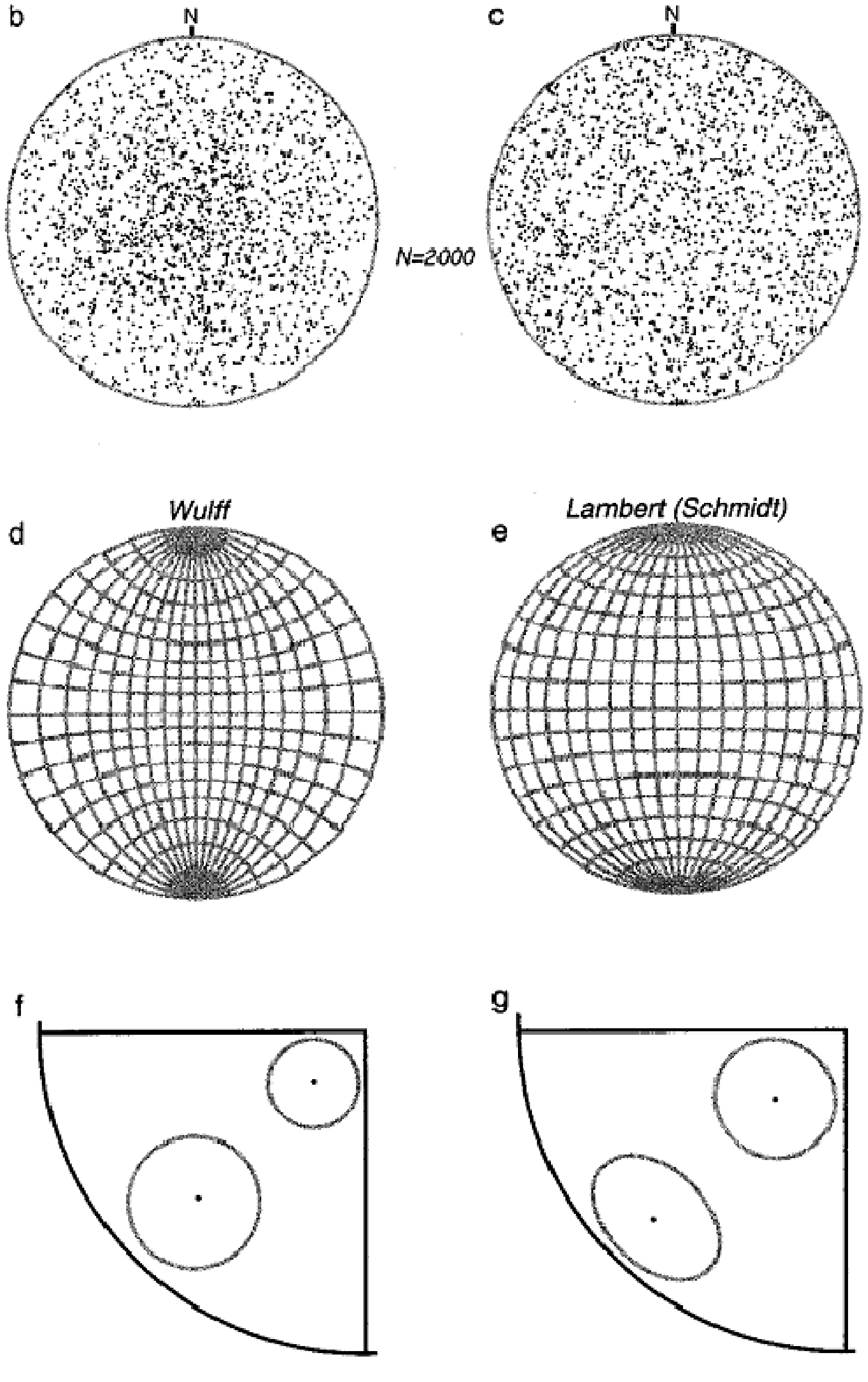
2) Stereographic (equal-angle, 등각) projection



Lower focal point (=upper hemisphere) projection

Advantage: Shape of a small circle (angle) is preserved (conformal).

Disadvantage: Area of a small circle changes according to its location on the sphere.


b
N
c
N
N=2000
Wulff
d
Lambert (Schmidt)
e
f
g

2) Stereographic projection of lines and planes

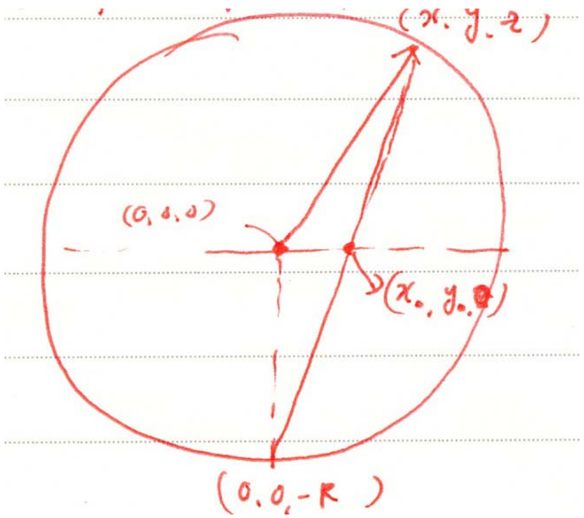
- Projection of a vector

Define a line intersecting a focal point and a point (x, y, z) on a sphere, and calculate t_0 when the line intersects a horizontal projection plane.

$$x' = xt, \quad y' = yt, \quad z' = -R + (z + R)t$$

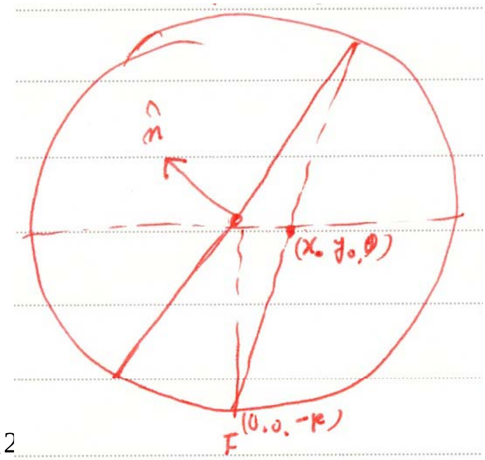
$$z' = 0 \text{ at the projection plane } \rightarrow t_0 = \frac{R}{z + R}$$

$$\therefore \quad x_0 = \frac{Rx}{z + R}, \quad y_0 = \frac{Ry}{z + R}$$



• Projection of a great circle

1. Define a line intersecting a focal point and a point $(x_0, y_0, 0)$ in a horizontal projection plane.
2. Obtain t when the line meets the sphere.
3. Obtain the relation between x_0 and y_0 when the line meets a joint plane.



Joint plane: $\sin\alpha\sin\beta x + \sin\alpha\cos\beta y + \cos\alpha z = 0$

Sphere: $x^2 + y^2 + z^2 = R^2$

Line $\left(0,0,\text{-R}\right) - \left(x_0, y_0, 0\right)$: $\quad x = x_0 t, \quad y = y_0 t, \quad z = R\left(t-1\right)$

Line - sphere: $x_0^2 t^2 + y_0^2 t^2 + R^2\left(t-1\right)^2 = R^2 \rightarrow \quad t = \dfrac{2R^2}{x_0^2 + y_0^2 + R^2}$

$$x = \frac{2R^2 x_0}{x_0^2 + y_0^2 + R^2}, \quad y = \frac{2R^2 y_0}{x_0^2 + y_0^2 + R^2}, \quad z = \frac{R\left(R^2 - x_0^2 - y_0^2\right)}{x_0^2 + y_0^2 + R^2}$$

Line - Joint plane:

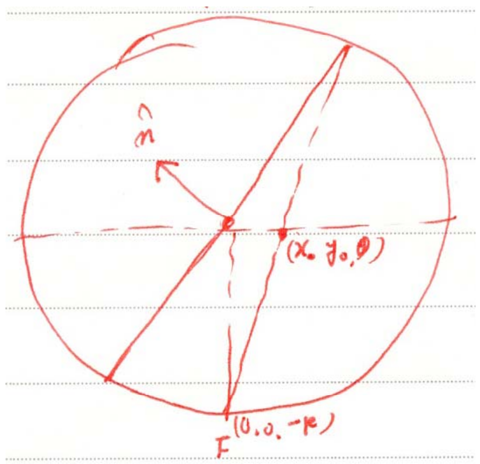
$$2R^2 \sin\alpha \sin\beta x_0 + 2R^2 \sin\alpha \cos\beta y_0 + R\left(R^2 - x_0^2 - y_0^2\right)\cos\alpha = 0$$

$$2R \tan\alpha \sin\beta x_0 + 2R \tan\alpha \cos\beta y_0 + \left(R^2 - x_0^2 - y_0^2\right) = 0$$

$$\left(x_0 - R\tan\alpha \sin\beta\right)^2 + \left(y_0 - R\tan\alpha\cos\beta\right)^2 - R^2\tan^2\alpha\left(\sin^2\beta + \cos^2\beta\right) = R^2$$

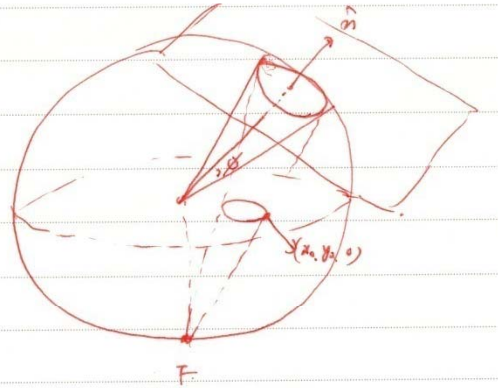
$$\left(x_0 - R\tan\alpha \sin\beta\right)^2 + \left(y_0 - R\tan\alpha\cos\beta\right)^2 = R^2\left(1 + \tan^2\alpha\right) = \frac{R^2}{\cos^2\alpha} \quad \rightarrow \text{Circle}$$

Circle: $\left(C_x, C_y\right) = \left(R\tan\alpha\sin\beta,\ R\tan\alpha\cos\beta\right), \quad \text{radius} = \dfrac{R}{\cos\alpha}$



- Projection of a small circle

Apply the same procedure for the great circle to deriving the small circle projected.



Joint plane: $\sin\alpha\sin\beta x + \sin\alpha\cos\beta y + \cos\alpha z = R\cos\phi$

Sphere: $x^2 + y^2 + z^2 = R^2$

Line: $x = x_0 t, \quad y = y_0 t, \quad z = R(t-1)$

Sphere - line: $x = \dfrac{2R^2 x_0}{x_0^2 + y_0^2 + R^2}, \quad y = \dfrac{2R^2 y_0}{x_0^2 + y_0^2 + R^2}, \quad z = \dfrac{R\left(R^2 - x_0^2 - y_0^2\right)}{x_0^2 + y_0^2 + R^2}$

Joint plane - line:

$$2\mathrm{R}^2 \sin\alpha\sin\beta x_0 + 2\mathrm{R}^2 \sin\alpha\cos\beta y_0 + R\left(R^2 - x_0^2 - y_0^2\right)\cos\alpha = R\cos\phi\left(R^2 + x_0^2 + y_0^2\right)$$

$$(\cos\alpha + \cos\phi)\, x_0^2 - 2R\sin\alpha\sin\beta x_0 + (\cos\alpha + \cos\phi)\, y_0^2 - 2R\sin\alpha\cos\beta y_0 = R^2(\cos\alpha - \cos\phi)$$

$$\left(x_0 - \frac{\mathrm{R}\sin\alpha\sin\beta}{\cos\alpha + \cos\phi}\right)^2 + \left(y_0 - \frac{\mathrm{R}\sin\alpha\cos\beta}{\cos\alpha + \cos\phi}\right)^2 = \left(\frac{R\sin\phi}{\cos\alpha + \cos\phi}\right)^2 \quad \rightarrow \quad \text{Circle}$$

Circle:

$$\left(C_x, C_y\right) = \left(\frac{\mathrm{R}\sin\alpha\sin\beta}{\cos\alpha + \cos\phi}, \frac{\mathrm{R}\sin\alpha\cos\beta}{\cos\alpha + \cos\phi}\right), \quad \text{radius} = \frac{R\sin\phi}{\cos\alpha + \cos\phi}$$

- H.W.2

Draw an equatorial-net by equal-angle projection. Set the angle between two adjacent longitudinal/latitudinal lines 10°.

3) Stereographic projection of a joint pyramid

- Projection of half spaces

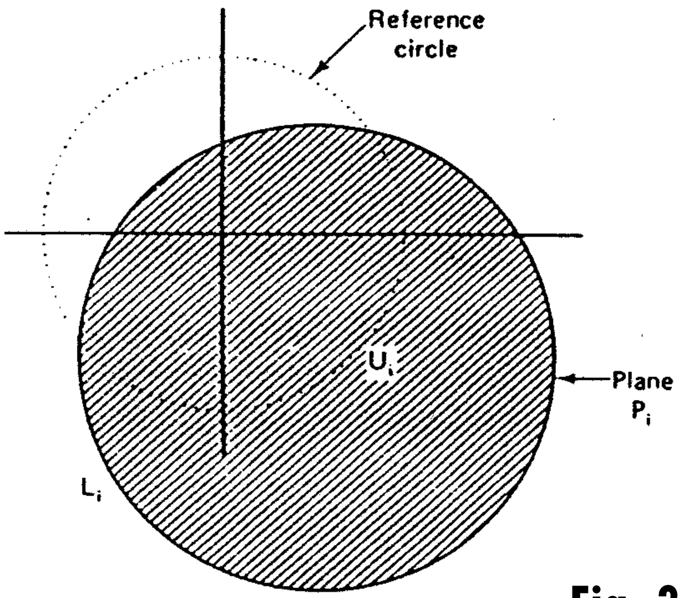


Fig. 3.16

- Joint pyramid: an intersect of joint half spaces shifted to the center of a projection sphere

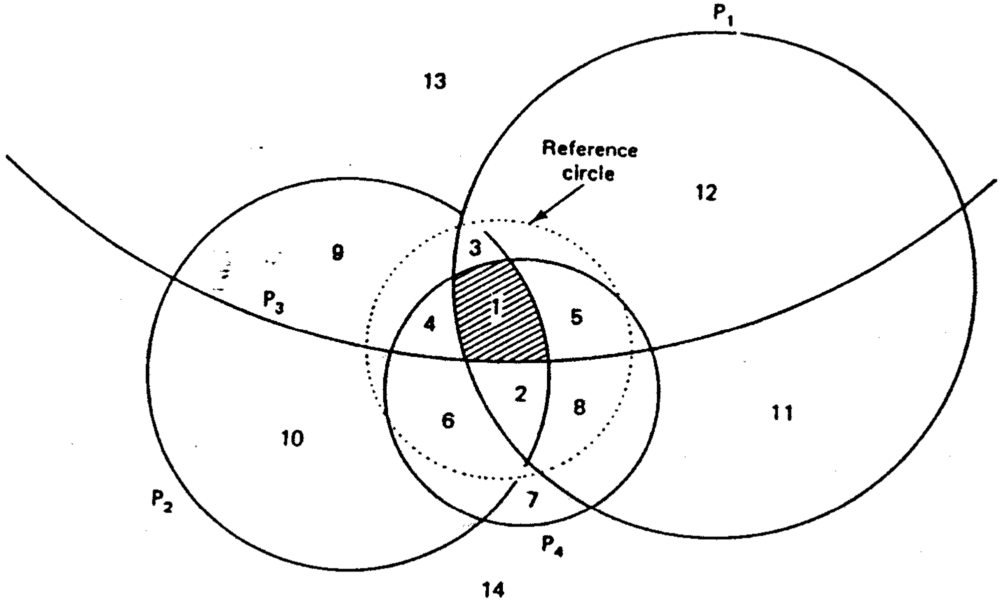


Fig. 3.17

- Intersection of two joints

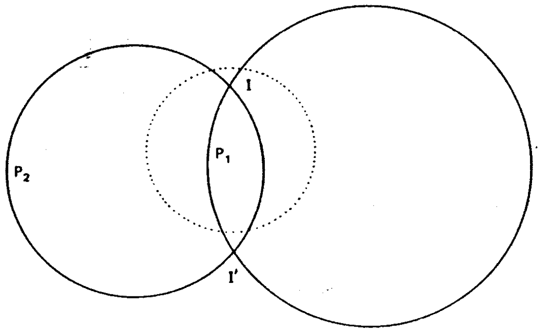


Fig. 3.12

- Arc of joint pyramid (Fig. 3.17)

- An arc represents a joint plane.

- A concave arc to the center of a joint pyramid indicates upper space of the joint arc when lower-focal-point projection adopted.

4) Additions

- Normal to a given plane

$$(x, y, z) \to (x_0, y_0)$$

$$x_0 = \frac{Rx}{R+z}, \quad y_0 = \frac{Ry}{R+z}$$

$$\vec{n} = (R\sin\alpha\sin\beta,\ R\sin\alpha\cos\beta,\ R\cos\alpha) \to \left(\frac{R\sin\alpha\sin\beta}{1+\cos\alpha},\ \frac{R\sin\alpha\cos\beta}{1+\cos\alpha}\right)$$

Distance from an origin to $(x_0, y_0) = \sqrt{x_0^2 + y_0^2} = \dfrac{R\sin\alpha}{1+\cos\alpha} = R\tan\dfrac{\alpha}{2}$

- Plane normal to a given vector (line)

$$\left(n_x, n_y, n_z\right) = \left(\sin\alpha\sin\beta, \sin\alpha\cos\beta, \cos\alpha\right) \rightarrow \alpha,\ \beta$$

$$\rightarrow \left(C_x, C_x\right) = \left(R\tan\alpha\sin\beta,\ R\tan\alpha\cos\beta\right),\quad \text{radius} = \frac{R}{\cos\alpha}\ \text{ from the great circle equation}$$

- $(X_0,\ Y_0) \rightarrow (X, Y, Z)$

From the procedure to obtain the great circle equation

Sphere: $x^2 + y^2 + z^2 = R^2$

Line: $x = x_0 t,\quad y = y_0 t,\quad z = R\left(t - 1\right)$

$$\rightarrow t = \frac{2R^2}{x_0^2 + y_0^2 + R^2},\quad x = \frac{2R^2 x_0}{x_0^2 + y_0^2 + R^2},\quad y = \frac{2R^2 y_0}{x_0^2 + y_0^2 + R^2},\quad z = \frac{R\left(R^2 - x_0^2 - y_0^2\right)}{x_0^2 + y_0^2 + R^2}$$

- Center of a great circle passing two points

-Vector analysis

$\left(X_1, Y_1\right)$ in a projection plane $\rightarrow \left(x_1, y_1, z_1\right)$ on a sphere, $\left(X_2, Y_2\right) \rightarrow \left(x_2, y_2, z_2\right)$

$$\hat{n} = \left(n_x, n_y, n_z\right) = \frac{\hat{x}_1 \times \hat{x}_2}{\left\| \hat{x}_1 \times \hat{x}_2 \right\|}$$

Center of a great circle:

$$\left(n_x, n_y, n_z\right) = \left(\sin\alpha \sin\beta, \sin\alpha \cos\beta, \cos\alpha\right) \rightarrow \left(R\tan\alpha \sin\beta, R\tan\alpha \cos\beta\right) = \left(\frac{Rn_x}{n_z}, \frac{Rn_y}{n_z}\right)$$

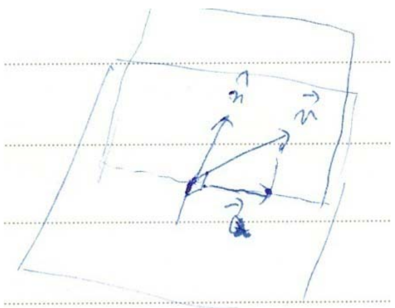
$$\left(\frac{Rn_x}{n_z}, \frac{Rn_y}{n_z}\right) = \left(\frac{R\left(y_1 z_2 - z_1 y_2\right)}{x_1 y_2 - y_1 x_2}, \frac{R\left(z_1 x_2 - x_1 z_2\right)}{x_1 y_2 - y_1 x_2}\right)$$

- Graphical procedure: refer to Fig.3.19 and Fig.3.9 with Eqn.(3.7)

1) Plot an opposite vector point to one of the predefined points.
2) Draw a circle passing through the 3 points.

• Orthographic projection of a vector on a plane

1) Draw a great circle of a plane.
2) Plot plane normal (**n**) and a vector (**v**).
3) Plot the opposite vector, **-v**.
4) Draw a circle passing through the 3 points (**n**, **V**, and-**V**)
5) Find out the intersection points of the two great circles.



Refer to Fig. 3.20

5) Projection of sliding direction

- Lifting

- The lifting direction is identical with the direction of resultant force: $\hat{s} = \hat{r}$

- Single-face sliding

1) Draw a great circle of a block sliding plane.
2) Draw a great circle passing through a normal to the sliding plane and a direction vector of resultant force.
3) Find out two intersection points (vectors) of the above two planes
4) Draw a great circle whose normal vector is identical with the direction vector of resultant force.
5) The sliding vector is one of the two intersection vectors which is located in the great circle above.

- Sliding in two planes simultaneously

1) Draw great circles of two block sliding planes.

2) Find out two intersection points (vectors) of the above two circles

3) Draw a great circle whose normal vector is identical with the direction vector of resultant force.

5) The sliding vector is one of the two intersection vectors which is located in the great circle above.