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457.643 Structural Random Vibrations
In-Class Material: Class 02

I. Basic Elements (Contd.)

◎ **Joint characteristic function:** alternative complete description to ______ PDF

$$M_{\mathbf{X}}(\boldsymbol{\theta}) \equiv \mathrm{E}_{\mathbf{X}}\{\exp[i\boldsymbol{\theta}^{\mathrm{T}}\mathbf{X}]\}$$
$$= \mathrm{E}_{\mathbf{X}}\{\exp[i(\theta_1 X_1 + \theta_2 X_2 + \cdots + \theta_n X_n)]\}$$
$$= \int \cdots \int \exp[i(\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n)]\, f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

→ m____variate F______ transform of _____ PDF

Therefore,

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(\quad)^n} \int \cdots \int \exp[-i(\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n)] \qquad d\boldsymbol{\theta}$$

Can show __________-generating property for the joint characteristic function, i.e.

$$\frac{1}{i^{m_1+\cdots+m_n}} \frac{\partial^{m_1+\cdots+m_n} M_{\mathbf{X}}(\boldsymbol{\theta})}{\partial\theta_1^{m_1}\cdots\theta_n^{m_n}}\bigg|_{\boldsymbol{\theta}=\mathbf{0}} = \mathrm{E}\left[X_1^{m_1} X_2^{m_2} \cdots X_n^{m_n}\right]$$

Some observations:

1) Consistency rule: $M_{\mathbf{X}}(\theta_1, \cdots, \theta_k, 0, \cdots, 0) =$

2) For statistically independent random variables, $M_{\mathbf{X}}(\boldsymbol{\theta}) =$

◎ **Joint log characteristic function**

Remember $M_X(\theta) = \mathrm{E}_X[\exp(i\theta X)] = \int_{-\infty}^{\infty} \exp(i\theta X)\, f_X(x) dx$

Joint log characteristic function $L_{\mathbf{X}}(\boldsymbol{\theta}) =$

Joint cumulant function

$$\kappa(\mathbf{X}) = \frac{1}{i^n} \frac{\partial^n L_{\mathbf{X}}(\boldsymbol{\theta})}{\partial\theta_1 \cdots \theta_n}\bigg|_{\boldsymbol{\theta}=\mathbf{0}}$$

- $\kappa(X_i) =$
- $\kappa(X_i, X_j) =$
- $\kappa(X_i, X_j, X_k) =$

Example: Characteristic function of $Y = X_1 + X_2 + \cdots + X_n$ (and PDF?)

◎ Multivariate normal (Gaussian) distribution

Joint PDF:

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{|\det \boldsymbol{\Sigma}|}} \exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{M})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{M})\right]$$

- completely determined by _____ and ______ order moments
- denoted by $\boldsymbol{X} \sim N(\boldsymbol{M}, \boldsymbol{\Sigma})$
- can show $\boldsymbol{Y} = \boldsymbol{AX} + \boldsymbol{b} \sim N\left(\boldsymbol{AM} + \boldsymbol{b}, \boldsymbol{A\Sigma A}^{\mathrm{T}}\right)$

e.g. $n = 1$, $X \sim N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

Can show

$$\mathrm{M}_{\mathbf{X}}(\boldsymbol{\theta}) = \exp\left(i\boldsymbol{M}^{\mathrm{T}}\boldsymbol{\theta} - \frac{1}{2}\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\Sigma}\boldsymbol{\theta}\right)$$

$$\mathrm{L}_{\mathrm{X}}(\boldsymbol{\theta}) =$$

- _________ function of $\boldsymbol{\theta}$
- Higher order ($n \geq$) cumulants are zero

Example: $\kappa(X_i, X_j)$ for bivariate normal random variables

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◎ Proof of Central Limit Theorem using characteristic functions

Consider $Z = X_1 + X_2 + \cdots + X_n$ where $X_i, i = 1, \ldots, n$ are statistically independent, identically distributed (SIID) random variables. Try

$$Z' = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)$$

where μ and σ respectively denote the common mean and standard deviation of X_i's.

Let $Y_i = \frac{X_i - \mu}{\sigma}$ Then, $Z' = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i$

The characteristic function of Z' is then derived as

$$M_{Z'}(\theta) = \mathrm{E}[\exp(\mathrm{i}\theta Z')] = \mathrm{E}\left\{ \exp\left[\frac{\mathrm{i}\theta}{\sqrt{n}} \sum_{j=1}^{n} Y_j \right] \right\} = \mathrm{E}\left[\prod_{j=1}^{n} \exp\left(\frac{\mathrm{i}\theta Y_j}{\sqrt{n}} \right) \right]$$

$$= \prod_{j=1}^{n} \mathrm{E}\left[\exp\left(\frac{\mathrm{i}\theta Y_j}{\sqrt{n}} \right) \right] = \prod_{j=1}^{n} M_{Y_j}\left(\frac{\theta}{\sqrt{n}} \right) = \left[M_Y\left(\frac{\theta}{\sqrt{n}} \right) \right]^n$$

statistically independent   identically distributed

Let us consider the characteristic function of Y, $M_Y(\theta)$. Note that the mean of Y is zero and its standard deviation is one. From the moment generating property of the characteristic function, we confirm

$$\left. \frac{dM_Y(\theta)}{d\theta} \right|_{\theta=0} = \mathrm{i}\mathrm{E}[Y] = i \cdot \mu_Y = 0$$

$$\left. \frac{d^2 M_Y(\theta)}{d\theta^2} \right|_{\theta=0} = \mathrm{i}^2 \mathrm{E}[Y^2] = -(\sigma_Y^2 + \mu_Y^2) = -1$$

Therefore, the characteristic function $M_Y(\theta)$ can be constructed by a Taylor series:

$$M_Y(\theta) = 1 - \frac{\theta^2}{2} + o(\theta^2)$$

$$M_Y\left(\frac{\theta}{\sqrt{n}} \right) = 1 - \frac{\theta^2}{2n} + o\left(\frac{\theta^2}{n} \right)$$

$$M_{Z'}(\theta) = \left[1 - \frac{\theta^2}{2n} + o\left(\frac{\theta^2}{n} \right) \right]^n$$

Since $\lim_{n\to\infty} \left(1 + \frac{x}{n} \right)^n = e^x$,

$$\lim_{n\to\infty} M_{Z'}(\theta) = \exp\left(-\frac{\theta^2}{2} \right)$$

The end result is the characteristic function of the standard ______ distribution. Thus, we hereby proved that Z' asymptotically follows the standard _______ distribution as $n \to \infty$. Since Z is a linear function of Z', Z also asymptotically follows a _______ distribution.

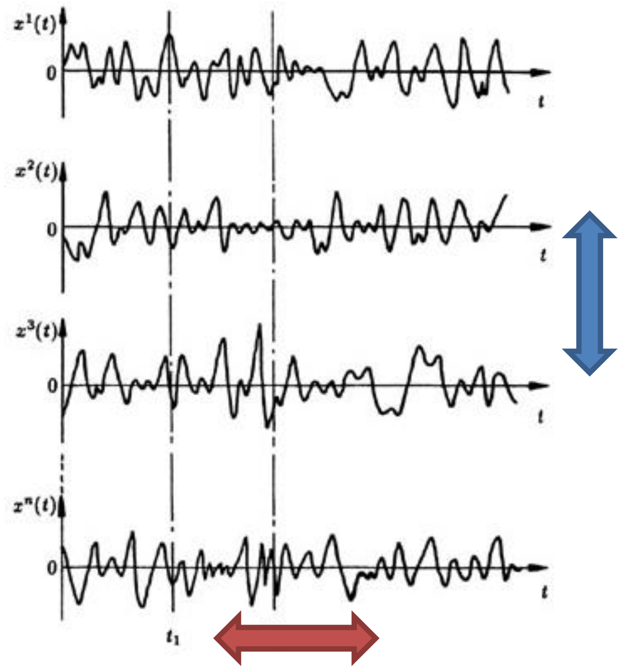
II. Introduction to Random Process

II-1. Random Process

◎ Definitions

Random (stochastic) process: $\{X(t)\}$ or $X(t)$ (cf. $x(t)$)

e.g. earthquake ground motion



Definition 1: Random process is an "e_______" (collection) of possible t____ h______ $\{x^{(1)}(t), x^{(2)}(t), \ldots\}$

Definition 2: "Continuously indexed" r_______ v________, or a family of random variables $\{X(0), \ldots X(t_k), \ldots X(t_m), \ldots\}$

Note: the concept of random process can be generalized

1) Random field $X(t, u, v)$, e.g. wind pressure at location (u, v) of the roof at time t
2) Vector random process:

$$\mathbf{X}(t) = \begin{Bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_n(t) \end{Bmatrix} \quad \text{e.g. } \mathbf{X}_g(t) = \begin{Bmatrix} x_g(t) \\ \dot{x}_g(t) \\ \ddot{x}_g(t) \end{Bmatrix}$$

3) Vector random field:

$$\mathbf{X}(t, u, v) = \begin{Bmatrix} X_1(t, u, v) \\ X_2(t, u, v) \\ \vdots \\ X_n(t, u, v) \end{Bmatrix}$$

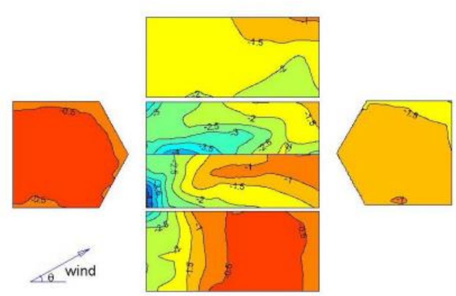


Figure credit: http://www.wind.arch.t-kougei.ac.jp/info_center/windpressure/lowrise/Introductionofthedatabase.pdf