Seoul National University
Dept. of Civil and Environmental Engineering

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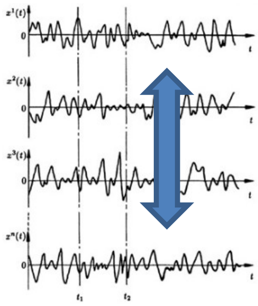
457.643 Structural Random Vibrations
In-Class Material: Class 03

II-1. Random Process (contd.)

◎ "Average" of random process

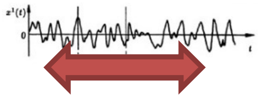


(a) "Ensemble" average: average over the ensemble

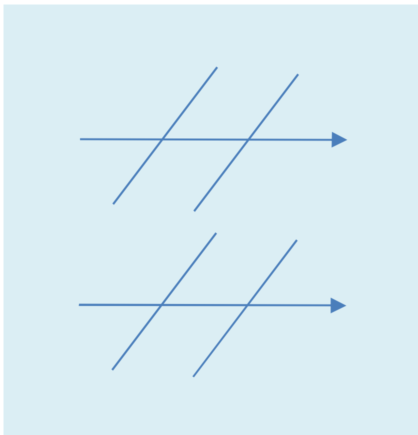
$$\mathrm{E}[X(t)] = \lim_{n\to\infty} \frac{\quad + \quad + \cdots + \quad}{n} = \int_{-\infty}^{\infty} \qquad dx$$

(b) "Temporal" average (for a specific time history)

$$\langle X(t)\rangle = \frac{\quad}{\quad}\int \quad x(t)dt$$



- Temporal average is another r_____ v______



◎ Specification of a random process

(a) By probabilistic distribution function

- $f_{X(t)}(x,t)$: 1st order "m_________" PDF
- $f_{X(t_1)X(t_2)}(x_1,t_1;x_2,t_2)$: 2nd order joint PDF
- ⋮
- $f_{X(t_1)\cdots X(t_n)}(x_1,t_1;\cdots;x_n,t_n)$: n^{th} order joint PDF

Theoretically, need the ____th order joint PDF for complete description of a random process

(b) By characteristic function

- $M_{X(t)}(\theta,t)$: 1st order characteristic function
- ⋮
- $M_{X(t_1)\cdots X(t_n)}(\theta_1,t_1;\cdots;\theta_n,t_n)$: n^{th} order joint characteristic function

(c) By moment functions (i.e. partial descriptors)
→ most common (because of lack of i_________)

- $\mathrm{E}[X(\mathrm{t})] = \mu_X(t)$ or $\mu(t)$: ________ function
- $\mathrm{E}[X(t_1)X(t_2)] = \phi_{XX}(t_1,t_2)$ or $\phi(t_1,t_2)$: auto ____________ function
- $\mathrm{E}\{[X(t_1)-\mu(t_1)][X(t_2)-\mu(t_2)]\} =$: auto __________ function

(d) By a function of random variables

- $X(t) = At + B$
- $X(t) = \sum_{i=1}^{n} A_i \cos(\omega_i t + \Theta_i)$

(e) Others: random pulses, log-characteristic function, cumulants, ARMA, etc.

◎ First & second order moment functions

$\mathrm{E}[X(\mathrm{t})] = \mu_X(t)$ or $\mu(\mathrm{t})$: (E__________) mean function

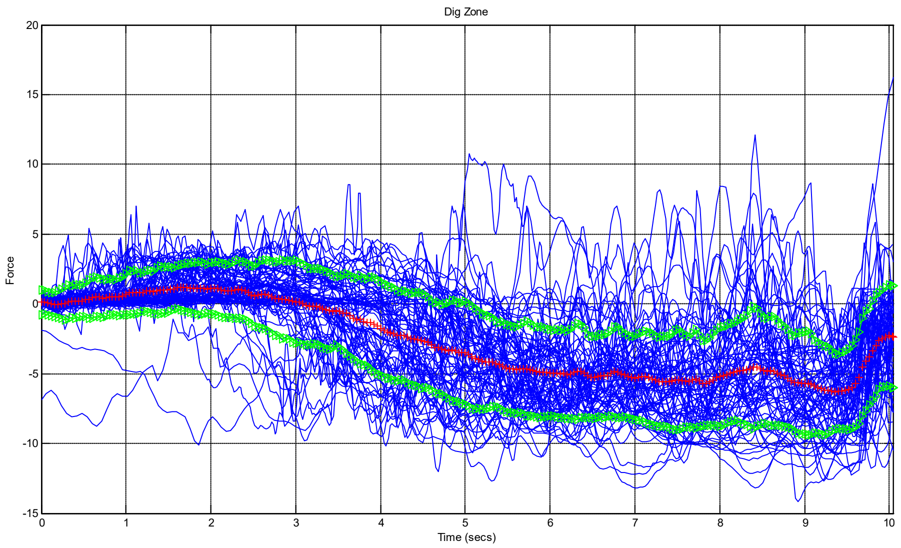
$\mathrm{E}[X(t_1)X(t_2)] = \phi_{XX}(t_1, t_2)$ or $\phi(t_1, t_2)$: Auto-*correlation* function

⋮

$\mathrm{E}\{[X(t_1) - \mu(t_1)][X(t_2) - \mu(t_2)]\} = \phi_{XX}(t_1, t_2) - \mu(t_1)\mu(t_2)$
$= \kappa_{XX}(t_1, t_2)$: Auto-*covariance* function

$\sigma_X(t) = \sqrt{\qquad\qquad}$: Standard deviation function

$\rho_{XX}(t_1, t_2) = \overline{\qquad\qquad\qquad}$: Auto-correlation-*coefficient* function



Example: 77 force time histories during “digging” tasks and their moment functions

Note:

If $\mu_X(t) = 0$ (zero-mean process),

$\phi_{XX}(t_1, t_2)$ $\qquad\qquad$ $\kappa_{XX}(t_1, t_2)$

One can transform a random process to a zero-mean process by

$Y(t) = X(t) -$

Why?

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• For a complex-valued random process,

$\phi_{XX}(t_1, t_2) = \mathrm{E}[X(t_1)X^*(t_2)]$

$\kappa_{XX}(t_1, t_2) = \mathrm{E}[(X(t_1) - \mu(t_1))(X^*(t_2) - \mu^*(t_2)]$

Note that $\phi_{XX}(t, t)$ and $\kappa_{XX}(t, t)$ are always _______-valued.

• More than one random process involved

$\phi_{XY}(t_1, t_2) = \mathrm{E}[X(t_1)Y^*(t_2)]$: _______ correlation function

$\kappa_{XY}(t_1, t_2) = \mathrm{E}[\left(X(t_1) - \mu(t_1)\right)\left(X^*(t_2) - \mu^*(t_2)\right)]$: _______ covariance function

$\rho_{XY}(t_1, t_2) = \dfrac{\qquad}{\qquad}$: ________ correlation coefficient function

• Importance of 1st and 2nd order moment functions

1) Most of the time, 1st and 2nd order moment functions are all one can get from data
2) For Gaussian, 1st and 2nd order moment functions are all you need for a complete description.
3) Using Chebyshev bounds, one can get upper bound estimate on the probability using moments

$$\mathrm{P}(|Z| > b) \leq \frac{\mathrm{E}[|Z|^c]}{b^c}$$

e.g. $c = 2$, $Z = X - \mu_X$

$$\mathrm{P}(|X - \mu_X| > b) \leq \frac{E[|X - \mu_X|^2]}{b^2} = \underline{\qquad}$$

◎ **Five important properties of** $\phi_{XY}(t_1, t_2)$ and $\kappa_{XY}(t_1, t_2)$

1) "Hermitian" ("Symmetric" for a real random process)

$\phi_{XY}(t_1, t_2) =$

$\kappa_{XY}(t_1, t_2) =$

2) Boundedness

Schwarz inequality $|\mathrm{E}[XY]| \leq \sqrt{\mathrm{E}[X^2]\mathrm{E}[Y^2]}$

Thus, $|\phi_{XY}(t_1, t_2)| \leq \sqrt{\phi_{XX}(\ ,\ )\phi_{YY}(\ ,\ )}$

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Also, $|\phi_{XX}(t_1, t_2)| \leq \sqrt{\phi_{XX}(\ ,\ )\phi_{XX}(\ ,\ )}$

Similarly, $|\kappa_{XY}(t_1, t_2)| \leq \sqrt{\kappa_{XX}(\ ,\ )\kappa_{YY}(\ ,\ )} = \sqrt{\sigma_X^2(\ )\sigma_Y^2(\ )}$

Note:

If $\mathrm{E}[X^2(t)]$ is bounded ($< \infty$) for $\forall t$,

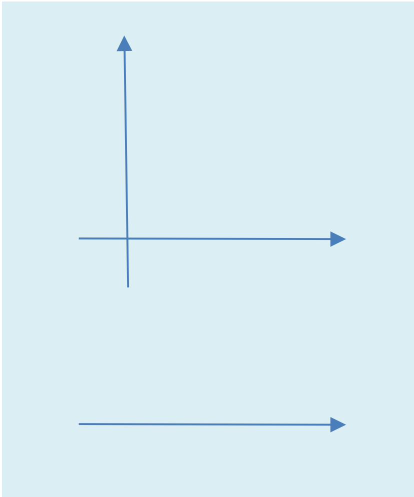
$|\quad_{XX}(t,s)| < \infty$

If $\sigma_X^2(t)$ is bounded ($< \infty$) for $\forall t$,

$|\quad_{XX}(t,s)| < \infty$

$X(t)$ is a "_______ ________" random process

if _________ is always finite



(Check L&S p.121. Later we will confirm that this means PSD exists)

3) Non-negative Definiteness

For an arbitrary function $h(t)$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \phi_{XX}(t_i, t_j) h(t_i) h^*(t_j) \geq$$

Proof:

$$
\begin{aligned}
(\mathrm{LHS}) &= \{h(t_1)\ \cdots h(t_n)\}\left[\phi_{XX}(t_i, t_j)\right]_{n\times n}\{h^*(t_1)\ \cdots h^*(t_n)\}^{\mathrm{T}} \\
&= \mathbf{h}^{\mathrm{T}}\mathrm{E}[\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}]\mathbf{h}^* \\
&= \mathrm{E}[\mathbf{h}^{\mathrm{T}}\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}\mathbf{h}^*] \\
&= \mathrm{E}[YY^*] \\
&= \mathrm{E}[\qquad]\quad 0
\end{aligned}
$$

Why is this property important?

Fourier transform of non-negative definite function is ____________
(Lin 1967, p.42 – Bochner's theorem)

Lin, Y.K. (1967) *Probabilistic Theory of Structural Dynamics*, McGraw-Hill, New York, NY.

Note: However,
$\phi_{XY}(t_1, t_2)$: NOT non-negative definite

$\because E[XY]$ can be _________
$\therefore$ Cross PSD can be __________