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457.643 Structural Random Vibrations
In-Class Material: Class 04

II-1. Random Process (contd.)

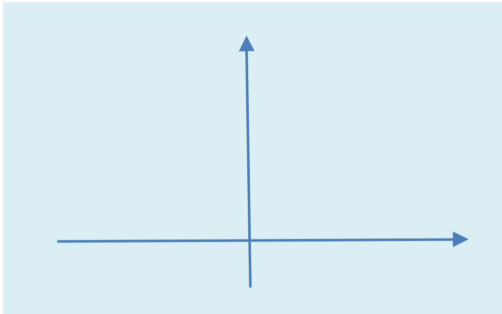
◎ **Five important properties of** $\phi_{XY}(t_1, t_2)$ and $\kappa_{XY}(t_1, t_2)$ **(contd.)**

4) For a process containing no periodic components,

   ________ diminishes as $|t_1 - t_2| \to \infty$

   $$\lim_{|t_1 - t_2| \to \infty} \kappa_{XX}(t_1, t_2) =$$

   $$\lim_{|t_1 - t_2| \to \infty} \phi_{XX}(t_1, t_2) =$$



5) Continuity property

   $\phi_{XY}(\cdot,\cdot)$ (or $\kappa_{XY}(\cdot,\cdot)$) must be continuous at (t_1, t_2) if $\phi_{XX}(\cdot,\cdot)$ and $\phi_{YY}(\cdot,\cdot)$ are continuous at (,) and (,) respectively.

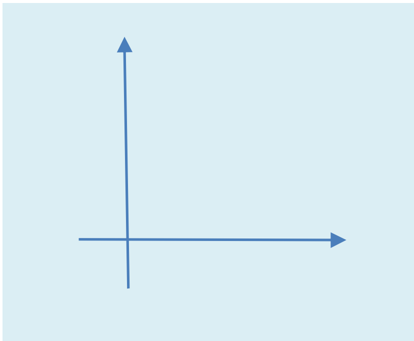
   i.e.

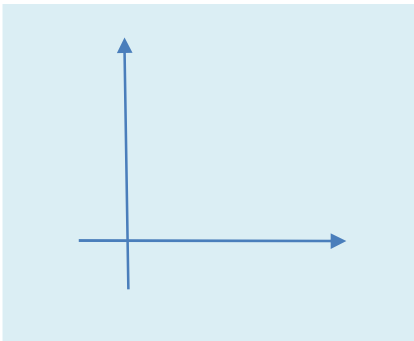
   $$\lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_2 \to 0}} \phi_{XY}(t_1 + \epsilon_1, t_2 + \epsilon_2) =$$

   if

   $$\lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_2 \to 0}} \phi_{XX}(t_1 + \epsilon_1, t_1 + \epsilon_2) = \qquad \text{and}$$

   $$\lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_2 \to 0}} \phi_{YY}(t_2 + \epsilon_1, t_2 + \epsilon_2) =$$





   Therefore, if $\phi_{XX}(t_1, t_2)$ and $\phi_{YY}(t_1, t_2)$ are continuous at all points on the diagonal $t_1 = t_2$, $\phi_{XY}(t_1, t_2)$ is continuous at all points in the 2D domain (t_1, t_2)

   Special case: $Y \to X$

   $\phi_{XX}(\cdot,\cdot)$ (or $\kappa_{XX}(\cdot,\cdot)$) must be continuous at (t_1, t_2) if $\phi_{XX}(\cdot,\cdot)$ is continuous at (,) and (,).

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※ **Proof of "Continuity Property"**

Consider

$$
\begin{aligned}
\phi_{XY}(t_1+\epsilon_1,t_2+\epsilon_2)-\phi_{XY}(t_1,t_2) &= \mathrm{E}[X(t_1+\epsilon_1)Y(t_2+\epsilon_2)]-\mathrm{E}[X(t_1)Y(t_2)] \\
&= \mathrm{E}[\{X(t_1+\epsilon_1)-X(t_1)\}\{Y(t_2+\epsilon_2)-Y(t_2)\}] \\
&+\mathrm{E}[\{X(t_1+\epsilon_1)-X(t_1)\}Y(t_2)] \\
&+\mathrm{E}[X(t_1)\{Y(t_2+\epsilon_2)-Y(t_2)\}]
\end{aligned}
\tag{1}
$$

Applying Schwarz's inequality to the first of the three expectations in the last line of Eq. (1), one can get

$$
\begin{aligned}
&|E[\{X(t_1+\epsilon_1)-X(t_1)\}\{Y(t_2+\epsilon_2)-Y(t_2)\}]| \\
&\qquad\le\sqrt{E[\{X(t_1+\epsilon_1)-X(t_1)\}^2]E[\{Y(t_2+\epsilon_2)-Y(t_2)\}^2]}
\end{aligned}
$$

The first term in the square root is expanded to

$$\phi_{XX}(t_1+\epsilon_1,t_1+\epsilon_1)-2\phi_{XX}(t_1+\epsilon_1,t_1)+\phi_{XX}(t_1,t_1)$$

This converges to zero if

$$\lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}}\phi_{XX}(t_1+\epsilon_1,t_1+\epsilon_2)=\phi_{XX}(t_1,t_2)$$

Therefore, the first expectation in Eq. (1) converges to zero.

Similarly, the other two expectations in Eq. (1) converge to zero if

$$\lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}}\phi_{XX}(t_1+\epsilon_1,t_1+\epsilon_2)=\phi_{XX}(t_1,t_1)$$

and

$$\lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}}\phi_{YY}(t_2+\epsilon_1,t_2+\epsilon_2)=\phi_{YY}(t_2,t_2)$$

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Example

$X(t) = A\cos\omega t + B\sin\omega t$

Given: $\mathrm{E}[A] = \mathrm{E}[B] = 0,\ \mathrm{E}[A^2] = \mathrm{E}[B^2] = \sigma^2$, $\mathrm{E}[AB] = \rho\sigma^2$

1) $\mathrm{E}[X(\mathrm{t})]$

2) $\phi_{XX}(t_1, t_2)$ and $\kappa_{XX}(t_1, t_2)$

Does $\kappa_{XX}(t_1, t_2)$ diminish as $|t_1 - t_2| \to \infty$? Why or Why not?

3) $\sigma_X^2(t)$

4) $\rho_{XX}(t_1, t_2)$

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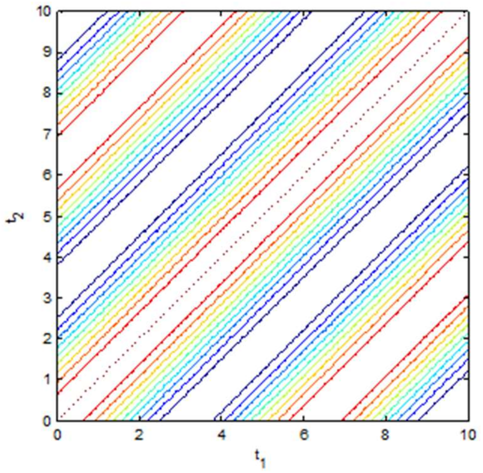
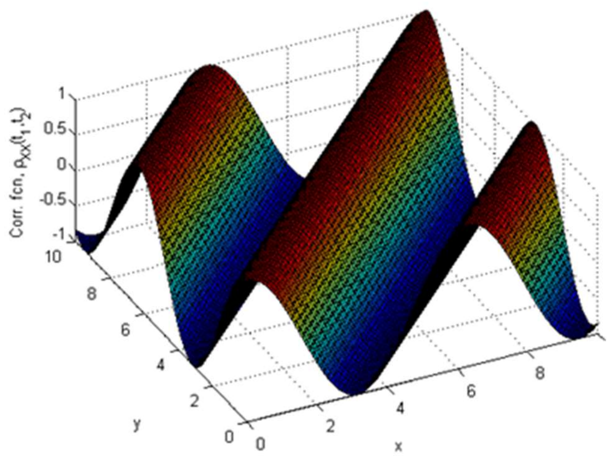
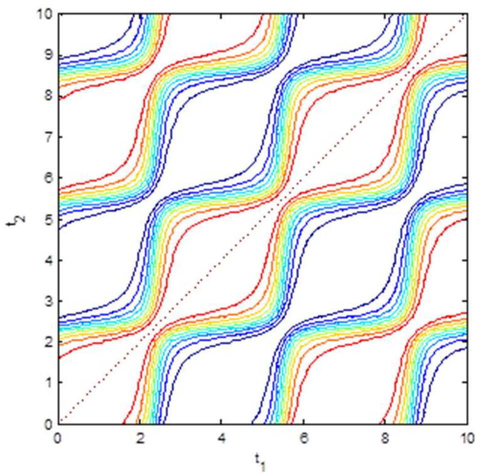
Correlation Coefficient Functions

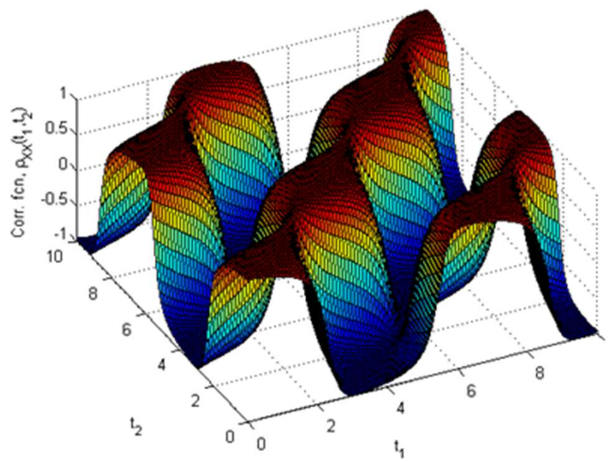
Case I: $\rho_{AB} = \rho = 0$





Case II: $\rho_{AB} = \rho = 0.8$



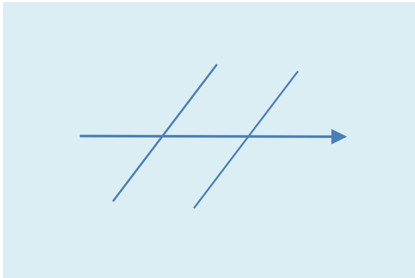


◎ Stationary process (cf. Homogeneous random field)

A R.P. is stationary if its "____________ description" is invariant to a __________ in the time parameter

(Strictly Stationary)

$f_{X\cdots X}(x_1,\cdots,x_n;t_1,\cdots,t_n) = f_{X\cdots X}(x_1,\cdots,x_n;t_1+h,\cdots,t_n+h)$



(1st Order Stationary)

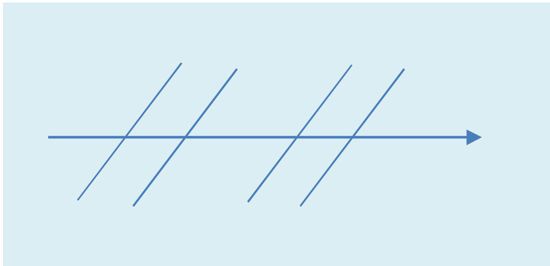
$f_X(x;t) = f_X(x;t+h) =$

Therefore, $\mu_X(t) =$, $\sigma_X(t) =$,⋯

(2nd Order Stationary)

$f_{XX}(x_1,x_2;t_1,t_2) = f_{XX}(x_1,x_2;\quad ,\quad)$

$= f_{XX}(x_1,x_2;\quad)$



Therefore,

$\phi_{XX}(t_1,t_2) = \phi_{XX}(t_1+h,t_2+h) \quad \forall(t_1,t_2)$

$= R_{XX}(\tau)$ where $\tau =$

$\kappa_{XX}(t_1,t_2) = \kappa_{XX}(t_1+h,t_2+h) \quad \forall(t_1,t_2)$

$= \Gamma_{XX}(\tau)$

"Weakly Stationary" or "Stationary in a Wide Sense" (Lin 1967)

When a random process satisfies

- $\mu_X(t) =$
- $\sigma_X(t) =$
- $\phi_{XX}(t_1,t_2) =$

Various Concepts of "Stationarity" in L&S

- Mean-value stationary
- Second-moment stationary
- *j*-th moment stationary
- *j*-th order stationary
- Strictly stationary

When ( ) and ( ) conditions above are satisfied, the random process is considered
________ __________