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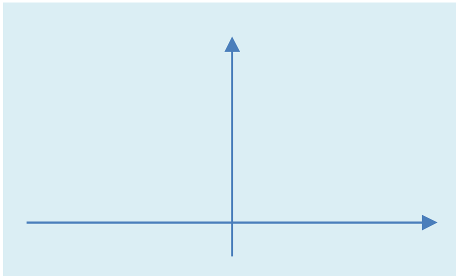
457.643 Structural Random Vibrations
In-Class Material: Class 05

II-1. Random Process (contd.)

◎ **Properties of** $R_{XX}(\tau)$ **and** $\Gamma_{XX}(\tau)$

(i.e. properties of second motion functions of ____________ process)

1) Hermitian (Symmetric)

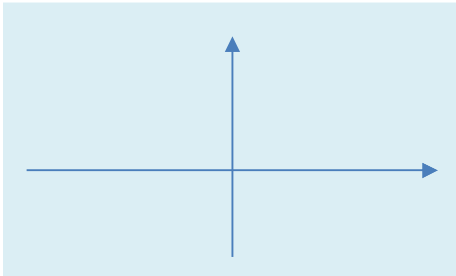


$R_{XX}(\tau) = R_{XX}^*(-\tau)$

$\Gamma_{XX}(\tau) = \Gamma_{XX}^*(-\tau)$

Real part, $\mathrm{Re}[R_{XX}(\tau)]$: _______ function

Imaginary part, $\mathrm{Im}[R_{XX}(\tau)]$: _______ function



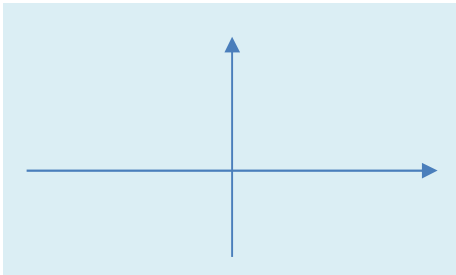
$R_{XY}(\tau) =$

$\Gamma_{XY}(\tau) =$

2) Boundedness

$|R_{XY}(\tau)| \leq \sqrt{R_{XX}(\ \ )R_{YY}(\ \ )}$

$|R_{XX}(\tau)| \leq \sqrt{R_{XX}(\ \ )R_{XX}(\ \ )} = R_{XX}(\ \ ) = \mathrm{E}[\qquad]$



Similarly,

$|\Gamma_{XY}(\tau)| \leq \sqrt{\Gamma_{XX}(\ \ )\Gamma_{YY}(\ \ )} =$

$|\Gamma_{XX}(\tau)| \leq$

3) Non-negative Definiteness

$$\sum_i \sum_j R_{XX}(t_i - t_j)h(t_i)h^*(t_j) \geq 0$$

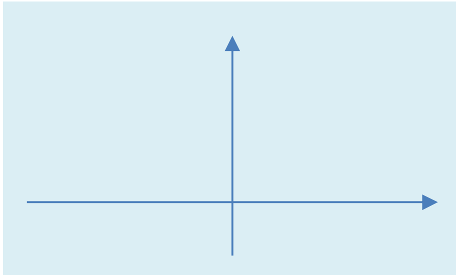
As the number of discretized points → ∞, the double summation becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(t_1 - t_2)h(t_1)h^*(t_2)dt_1dt_2 \geq 0$$

Substituting $t_1 = t_2 + \tau$, the integral becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau)h(t_2+\tau)h^*(t_2)\,d\tau dt_2 = \int_{-\infty}^{\infty} R_{XX}(\tau)H(\tau)d\tau \geq 0$$

4) Continuity



$R_{XX}(\tau)$ must be continuous at all τ

if $R_{XX}(\tau)$ is continuous at $\tau =$

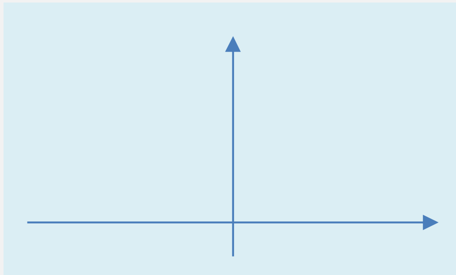
5) $\kappa_{XX}(\tau)$ diminishes for r.p with no periodic components as $|\tau| \rightarrow$

$$\lim_{|\tau|\rightarrow\infty} \Gamma_{XX}(\tau) =$$

$$\lim_{|\tau|\rightarrow\infty} R_{XX}(\tau) =$$

Example

$$R_{XX}(\tau) = \begin{cases} 1 - \frac{|\tau|}{a} & 0 \leq |\tau| \leq ka \\ 0 & elsewhere \end{cases} \qquad 0 < k < 1,\ a > 0$$



Check if the auto-correlation model is valid in terms of the important properties.

Example

Recall $X(t) = A\cos\omega t + B\sin\omega t$ in an earlier example.

We derived $\phi_{XX}(t_1, t_2) = \sigma^2[\cos\omega(t_1 - t_2) + \rho\sin\omega(t_1 + t_2)]$ and $\mu_X(t) = 0$

1) Condition(s) to make $X(t)$ a weakly stationary process:

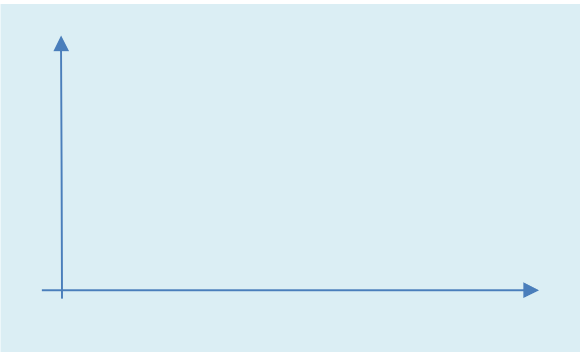
2) Suppose $\rho = 0$, and A and B are jointly Gaussian. Then, the process $X(t)$ is __________ ____________ ____________ process

◎ Poisson process

i) Example to demonstrate/review important concepts of random processes

ii) Introduction to an important class of random processes

$N(t)$: Number of _____________ in $(0, t]$



- C__________ index parameter
- D________-valued process
- Inherently stationary/non-stationary process
- Examples:

◎ Basic assumptions of Poisson random process

1) There exists m______ o________ rate (or intensity function), defined as

$$\lim_{\Delta t \to 0} \frac{\text{Average No. of Occurrences in } (t, t + \Delta t)}{\Delta t} =$$

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2) "Probability of two or more occurrences in Δt" $\ll$

Therefore,

$$\text{Average No. of Ocurrences in } (t, t+\Delta t)$$
$$= \nu(t) \cdot$$
$$= \sum_{n=0}^{\infty} n \cdot P\big(n \text{ occurrences in } (t, t+\Delta t)\big)$$
$$= 1 \cdot P\big(1 \text{ occurrence in } (t, t+\Delta t)\big) + 2 \cdot P\big(2 \text{ occurrences in } (t, t+\Delta t)\big) + \cdots$$
$$\cong$$

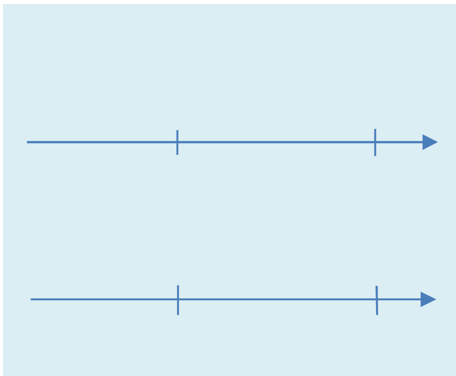
3) No. of occurrences in two non-overlapping intervals are ______________ ________________

◎ Probability functions and partial descriptors of Poisson process

1) Probability mass function (PMF) of $N(t)$

$$P_{N(t)}(n; t) \equiv P(N(t) = \quad)$$
$$= P_n(t)$$
$$= P_n(t-\Delta t) \cdot (1 - \nu \cdot \Delta t) + P_{n-1}(t-\Delta t) \cdot \nu \cdot \Delta t$$

"scenario 1" "scenario 2"



Thus,

$$\frac{P_n(t) - P_n(t-\Delta t)}{\Delta t} + \nu \cdot P_n(t-\Delta t) = \nu \cdot P_{n-1}(t-\Delta t)$$

As $\Delta t \to 0$, we get a recursive ODE:

$$\frac{d}{dt} P_n(t) + \nu(t) \cdot P_n(t) = \nu(t) \cdot P_{n-1}(t)$$

Solution: $P_n(t) \cdot \exp\left(\int_0^t \nu(t) dt\right) = \int_0^t \nu(t) \cdot P_{n-1}(t) \cdot \exp\left(\int_0^t \nu(t) dt\right) dt + C_n$

$= m(t)$

i) $n = 0$

$$P_0(t) \cdot e^{m(t)} = \int_0^t \nu(t) P_{-1}(t) e^{m(t)}\, dt + C_0$$

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$P_0(t) = C_0 \cdot e^{-m(t)}$

Initial condition $P_0(0) = 1$. Therefore, $C_0 =$

$P_0(t) =$

ii) $n = 1$

$P_1(t) \cdot e^{m(t)} = \int_0^t \nu(t) P_0(t) e^{m(t)}\, dt + C_1$

$=$

Initial condition $P_1(0) =$. Therefore, $C_1 =$

$P_1(t) =$

Solving recursively, one can get

$$P_n(t) = P_N(n;t) = \frac{[m(t)]^n \exp[-m(t)]}{n!}, \quad n = 0,1,2,\cdots$$

PMF of Poisson process $N(t)$ ("Poisson distribution" PMF)