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457.643 Structural Random Vibrations
In-Class Material: Class 06

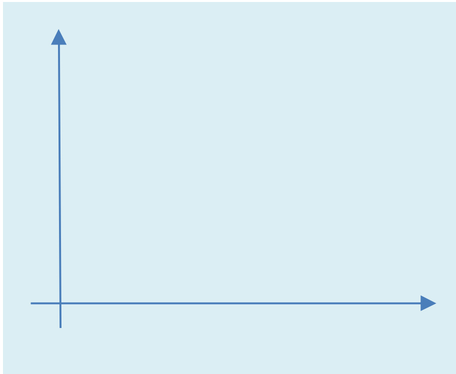
II-1. Random Process (contd.)

◎ Probability functions and partial descriptors of Poisson process (contd.)

2) "Homogeneous" Poisson process (HPP)

Definition: $\nu(t) =$

$$\therefore m(t) = \int_0^t \nu(t)dt =$$



Continuous change of ________
over time duration length t

PMF of HPP:

$$P_N(t) = \frac{[\quad]^n \exp(-\qquad)}{n!}$$

3) First-order characteristic function

$$M_N(\theta, t) = \mathrm{E}\big[\exp\big(\mathrm{i}\theta N(t)\big)\big]$$

$$= \sum_{n=0}^{\infty} \exp(\mathrm{i}\theta n) \cdot \frac{[\qquad]^n \cdot \exp[\qquad]}{n!}$$

$$= \exp[\qquad] \sum_{n=0}^{\infty} \frac{[\qquad\qquad]^n}{n!}$$

$$= \exp[\qquad] \cdot \exp[\qquad\qquad\qquad]$$

$$= \exp[-m(t) \cdot (1 - \exp(\mathrm{i}\theta))]$$

Note: $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

4) Mean

$$\mathrm{E}[N(t)] = \frac{1}{i} \cdot \frac{dM}{d\theta}\bigg|_{\theta=0}$$

$$\frac{dM}{d\theta} =$$

$$\mathrm{E}[N(t)] = \mu_N(t) = m(t) = \int_0^t \nu(t)dt$$

5) Standard deviation

$$\mathrm{E}[N^2(t)] = \frac{1}{i^2} \cdot \frac{d^2 M}{d\theta^2}\bigg|_{\theta=0}$$
$$= m(t) + m^2(t)$$

$$\therefore\ Var[N(t)] =$$

$$\therefore\ \sigma_{N(t)} = \sqrt{\qquad\qquad} = \sqrt{\qquad\qquad\qquad}$$

6) Mean and standard deviation for HPP

$$\mathrm{E}[N(t)] =$$

$$\mathrm{E}[N^2(t)] =$$

$$\sigma_{N(t)} =$$

Question: Is HPP a stationary process?

7) 2nd order joint PMF

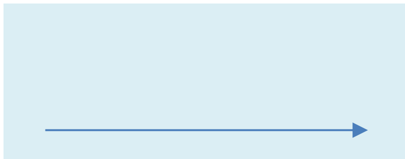
$$P_{NN}(n_1, n_2; t_1, t_2) = P(\qquad\qquad\qquad\qquad\qquad\qquad)$$

$$= P(\qquad\qquad\qquad\qquad\qquad\qquad\qquad)$$

$$= P(N(t_2) = n_2) \times$$

$$= \frac{[m(t_2)]^{n_2} \cdot \exp[-m(t_2)]}{n_2!} \times \frac{[\qquad\qquad]^{n_1-n_2} \cdot \exp[\qquad\qquad\qquad]}{(n_1-n_2)!}$$

$$= \frac{[m(t_2)]^{n_2} \cdot [m(t_1) - m(t_2)]^{n_1-n_2} \cdot \exp[-m(t_1)]}{n_2!\,(n_1-n_2)!}$$



Note: Set $t_1 > t_2$ and $n_1 \geq n_2$
The derivation depends on this convention

8) Joint characteristic function

$$M_{NN}(\theta_1, \theta_2; t_1, t_2) = \mathrm{E}\{\exp[\mathrm{i}(\theta_2 N(t_2) + \theta_1 N(t_1))]\}$$

→ This is not the same as the product of the two expectations, i.e. two marginal characteristic functions. Why?

$$= \mathrm{E}\{\exp[\mathrm{i}(\theta_1 + \theta_2) N(t_2)] \cdot \exp[\mathrm{i}\theta_1 (N(t_1) - N(t_2))]\}$$

$$= \mathrm{E}\{\exp[\mathrm{i}(\theta_1 + \theta_2) N(t_2)]\} \cdot \mathrm{E}\{\exp[\mathrm{i}\theta_1 (N(t_1) - N(t_2))]\}$$

→ Why?

$$= \exp[-m(t_2) \cdot (1 - \exp(i(\theta_1 + \theta_2)))]$$
$$\times \exp[-(m(t_1) - m(t_2)) \cdot (1 - \exp(i(\theta_1)))]$$

$$= \exp\{-m(t_2)[1 - \exp(\mathrm{i}(\theta_1 + \theta_2))] - (m(t_1) - m(t_2))(1 - \exp(\mathrm{i}\theta_1))\}$$

9) Auto correlation function

$$\begin{aligned}
\phi_{NN}(t_1, t_2) &= \mathrm{E}[N(t_1)\cdot N(t_2)] \\
&= \mathrm{E}\{N(t_2)\cdot[N(t_1) - N(t_2) + N(t_2)]\} \\
&= \mathrm{E}\{N(t_2)\cdot[N(t_1) - N(t_2)]\} + \mathrm{E}[N^2(t_2)] \\
&= \mathrm{E}[N(t_2)]\cdot \mathrm{E}[N(t_1) - N(t_2)] + \mathrm{E}[N^2(t_2)] \\
&= m(t_2)\cdot[m(t_1) - m(t_2)] + m^2(t_2) + m(t_2) \\
&= m(t_2) + m(t_1)\cdot m(t_2)
\end{aligned}$$

→ Violating symmetry?

10) Auto covariance function

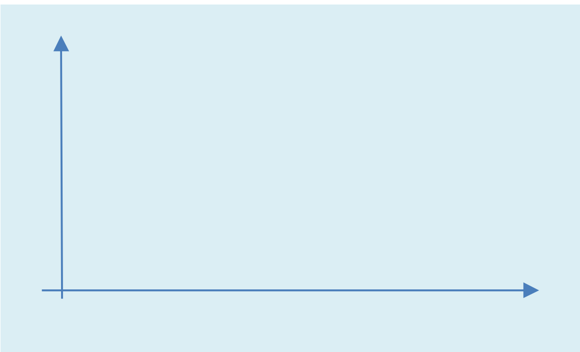
$$\begin{aligned}
\kappa_{NN}(t_1, t_2) &= \phi_{NN}(t_1, t_2) - \\
&= m(t_2) + m(t_1)\cdot m(t_2) - \\
&=
\end{aligned}$$

→ Violating symmetry?

11) Auto correlation coefficient function

$$\rho_{NN}(t_1, t_2) = \frac{\qquad}{\qquad} = \frac{\qquad}{\sqrt{\qquad}\cdot\sqrt{\qquad}} = \sqrt{\frac{\qquad}{\qquad}} \le$$

◎ Waiting time until the n^{th} occurrence of a Poisson process W_n



W_n: Waiting time until the n^{th} occurrence

$T_n = W_n - W_{n-1}$: ____________ time

1) Probability ____________ function of W_n

$f_{W_n}(t)dt = P(t < W_n \le t + dt)$ ~ Definition of PDF

Therefore,

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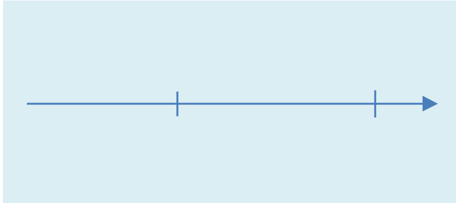
$$f_{W_n}(t) = \lim_{\Delta t \to 0} \frac{P(t < W_n \leq t + \Delta t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{P_N(n-1, t) \times \nu(t) \times \Delta t + o(\Delta t)}{\Delta t}$$

$$= \nu(t) \cdot \frac{[\qquad]^{n-1} \cdot \exp[\qquad]}{(\qquad)!}, \qquad t > 0$$



Note: $(n-1)$ occurrences up to time t and ________ occurrence during ______

2) PDF of W_n for HPP

$f_{W_n}(t) =$ ________________________, $t > 0$

$=$ ________________________ "   " **PDF:** n is a real number

$n = 1$: ________________________ distribution

$f_{W_1}(t) =$

Note: For HPP, PDF of waiting time until __________ occurrence = PDF of ____________ time, T_n (will be shown below)

3) Distribution functions of interarrival time $T_n = W_n - W_{n-1}$

CDF

$$F_{T_n}(t) = P(T_n \leq t)$$

$$= 1 - P(T_n > t)$$

$$= 1 - \int_0^{\infty} P(T_n > t | W_{n-1} = w) f_{W_{n-1}}(w) dw$$

Here,

$P(T_n > t | W_{n-1} = w) = P(\qquad$ events in $(w, w+t))$

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Using Poisson distribution,

$P(T_n > t | W_{n-1} = w) = \exp[-m(w+t) + m(w)]$

Therefore,

$$F_{T_n}(t) = 1 - \int_0^\infty \exp[-m(w+t) + m(w)] \times \frac{\nu(w)[m(w)]^{n-2} \exp[-m(w)]}{(n-2)!} dw$$
$$= 1 - \int_0^\infty \frac{\nu(w) m(w)^{n-2} \exp[-m(w+t)]}{(n-2)!} dw$$

For HPP,

$$F_{T_n}(t) = 1 - \int_0^\infty \frac{\nu(\nu w)^{n-2} \exp[-\nu(w+t)]}{(n-2)!} dw = 1 - \exp(-\nu t)$$

$$f_{W_n}(t) = \frac{dF_{T_n}(t)}{dt} = \nu \cdot \exp(-\nu t) = f_{T_1}(t)$$