Seoul National University
Dept. of Civil and Environmental Engineering

Instructor: Junho Song
junhosong@snu.ac.kr

457.643 Structural Random Vibrations
In-Class Material: Class 08

II-2. Stochastic Calculus (contd.)

◎ Mean-square derivative (derivative of r.p. in mean square sense) (contd.)

Note: Deterministic:

$$\dot{x}(t) = \lim_{h\to 0} \frac{x(t+h) - x(t)}{h}$$

Definition of "mean-square" derivative of a random process:

$$\dot{X}(t) \equiv \underset{h\to 0}{\mathrm{l.i.m}} \frac{X(t+h) - X(t)}{h}$$

When is a random process "mean-square differentiable"? (or when does the limit exist in the mean square sense?)

Recall Theorem 2 with $X(t)$ replaced by $Y(t)$:

> **Theorem 2:**
>
> $\underset{t\to t_0}{\mathrm{l.i.m.}} Y(t) = Y$ iff $\phi_{YY}(t,s)$ is continuous at (t_0, t_0) no matter how (t,s) approaches (t_0, t_0)

Substituting $Y(t) = \frac{X(t+h)-X(t)}{h}$ above, we need to check the limit of $\phi_{YY}(t,s)$ at the diagonal, i.e. $t = s$. Consider

$$\begin{aligned}\lim_{h\to 0, h'\to 0} \phi_{YY}(t,s) &= \lim_{h\to 0, h'\to 0} \mathrm{E}\left[\frac{X(t+h)-X(t)}{h} \cdot \frac{X(s+h')-X(s)}{h'}\right] \\ &= \lim_{h\to 0, h'\to 0} \frac{1}{h}\left[\frac{\phi_{XX}(t+h, s+h') - \phi_{XX}(t+h, s)}{h'} - \frac{\phi_{XX}(t, s+h') - \phi_{XX}(t,s)}{h'}\right] \\ &= \frac{\partial^2}{\partial\ \ \partial\ \ }\end{aligned}$$

Therefore, $X(t)$ is mean-square differentiable iff $\phi_{XX}(t,s)$ is _____-______ _________ at $t = s$

Instructor: Junho Song
junhosong@snu.ac.kr

In summary,

- $\phi_{XX}(t,s)$ is continuous at $t = s = t_0$ iff $\underset{t \to t_0}{\text{l.i.m}}\, X(t) = X$ (Theorem 2)
- $\phi_{XX}(t,s)$ is second-order differentiable at $t = s = t_0$ iff $\dot{X}(t)$ exists at $t = t_0$ (mean-square differentiable)

◎ Properties of $\dot{X}(t)$

1) $\text{E}[\dot{X}(t)] = \text{E}\left[\underset{h \to 0}{\text{l.i.m.}} \frac{X(t+h)-X(t)}{h}\right]$

$$= \lim_{h \to 0} \text{E}\left[\frac{X(t+h)-X(t)}{h}\right]$$

$$= \lim_{h \to 0}$$

$$= \frac{d}{d}$$

The mean of the (mean-square) derivative of a r.p. is the derivative of the mean function

2) $\text{E}[X(t) \cdot \dot{X}(s)] = \phi_{X\dot{X}}(t,s)$

$$= \text{E}\left[X(t) \cdot \underset{h \to 0}{\text{l.i.m.}} \frac{X(s+h)-X(s)}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{\qquad}{}\right]$$

$$= \frac{\partial}{\partial}$$

$$\therefore \text{E}[\dot{X}(t) \cdot X(s)] = \phi_{\dot{X}X}(t,s) = \frac{\partial}{\partial}$$

3) $\text{E}[\dot{X}(t) \cdot \dot{X}(s)] = \phi_{\dot{X}\dot{X}}(t,s)$

$$= \text{E}\left[\underset{h_1 \to 0}{\text{l.i.m.}} \frac{X(t+h_1)-X(t)}{h_1} \cdot \underset{h_2 \to 0}{\text{l.i.m.}} \frac{X(s+h_2)-X(s)}{h_2}\right]$$

$$= \frac{\partial^2}{\partial \ \partial}$$

◎ Mean-square derivative $\dot{X}(t)$ for a stationary r.p. $X(t)$

- $\mu_X(t) = \mu$
- $\phi_{XX}(t,s) = R_{XX}(\tau),\ \ \tau = t - s$

1) $X(t)$ is mean-square continuous iff $R_{XX}(\tau)$ is continuous at $\tau =$

2) $X(t)$ is mean-square differentiable iff $\frac{\partial^2 \phi_{XX}(t,s)}{\partial t \partial s} =$ is unique and finite at $\tau =$

- $\frac{\partial \phi_{XX}(t,s)}{\partial s} =$
- $\frac{\partial^2 \phi_{XX}(t,s)}{\partial t \partial s} =$

3) $\mu_{\dot{X}}(t) = \mathrm{E}[\dot{X}(t)] = \frac{d}{d} =$

The mean of the time rate of a stationary r.p. is ____

4) Cross correlation between $X(t)$ and $\dot{X}(t)$

$$R_{X\dot{X}}(\tau) = \frac{\partial \phi_{XX}(t,s)}{\partial} = -\_\_\_\_$$

$$R_{\dot{X}X}(\tau) = \frac{\partial \phi_{XX}(t,s)}{\partial} = \_\_\_\_$$

At $\tau = 0$,

$$R_{X\dot{X}}(0) = \mathrm{E}[\quad \cdot \quad] = -\frac{d}{d\tau} R_{XX}(\tau)\Big|_{\tau=0}$$

$$R_{\dot{X}X}(0) = \mathrm{E}[\quad \cdot \quad] = \frac{d}{d\tau} R_{XX}(\tau)\Big|_{\tau=0}$$

$\therefore R_{X\dot{X}}(0) = R_{\dot{X}X}(0) =$

When $X(t)$ is stationary r.p. and mean-square differentiable,

- $X(t)$ and $\dot{X}(t)$ are ____________, i.e. $\mathrm{E}[X\dot{X}] = 0$
- $X(t)$ and $\dot{X}(t)$ are ____________ as well because

5) $R_{\dot{X}X}(\tau) = \mathrm{E}[\dot{X}(t+\tau) \cdot X(t)]$

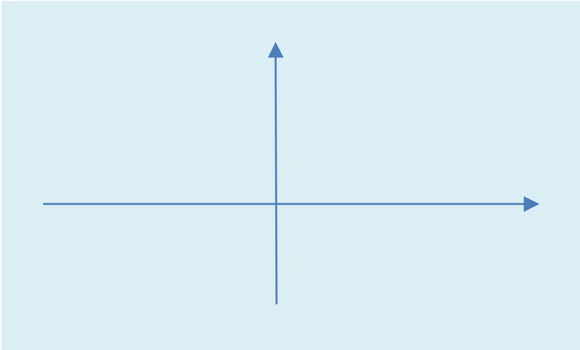
$R_{X\dot{X}}(-\tau) = \mathrm{E}[X(t-\tau) \cdot \dot{X}(t)] = \mathrm{E}[X(t) \cdot \dot{X}(t+\tau)]$

Therefore,

$R_{X\dot{X}}(-\tau) = R_{\dot{X}X}(\tau)$
$\quad = -R_{X\dot{X}}(\tau)$

Note: $R_{\dot{X}X}(\tau) = dR_{XX}(\tau)/d\tau$ and $R_{X\dot{X}}(\tau) = -dR_{XX}(\tau)/d\tau$

$R_{X\dot{X}}(\tau)$ is an ________ function (_____symmetric around $\tau =$)



◎ **Example:** $R_{XX}(\tau) = \frac{n\sigma^2}{2} \cdot \frac{sin\omega\tau}{\omega\tau}$

➔ See next page for plots

1) Is the random process $X(t)$ mean-square continuous?

$\lim_{\tau \to 0} R_{XX}(\tau) =$

$R_{XX}(\tau)$ is ______________ at $\tau =$. Therefore, $X(t)$ is ____________________

2) Is the random process $X(t)$ mean-square differentiable?

$$\frac{dR_{XX}(\tau)}{d\tau} = \frac{n\sigma^2\omega}{2} \cdot \frac{\omega\tau \cdot \cos\omega\tau - \sin\omega\tau}{(\omega\tau)^2}$$

(Is $R_{\dot{X}X}(\tau)$ anti-symmetric around $\tau = 0$?)

$$\lim_{\tau \to 0} \frac{dR_{XX}(\tau)}{d\tau} = \lim_{\tau \to 0} \frac{n\sigma^2\omega}{2} \cdot \frac{\omega\tau\left(1 - \frac{1}{2}(\omega\tau)^2 + \cdots\right) - \left(\omega\tau - \frac{1}{6}(\omega\tau)^3 + \cdots\right)}{(\omega\tau)^2} = \quad <$$

$$\frac{d^2R_{XX}(\tau)}{d\tau^2} = -\frac{n\sigma^2\omega^2}{2} \cdot \frac{(\omega\tau)^2 \sin\omega\tau + 2\omega\tau\cos\omega\tau - 2\sin\omega\tau}{(\omega\tau)^3}$$

$$\lim_{\tau \to 0} \frac{d^2R_{XX}(\tau)}{d\tau^2} = -\frac{n\sigma^2\omega^2}{6} <$$

Therefore, $X(t)$ is ______________________

$$R_{XX}(\tau) = \frac{n\sigma^2}{2} \cdot \frac{\sin \omega_0 \tau}{\omega_0 \tau}$$

$$R_{X\dot{X}}(\tau) = -\frac{d}{d\tau} R_{XX}(\tau) = -\frac{n\sigma^2 \omega_0}{2} \cdot \frac{\omega_0 \tau \cos \omega_0 \tau - \sin \omega_0 \tau}{(\omega_0 \tau)^2}$$

$$R_{\dot{X}X}(\tau) = \frac{d}{d\tau} R_{XX}(\tau) = \frac{n\sigma^2 \omega_0}{2} \cdot \frac{\omega_0 \tau \cos \omega_0 \tau - \sin \omega_0 \tau}{(\omega_0 \tau)^2}$$

) odd fcn

$$R_{\dot{X}\dot{X}}(\tau) = -\frac{d^2}{d\tau^2} R_{XX}(\tau) = \frac{\omega_0^2 n\sigma^2}{2} \cdot \frac{\omega_0^2 \tau^2 \sin \omega_0 \tau + 2\omega_0 \tau \cos \omega_o \tau - 2 \sin \omega_o \tau}{(\omega_0 \tau)^3}$$

even