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Dept. of Civil and Environmental Engineering

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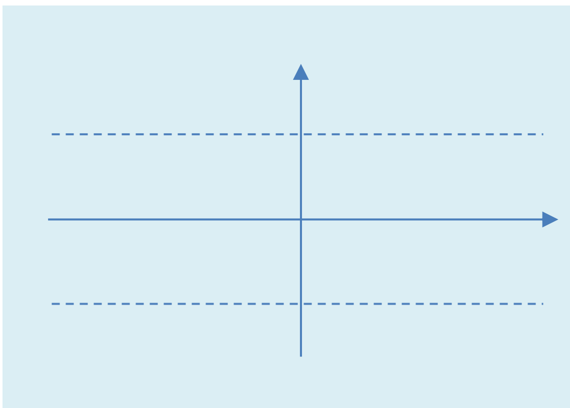
457.643 Structural Random Vibrations
In-Class Material: Class 11

II-2. Stochastic Calculus (contd.)

◎ Example

$X(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$ where $\mathrm{E}[A] = \mathrm{E}[B] = 0$, $\mathrm{E}[A^2] = \mathrm{E}[B^2] = \sigma^2$, and $\rho_{AB} = 0$

It was shown that $\mu_X(t) = 0$ and $\phi_{XX}(t_1, t_2) = R_{XX}(\tau) = \sigma^2 \cos\omega_0\tau$
(“   ” process)



$$\Phi_{XX}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XX}(\tau)\exp(-i\omega\tau)d\tau$$

$$= \frac{\sigma^2}{2\pi}\int_{-\infty}^{\infty} \cos\omega_0\tau \cdot \cos\omega\tau\, d\tau$$

$$= \frac{\sigma^2}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2}[\qquad\qquad\qquad\qquad\qquad\qquad]\, d\tau$$

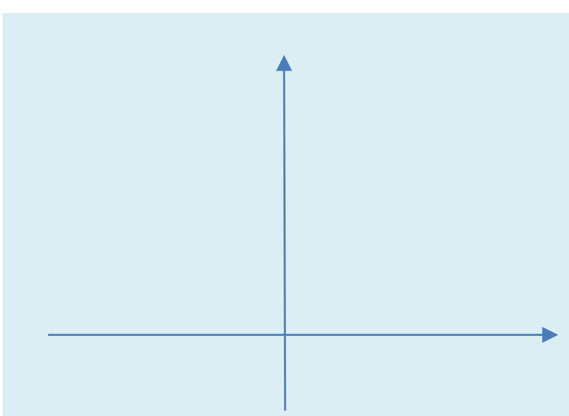
$$= \frac{\sigma^2}{2\pi}\int_{0}^{\infty} [\qquad\qquad\qquad\qquad\qquad\qquad]\, d\tau$$

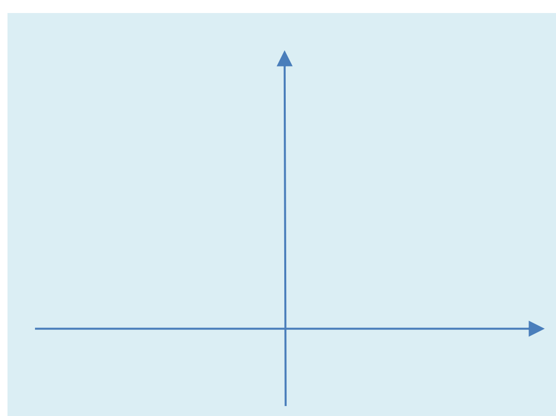
$$= \frac{\sigma^2}{2\pi}\left[\frac{\qquad\qquad}{} + \frac{\qquad\qquad}{}\right]_{\tau=0}^{\infty}$$

Note: $\left.\frac{\sin\omega t}{\omega}\right|_{t=0}^{\infty} = \pi \cdot \delta(\omega)$

Therefore,

$$\Phi_{XX}(\omega) = \frac{\sigma^2}{2\pi}[\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)] = \frac{\sigma^2}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$





$\int_{-\infty}^{\infty} \Phi_{XX}(\omega)d\omega = \mathrm{E}[\quad] =$

$\int_{0}^{\infty} G_{XX}(\omega)d\omega = \mathrm{E}[\quad] =$

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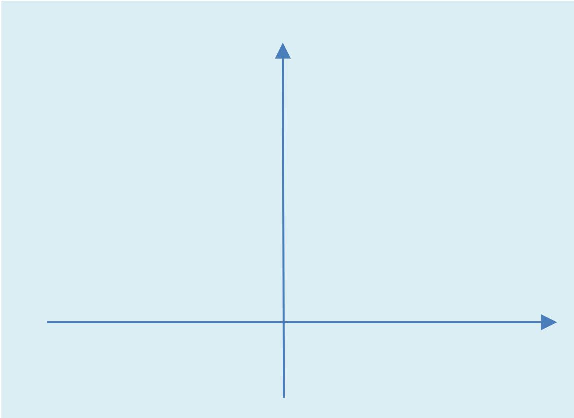
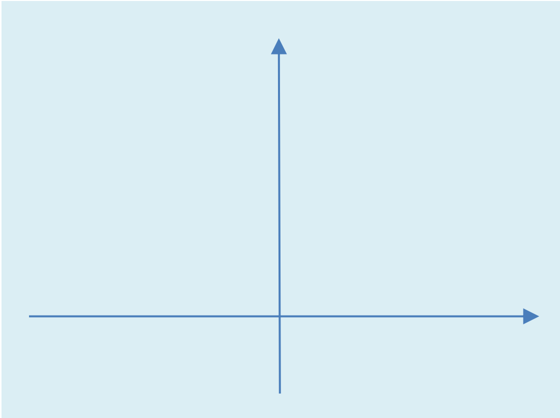
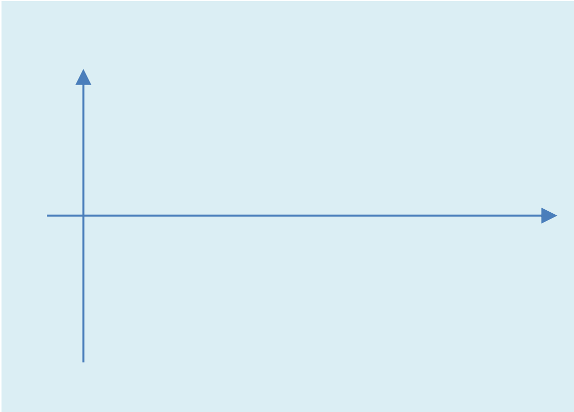
◎ Special processes

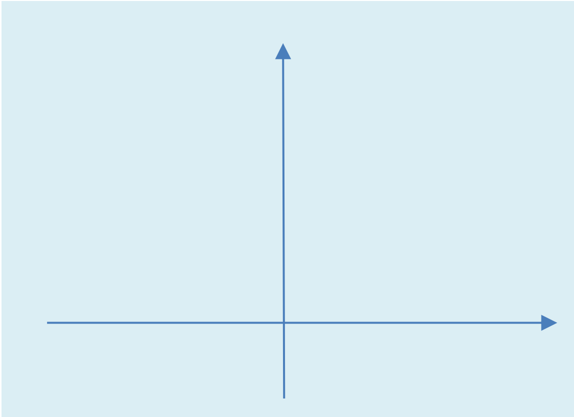
1) Narrow-band process (Example above is the ideal narrow-band process)

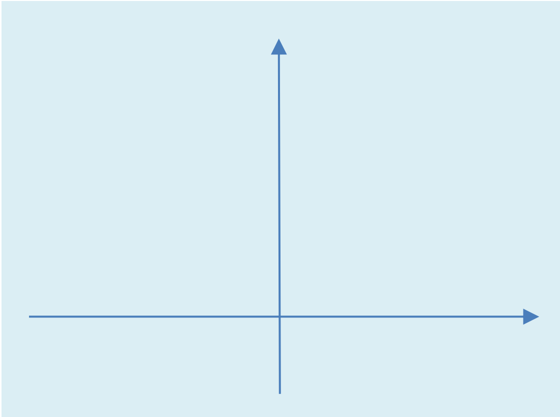






2) Wide-band process





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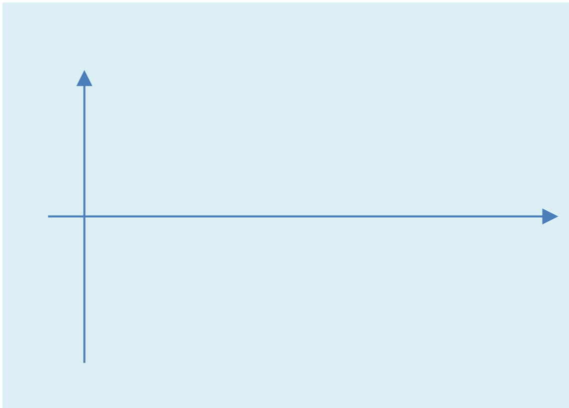
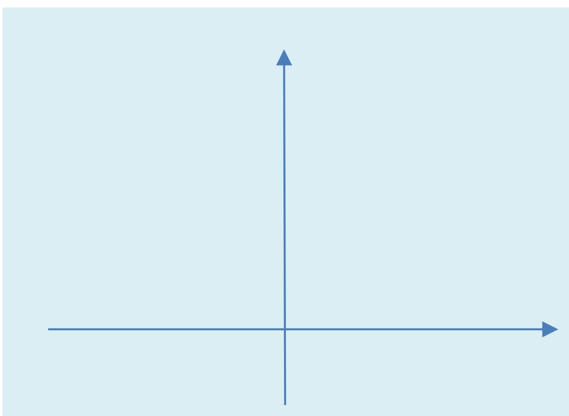


3) White noise (ideal wide-band process)



$$R_{XX}(\tau) = \int_{-\infty}^{\infty} \Phi_{XX}(\omega)\exp(\mathrm{i}\omega\tau)d\omega = \int_{-\infty}^{\infty} \Phi_0 \exp(\mathrm{i}\omega\tau)\, d\omega = 2\pi\Phi_0\delta(\tau)$$

Note: $1 = \int_{-\infty}^{\infty} \delta(\tau)e^{-i\omega\tau}\, d\tau$ and thus $\delta(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1 \cdot e^{i\omega\tau} d\omega$

※ "Shot Noise"

$\mu_X(t) = 0$ and

$\kappa_{XX}(t_1, t_2) = \phi_{XX}(t_1, t_2) = I(t_1) \cdot \delta(t_1 - t_2) = I(t_1) \cdot \delta(\tau)$

Here $I(t)$ is time-varying "intensity function."

Therefore, a ___________ ____________ shot noise is a ___________ ___________

That is, $I(t_1) = I = 2\pi\Phi_0$ for WN

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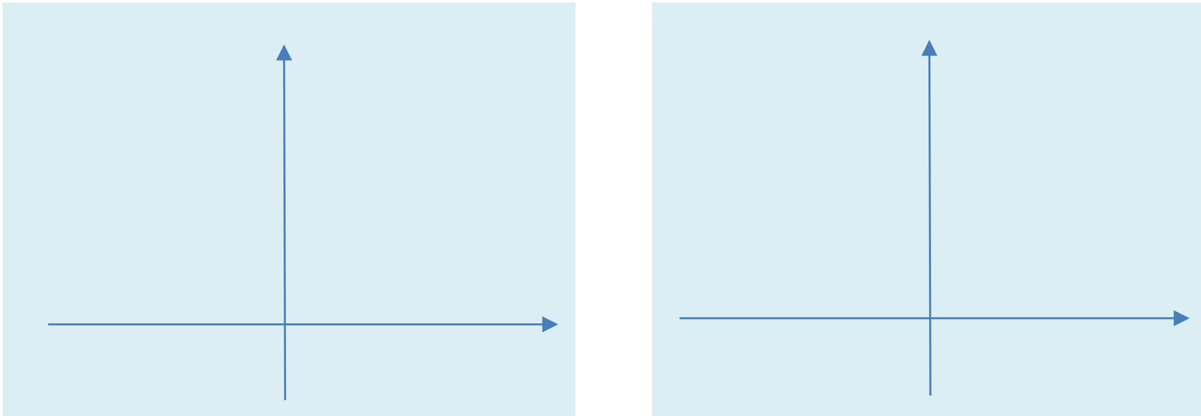
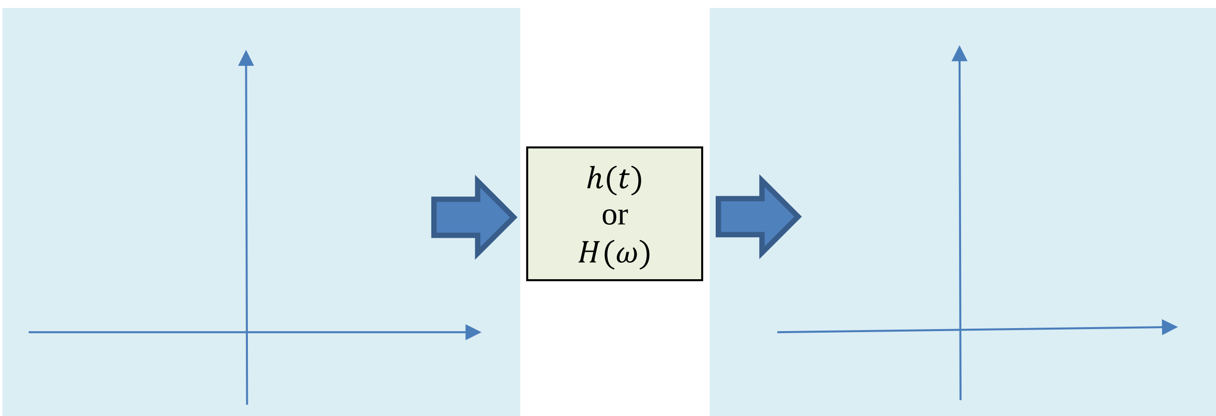
4) “Banded” white noise (more realistic WN)



5) “Filtered” white noise



e.g. SDOF oscillator (Kanai-Tajimi filter)

◎ Cross PSD

Consider jointly stationary processes $X(t)$ and $Y(t)$, i.e.

$$\phi_{XY}(t_1, t_2) = R_{XY}(\tau)$$

Cross PSD of $X(t)$ and $Y(t)$ is defined as

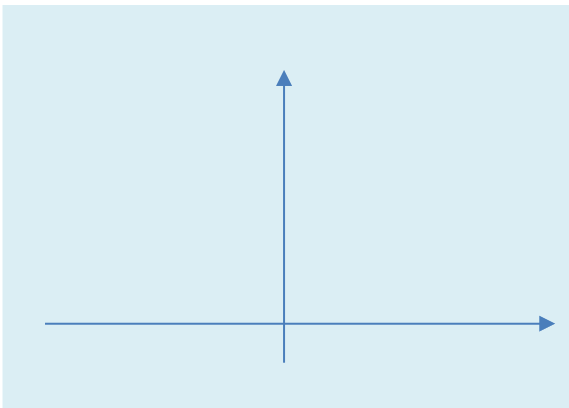
$$\Phi_{XY}(\omega) \equiv \lim_{T\to\infty} \frac{2\pi}{T} E[\bar{X}(\omega, T)\bar{Y}^*(\omega, T)]$$

One can show

$$\Phi_{XY}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XY}(\tau)\exp(-\mathrm{i}\omega\tau)d\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XY}(\tau) \qquad\qquad d\tau - i\frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XY}(\tau) \qquad\qquad d\tau$$

"co-spectrum" "quad-spectrum"



Properties of Cross PSD $\Phi_{XY}(\omega)$

1) Hermitian

   $\Phi_{XY}(\omega) = \Phi_{YX}^{*}(\omega)$

   Note: $\mathrm{Re}\,\Phi_{\mathrm{XY}}(\omega) = \mathrm{Re}\Phi_{XY}(-\omega)$, $\mathrm{Im}\,\Phi_{\mathrm{XY}}(\omega) = -\mathrm{Im}\,\Phi_{XY}(-\omega)$

   Note: $\mathrm{E}[X(t)\cdot Y(t)] = R_{XY}(\quad) = \int_{-\infty}^{\infty} \qquad\qquad d\omega$

   cf. $\mathrm{E}[X^2(t)] =$

2) If $\lim_{\omega\to\infty} \omega \cdot \mathrm{Re}\,\Phi_{XY}(\omega) = 0$, $\mathrm{E}[X(t)\cdot Y(t)]$ is ____________.

3) $\mathrm{Im}\,\Phi_{XY}(0) =$

   $$\mathrm{Re}\,\Phi_{XY}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XY}(\tau)\exp(-i\cdot 0\cdot\tau)\,d\tau =$$