Instructor: Junho Song
junhosong@snu.ac.kr

457.212 Statistics for Civil & Environmental Engineers
In-Class Material: Class 12
Useful Distribution Models (A&T: 3.2)

<< Bernoulli Trials >>

- Each trial has only (   ) possible **outcomes**
  ~ 'success'/'failure', 'occur'/'do not occur'
- Probability of occurrence of the event in each trial is (   ).
- The probabilities in different trials are **statistically** (   ).
  e.g. tossing coins repeatedly.
  occurrence of flooding each year.

1. **Binomial** distribution

(a) X ~ Binomial (n, p)

- **number of occurrences** of an event of specified probability p during n Bernoulli trials.

(b) PMF

$$p_X(x) = \binom{n}{x} \cdot p^x (1-p)^{n-x}$$

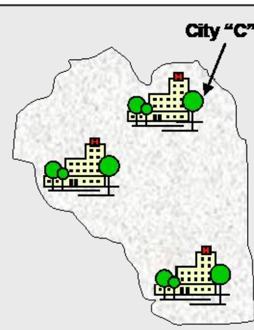
where $\binom{n}{x}$ is the binomial coefficient, i.e. $\dfrac{n!}{(n-x)!x!}$

(c) Mean: $\mathrm{E}[X] = np$

(d) Variance: $\sigma_X^2 = np(1-p)$

```
dbinom(1, size=3, prob=0.1) # PMF of binomial dist.
# probability of x=1 occurrence from n=3 trials with p = 0.1
```

Example 1: Three hospitals in City "C." The probability that each hospital will experience the shortage of electricity is 0.1 and they are statistically independent. X is the number of hospital(s) that has electricity shortage.



$P(X=0) =$
$P(X=1) =$
$P(X=2) =$
$P(X=3) =$

X is a Binomial random variable with $n =$, $p =$.
PMF:

Mean, $\mu_X =$, and standard deviation, $\sigma_X =$

Instructor: Junho Song
junhosong@snu.ac.kr

2. **Geometric** distribution

(a) $X \sim$ Geometric (p)

- number of trials **until the** (   ) or (   ) occurrence of an event of specified probability p

(b) PMF: $p_X(x) = (1-p)^{x-1} p$ because (x–1) no occurrences and then one occurrence

(c) Mean: $\mu_X = \mathrm{E}[X] = \sum_{x=1}^{\infty} x \cdot (1-p)^{x-1} p = \dfrac{1}{p}$

→ **Return period**: average number of trials until the first (next) occurrence

(d) Variance: $\sigma_X^2 = (1-p)/p^2$

3. **Negative Binomial** distribution

(a) $X \sim$ Negative Binomial (k, p)

- number of trials **until the** (   ) occurrence of an event of specified probability p

(b) PMF

$$p_X(x) = \binom{x-1}{k-1} p^{k-1} (1-p)^{x-k} \times p = \binom{x-1}{k-1} p^k (1-p)^{x-k}$$

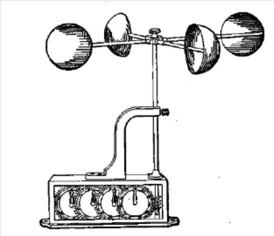
(c) Mean: $\mu_X = \mathrm{E}[X] = \sum_{x=1}^{\infty} x \cdot p_X(x) = \dfrac{k}{p}$

(d) Variance: $\sigma_X^2 = k \cdot (1-p)/p^2$

```
dgeom(3, prob=0.1) # PMF of geometric dist.
# probability that the event occurs for the first time in the x=3rd trial

dnbinom(3, size=2, prob=0.1) # PMF of negative binomial dist.
# probability that the event occurs for the second time(k=2) in the x=3rd
trial
```

Example 2: Event A: the annual maximum wind velocity is greater than 100 mph. P(A) = 0.2. The outcomes of all different years are statistically independent of each other.



(a) Bernoulli trials? Why? p = ?

(b) Probability that A occurs twice for a five-year duration?

(c) Probability that A occurs for the first time at the 7th year?

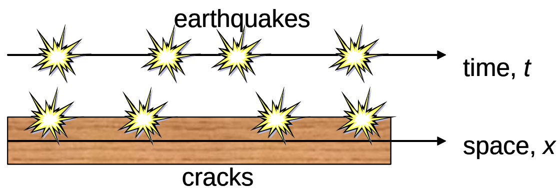
(d) Return period?

(e) Probability that it will take 10 years until the 3rd occurrence?

Instructor: Junho Song
junhosong@snu.ac.kr

<< Poisson Process >>

- An event can occur **at random** at any time or **any point** in space. (limiting case of Bernoulli trials as $n \to \infty$)
- The occurrence of an event in a given time (or space) interval is **statistically** (   ) of that in any other non-overlapping interval.
- The probability of occurrences of an event in a small interval Δt is proportional to Δt, i.e. **Prob.** = $\nu\Delta t$ where ν is the **constant** mean occurrence rate, i.e. (average # of occurrences)/(length)



4. **Poisson** distribution

(a) $X \sim$ Poisson (ν)

- **number of the (   )** of an event in time duration t for a Poisson process with ν.

(b) PMF

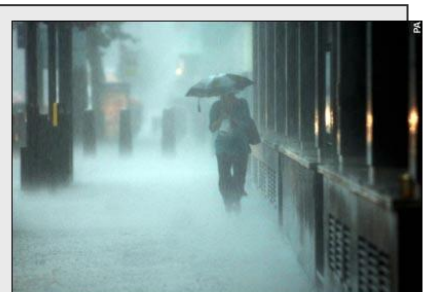
$$p_X(x) = \frac{(\nu t)^x}{x!}\exp(-\nu t), \quad x = 0,1,\ldots$$

(c) Mean: $\mu_X = \text{E}[X] = \sum_{x=0}^{\infty} x \cdot p_X(x) = \nu \cdot t$

(d) Variance: $\sigma_X^2 = \nu \cdot t$

```
dpois(2, lambda=3) # PMF of Poisson dist.
# Probability that the event occurs twice during the interval t for a
Poisson process with the occurrence rate nu (lambda = nu*t)
```

Example 3: Assume the occurrences of “heavy” rainstorms follow a Poisson process. From historical data, four “heavy” rainstorms occur per year in average.



(a) The mean occurrence rate?

(b) Probability of three storms for two years?

(c) Probability of at least one rainstorm for two years?

Instructor: Junho Song
junhosong@snu.ac.kr

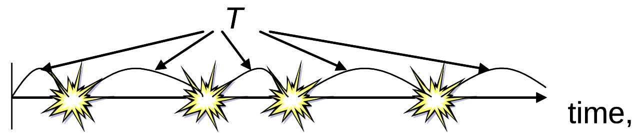
5. **Exponential** distribution

(a) T ~ Exponential (ν)

- **waiting time until the (   ) or (   ) occurrence** in a Poisson process with ν.

(b) CDF

$$F_T(t) = P(T \le t)$$
$$= P(\text{at least one occurrence in } t)$$
$$= 1 - \frac{(\nu t)^0}{0!}\exp(-\nu t)$$
$$= 1 - \exp(-\nu t)$$

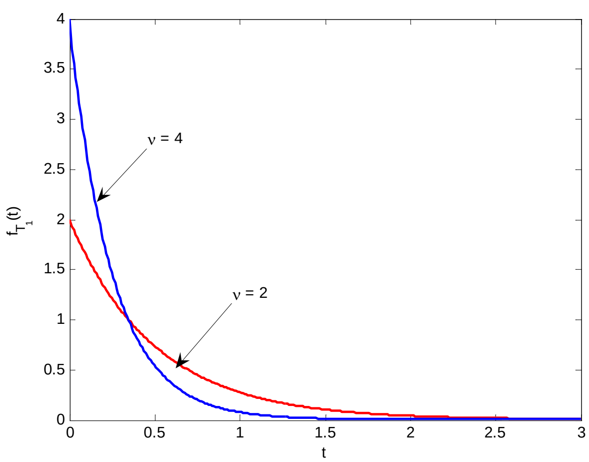


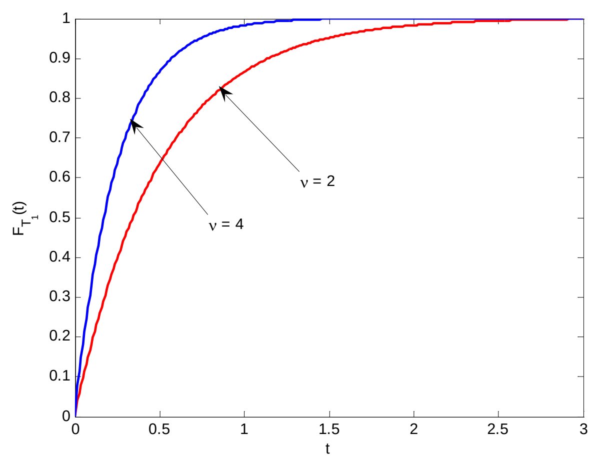
(c) PDF

$$f_T(t) = \frac{dF_T(t)}{dt}$$
$$= \nu \exp(-\nu t)$$

(c) Mean: $\mu_T = \mathrm{E}[T] = \int_0^\infty t \cdot \nu \exp(-\nu t) dt = \frac{1}{\nu}$ → **Return Period** (c.f. 1/p for Bernoulli)

(d) Variance: $\sigma_T^2 = \frac{1}{\nu^2}$






```
dexp(1.5, rate=2) # PDF of Exponential dist.
# Probability density at the waiting time = 1.5 for a Poisson process with
the mean occurrence rate 2
```

Instructor: Junho Song
junhosong@snu.ac.kr

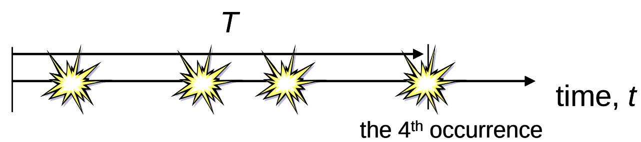
6. **Gamma** distribution

(a) T ~ Gamma (ν, k)

- **waiting time until the (   ) occurrence** in a Poisson process with ν.

(b) CDF:

$$F_T(t) = 1 - \sum_{x=0}^{k-1} \frac{(\nu t)^x}{x!} \exp(-\nu t)$$

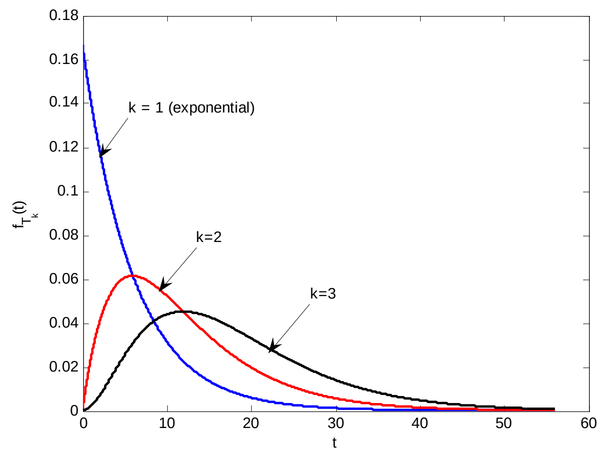


(c) PDF:

$$f_T(t) = \frac{\nu(\nu t)^{k-1}}{\Gamma(k)} \exp(-\nu t)$$

(d) Mean: $\mu_T = \mathrm{E}[T] = \frac{k}{\nu}$

(d) Variance: $\sigma_T^2 = \frac{k}{\nu^2}$




```
dgamma(1.5, shape=3, rate=2) # PDF of Gamma dist.
# Probability density at the waiting time until the 3rd occurrence(k=3) =
1.5 for a Poisson process with the mean occurrence rate 2
```

<< Summary >>

	Bernoulli Trials	Poisson Process
No. of **total trials** (or time intervals Δt)	*n* trials, or until the *k*-th occurrence	$n \rightarrow \infty$, $\Delta t \rightarrow 0$ any point on a continuous axis (time, space)
P(**no. of occurrence** = *x*)	**Binomial** (*n*, *p*) during *n* trials	**Poisson** (ν) during time period *t*
P(no. of trials or waiting time **until the first** (next) occurrence = *t*)	**Geometric** (*p*)	**Exponential** (ν)
P(no. of trials or waiting time **until the *k*-th** occurrence = *t*)	**Negative Binomial** (*k*, *p*)	**Gamma** (*k*, ν)
Return period: average required no. of trials or waiting time until the next occurrence	1/*p* where *p* is the probability of the event	1/ν where ν is the mean occurrence rate

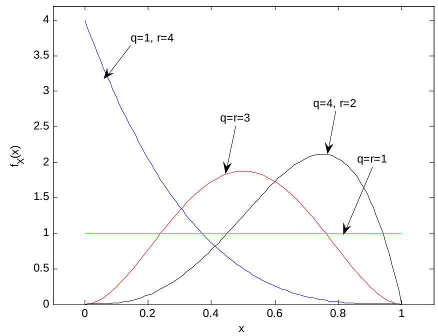
Instructor: Junho Song
junhosong@snu.ac.kr

7. **Beta** distribution

(a) X ~ Beta (a, b, q, r)

A random variable whose values are **bounded** between finite limits *a* and *b*. Its shape is determined by the combination of *q* and *r*.

(b) PDF



$$f_X(x) = \frac{1}{B(q,r)} \frac{(x-a)^{q-1}(b-x)^{r-1}}{(b-a)^{q+r-1}}, \quad a \le x \le b$$

$$= 0 \qquad \text{elsewhere}$$

where $B(q,r) = \int_0^1 x^{q-1}(1-x)^{r-1}dx = \frac{\Gamma(q)\Gamma(r)}{\Gamma(q+r)}$

```
dbeta(0.2, shape1=1, shape2=4) # PDF of Beta dist.
# shape1 and shape2 are q and r parameters, respectively
# R base package only support Beta distributions with (a,b)=(0,1) only
```