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457.643 Structural Random Vibrations
In-Class Material: Class 12

II-2. Stochastic Calculus (contd.)

◎ Cross PSD (contd.)

※ Application example of cross PSD

Der Kiureghian, A. (1996). A coherency model for spatially varying ground motions. *Earthquake Engineering and Structural Dynamics*, 25:99-111.

Coherency function of ground acceleration processes $a_k(t)$ and $a_l(t)$ at stations k and l:

$$\gamma_{kl}(\omega) = \frac{G_{a_k a_l}(\omega)}{\sqrt{G_{a_k a_k}(\omega) \cdot G_{a_l a_l}(\omega)}}$$

Using the coherency function, one can characterize

(1) Incoherence effect: scattering of waves in the heterogeneous medium and differential superpositioning of waves

(2) Wave passage effect: delay in the arrival of the wave

(3) Attenuation effect: amplitude decreases due to geometric spreading, material damping and wave scattering

◎ PSD of derivative process

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} \Phi_{XX}(\omega) e^{i\omega\tau} d\omega$$

$$R_{\dot{X}X}(\tau) = \frac{dR_{XX}(\tau)}{d\tau}$$

$$= \int_{-\infty}^{\infty} (\quad) \Phi_{XX}(\omega)\, e^{i\omega\tau} d\omega$$

Comparing the two equations above, we note

$$\Phi_{\dot{X}X}(\omega) =$$

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It is also seen that

$\Phi_{X\dot{X}}(\omega) =$

We also know that

$$R_{\dot{X}\dot{X}}(\tau) = -\frac{d^2 R_{XX}(\tau)}{d\tau^2}$$

$$= -\int_{-\infty}^{\infty} (\qquad)^2 \Phi_{XX}(\omega) e^{i\omega\tau} d\omega$$

$$= \int_{-\infty}^{\infty} \qquad\qquad e^{i\omega\tau} d\omega$$

Therefore,

$\Phi_{\dot{X}\dot{X}}(\omega) =$

In general,

$$\Phi_{X^{(m)}Y^{(n)}}(\omega) = (i\omega)^m (-i\omega)^n \Phi_{XY}(\omega)$$

◎ Generation of artificial time histories by PSD

e.g. "Spectral representation" method

Shinozuka & Deodatis 1991: Stationary & Gaussian

Deodatis & Micaletti 2001: Non-Gaussian

$$X(t) = \sum_{i=1}^{n} a_i \cos(\omega_i t + \theta_i)$$

- a_i: contribution from the frequency ω_i ~ determined by ______
- ω_i: closely-spaced frequency values (>0), deterministic.
- θ_i: random phase angle ~ $\mathrm{U}(0,2\pi]$
- θ_i and θ_j are statistically independent $(i \neq j)$

Check

1) $\mathrm{E}[X(t)]$

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$$\mathrm{E}[\cos(\omega_i t + \theta_i)] = \int_0^{2\pi} \cos(\omega_i t + \theta_i) \qquad d\theta$$

$$=$$

Therefore, $X(t)$ is a _____-______ process

2) $\phi_{XX}(t_1, t_2)$

$$\phi_{XX}(t_1, t_2) = \sum_{i=1}^{n}\sum_{i=1}^{n} a_i a_j \mathrm{E}\left[\cos(\omega_i t_1 + \theta_i)\cdot\cos\left(\omega_j t_2 + \theta_j\right)\right]$$

$(i \neq j)$

$$\mathrm{E}\left[\cos(\omega_i t_1 + \theta_i)\cdot\cos\left(\omega_j t_2 + \theta_j\right)\right] = \mathrm{E}[\cos(\omega_i t_1 + \theta_i)]\cdot\mathrm{E}\left[\cos\left(\omega_j t_2 + \theta_j\right)\right]$$

$$=$$

$(i = j)$

$$\mathrm{E}[\cos(\omega_i t_1 + \theta_i)\cdot\cos(\omega_i t_2 + \theta_i)] = \frac{1}{2}\left\{\mathrm{E}[\cos(\omega_i(t_1 + t_2) + 2\theta_i)] + \mathrm{E}\left[\cos\left(\omega_i(t_1 - t_2)\right)\right]\right\}$$

$$= \frac{1}{2}\cos\left(\omega_i(t_1 - t_2)\right)$$

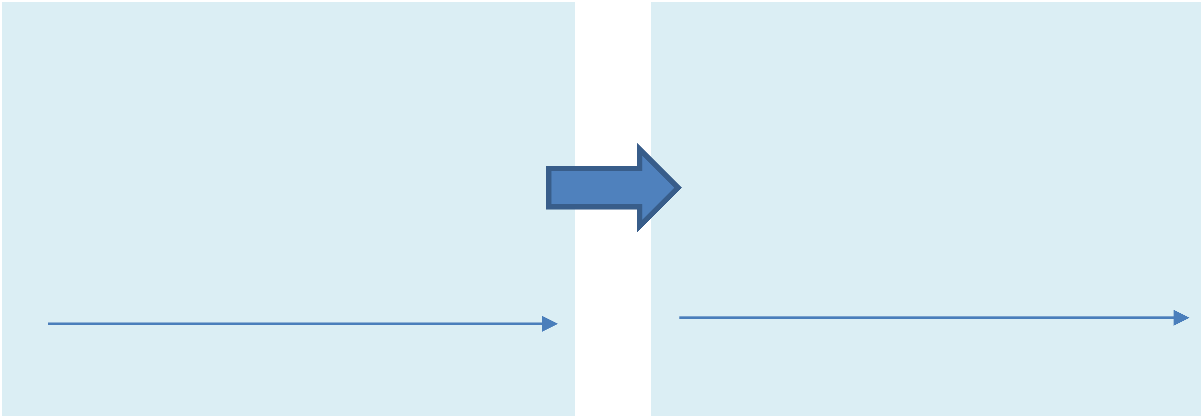
$= \frac{1}{2}\cos(\omega_i \quad)$ $\rightarrow X(t)$ is a __________ process

Therefore,

$$R_{XX}(\tau) = \frac{1}{2}\sum_{i=1}^{n} a_i^2\cos(\omega_i \tau)$$

$$\Phi_{XX}(\omega) = \frac{1}{4}\sum_{i=1}^{n} a_i^2[\delta(\omega + \omega_i) + \delta(\omega - \omega_i)]$$

Recall $R_{XX}(\tau) = \sigma^2 cos\omega_o\tau$ → $\Phi_{XX}(\omega) = \frac{\sigma^2}{2}[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)]$



Given PSD Spectral Representation

How to determine a_i?

… such that the powers $\mathrm{E}[X^2]$ in the corresponding intervals are equivalent

$$\int_{\frac{\omega_{i-1}+\omega_i}{2}}^{\frac{\omega_i+\omega_{i+1}}{2}} \Phi_{XX}(\omega)d\omega =$$

When $\Delta\omega$ is small,

$$\Phi_{\mathrm{XX}}(\omega_i) \cdot \frac{\omega_{i+1}-\omega_{i-1}}{2} =$$

Therefore,

$$\therefore \quad a_i = 2\sqrt{\Phi_{XX}(\omega_i) \cdot \frac{\omega_{i+1}-\omega_{i-1}}{2}}$$
$$= 2\sqrt{\Phi_{XX}(\omega_i) \cdot \quad}$$

The generated process has Gaussianity and Ergodicity (proof: Deodatis 2001)

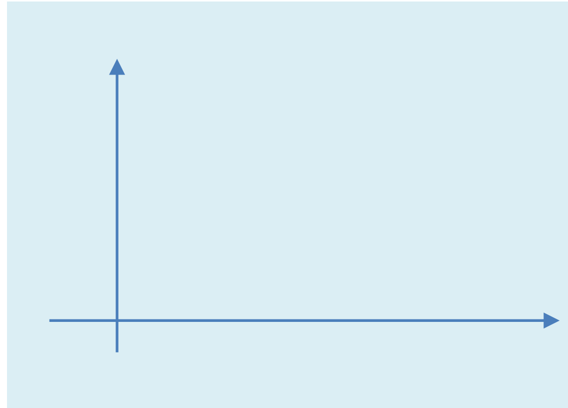
◎ Spectral moments

- VanMarcke (1972, ASME JEM): first introduced
- Michaelov et al. (1999, Structural Safety): a good summary and extension to nonstationary case

➔ Partial descriptors of PSD (cf. partial descriptors of PDF)

The m-th order spectral moment is defined as

$$\lambda_m = \int_0^\infty \omega^m G_{XX}(\omega)d\omega$$



1) Help compute variances easily

$$\lambda_0 = \int_0^\infty G_{XX}(\omega)d\omega =$$

$$\lambda_2 = \int_0^\infty \omega^2 G_{XX}(\omega)d\omega =$$

$$\lambda_4 = \int_0^\infty \omega^4 G_{XX}(\omega)d\omega =$$

$$\lambda_{2n} = \int_0^\infty \omega^{2n} G_{XX}(\omega)d\omega =$$

2) Capture frequency-related characteristics of a random process

(in analogy to spectral moments $\mathrm{E}[X^m]$ capturing characteristics of a random process)

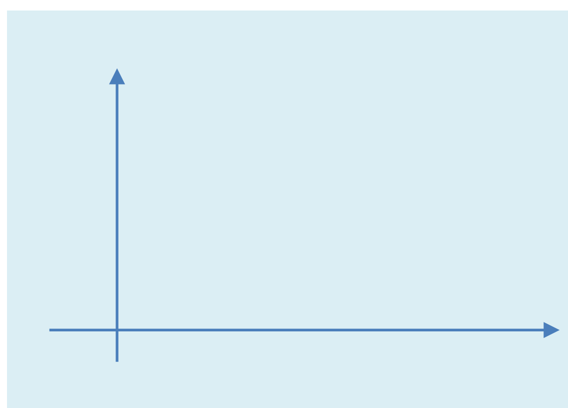
- Central frequency

$$\omega_c = \frac{\lambda_1}{\lambda_0} = \frac{\int_0^\infty \omega G_{XX}(\omega)d\omega}{\int_0^\infty G_{XX}(\omega)d\omega}$$

In analogy to the mean

$$\mathrm{E}[X] = \frac{\int_{-\infty}^\infty x \cdot f_X(x)dx}{\int_{-\infty}^\infty f_X(x)dx} =$$

Geometric "center" of probability density function



Central frequency: centroid of PSD

- Normalized radius of gyration of PSD

$$s = \frac{\sqrt{\frac{\lambda_2}{\lambda_0} - \left(\frac{\lambda_1}{\lambda_0}\right)^2}}{\omega_c}$$

$$= \frac{\sqrt{\frac{\lambda_2}{\lambda_0} - \omega_c^2}}{\omega_c}$$

$$= \boxed{\sqrt{\frac{\lambda_0 \lambda_2}{\lambda_1^2} - 1}}, \quad 0 \le s < \infty$$

- Bandwidth factor

$$\delta = \frac{s}{\sqrt{s^2 + 1}}$$

If $s \to 0, \delta =$

$s \to \infty, \delta =$

Therefore, $0 \le \delta \le 1$

If $s = \sqrt{\frac{\lambda_0 \lambda_2}{\lambda_1^2} - 1}$ is substituted,

$$\boxed{\delta = \sqrt{1 - \frac{\lambda_1^2}{\lambda_0 \lambda_2}}}$$