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457.643 Structural Random Vibrations
In-Class Material: Class 16

III-2. Random Vibration Analysis of Linear Structures

◎ Response of a linear system to a stochastic input process

Deterministic input

$$a_n \frac{d^n x}{dt^n} + \cdots + a_1 \frac{dx}{dt} + a_0 x = p(t)$$

$$x(t) = \sum_{i=0}^{n-1} x^{(i)}(0) g_i(t) + \int_0^t p(\tau) h_p(t-\tau) d\tau$$

$$\left(x(t) = \sum_{i=0}^{n-1} x^{(i)}(0) g_i(t) + \int_0^t f(\tau) h_f(t-\tau) d\tau \right)$$

Stochastic input

$$a_n \frac{d^n X}{dt^n} + \cdots + a_1 \frac{dX}{dt} + a_0 X = P(t)$$

$$X(t) = \sum_{i=0}^{n-1} X^{(i)}(0) g_i(t) + \int_0^t P(\tau) h_p(t-\tau) d\tau$$

◎ Example: stochastic response of standard SDOF oscillator

$$\ddot{x}(t) + 2\xi\omega_0 \dot{x}(t) + \omega_0^2 x = f(t)$$

$$x(t) = x(0) g(t) + \dot{x}(0) h(t) + \int_0^t f(\tau) h(t-\tau) d\tau$$

- $g(t) = e^{-\xi\omega_0 t} \cdot \left(\cos\omega_D t + \frac{\xi}{\sqrt{1-\xi^2}} \sin\omega_D t \right) \cdot U(t)$ → $g_0(t)$ above
- $h(t) = \frac{1}{\omega_D} e^{-\xi\omega_0 t} \cdot \sin\omega_D t \cdot U(t)$ → $g_1(t)$ above

When there exists randomness in the initial conditions and the external force, the response is a stochastic process, i.e.

$$X(t) = S_1 \cdot g(t) + S_2 \cdot h(t) + \int_0^t F(\tau)h(t-\tau)d\tau$$

Question: $\mu_X(t)$, $\phi_{XX}(t_1, t_2)$, …?

Assuming the integral (↗) exists in the mean-square sense, we can derive the moment functions as follows.

1) $\mu_X(t) = \mathrm{E}[X(t)]$

$$= \quad \cdot g(t) + \quad \cdot h(t) + \int_0^t \quad h(t-\tau)d\tau$$

2) $\phi_{XX}(t_1, t_2) = \mathrm{E}[X(t_1)X(t_2)]$

$$= \quad \cdot g(t_1)g(t_2) + \quad \cdot h(t_1)h(t_2) + \quad \cdot \{g(t_1)h(t_2) + g(t_2)h(t_1)\}$$

$$+ \int_0^{t_1}\int_0^{t_2} \phi_{FF}(\tau_1, \tau_2)h(t_1-\tau_1)h(t_2-\tau_2)d\tau_2 d\tau_1$$

$$+\mathrm{E}\left\{[S_1 g(t_1) + S_2 h(t_1)]\int_0^{t_2} F(\tau_2)h(t_2-\tau_2)d\tau_2\right\}$$

$$+\mathrm{E}\left\{[S_1 g(t_2) + S_2 h(t_2)]\int_0^{t_1} F(\tau_1)h(t_1-\tau_1)d\tau_1\right\}$$

3) $\kappa_{XX}(t_1, t_2) = COV[X(t_1), X(t_2)] = \mathrm{E}\left[\left(X(t_1) - \mu_X(t_1)\right) \cdot \left(X(t_2) - \mu_X(t_2)\right)\right]$

$$= \int_0^{t_1}\int_0^{t_2} \kappa_{FF}(\tau_1, \tau_2)h(t_1-\tau_1)h(t_2-\tau_2)d\tau_2 d\tau_1$$

$$+\sigma_{S_1}^2 g(t_1)g(t_2) + \sigma_{S_2}^2 h(t_1)h(t_2) + COV[S_1, S_2] \cdot [g(t_1) \cdot h(t_2) + g(t_2) \cdot h(t_1)]$$

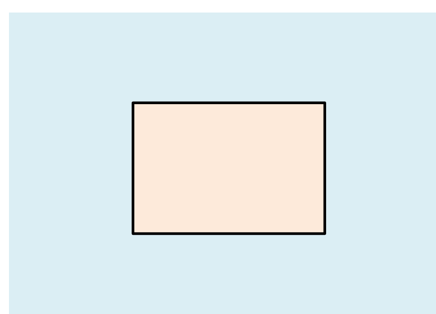
$+$ terms involving covariances between S_1 and F, and those between S_2 and F (usually zero)

◎ Response of a linear system under multiple stochastic inputs

Assuming zero IC's for simplicity, suppose a linear system is subjected to multiple stochastic loads

$F_1(t), F_2(t), \cdots$

Then, the stochastic response and its moment functions are



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- $X(t) = \sum_{i=1}^{n} \int_0^t F_i(\tau) h_i(t-\tau) d\tau$
- $\mu_X(t) = \sum_{i=1}^{n} \int_0^t \mu_{F_i}(\tau) h_i(t-\tau) d\tau$
- $\phi_{XX}(t_1, t_2) = \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^{t_1} \int_0^{t_2} \phi_{F_i F_j}(\tau_1, \tau_2) h_i(t_1-\tau_1) h_j(t_2-\tau_2) d\tau_2 d\tau_1$
- $\kappa_{XX}(t_1, t_2) = \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^{t_1} \int_0^{t_2} \kappa_{F_i F_j}(\tau_1, \tau_2) h_i(t_1-\tau_1) h_j(t_2-\tau_2) d\tau_2 d\tau_1$

If $F_i(t)$ and $F_j(t)$ $(i \neq j)$ are statistically independent of each other, the double summation becomes

◎ Cross covariance between response and excitation

$$\kappa_{XF}(t_1, t_2) = \mathrm{E}[(X(t_1) - \qquad)(F(t_2) - \qquad)]$$

$$X(t_1) - \mu_X(t_1) = \int_0^{t_1} F(\tau) h(t_1 - \tau) d\tau - \int_0^{t_1} \mu_F(\tau) h(t_1 - \tau) d\tau =$$

$$\therefore \kappa_{XF}(t_1, t_2) = \int_0^{t_1} \mathrm{E}\{[F(\tau) - \mu_F(\tau)][F(t_2) - \qquad]\} h(t_1 - \tau) d\tau$$

Therefore,

$$\kappa_{XF}(t_1, t_2) = \int_0^{t_1} \kappa_{FF}(\quad , \quad) h(t_1 - \tau) d\tau$$

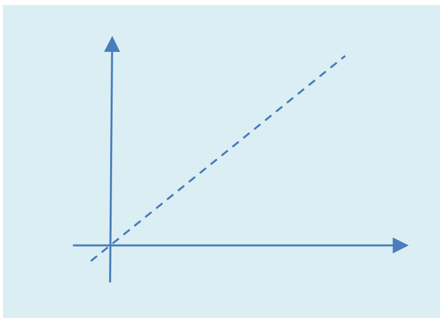
for $t_1 \geq t_2$

When $t_1 < t_2$, $\kappa_{XF}(t_1, t_2) =$

◎ Example: Response to shot noise (Delta-correlated process)

Recall: Shot noise is white noise with time-varying intensity

- $\mu_F(t) =$
- $\kappa_{FF}(t_1, t_2) = I(t_1)\delta(t_1 - t_2)$



When $I(t) = I$, i.e. constant, the shot noise becomes __________ noise

Assuming zero IC's

$$\kappa_{XX}(t_1,t_2) = \int_0^{t_2}\int_0^{t_1} \kappa_{FF}(\tau_1,\tau_2)h(t_1-\tau_1)h(t_2-\tau_2)d\tau_1 d\tau_2$$

$$= \int_0^{t_2}\int_0^{t_1} \qquad\qquad h(t_1-\tau_1)h(t_2-\tau_2)d\tau_1 d\tau_2$$

$$= \int_0^{\min(t_1,t_2)} I(\tau)h(t_1-\tau)h(t_2-\tau)d\tau$$

Example: Massless SDOF oscillator under shot noise

Equation of motion:

$$c\dot{x} + kx = p(t)$$

$$\dot{x} + \alpha x = f(t)$$

where $\alpha = k/c$ and $f(t) = p(t)/c$

Characterization of the system: impulse response function?

$$\dot{h}(t) + \alpha h(t) = \delta(\mathrm{t})$$

Initial condition: $h(0^+) = \frac{1}{1} =$

Homogeneous solution:

Set $h(t) = e^{rt}$

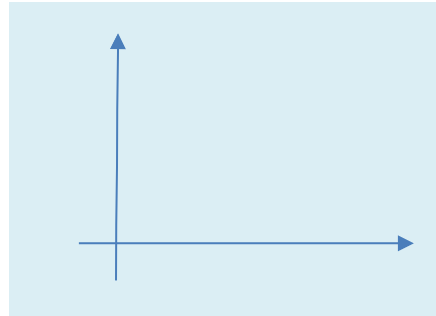
$r =$

Therefore,

$$h(t) = A \cdot e^{-\alpha t}$$

Applying IC, the impulse response function is $h(t) =$ for t 0

Suppose the intensity function of the shot noise $F(t)$ is given as



$I(t) = I$ for $0 < t \leq t_0$ and 0 otherwise

Then,

$$\kappa_{XX}(t_1, t_2) = \int_0^{\min(t_1,t_2)} I(\tau)h(t_1-\tau)h(t_2-\tau)d\tau$$

$$= I\int_0^{\min(t_0,t_1,t_2)} e^{-\alpha(t_1-\tau)} \cdot e^{-\alpha(t_2-\tau)}d\tau$$

$$= I \cdot e^{-\alpha(t_1+t_2)} \int_0^{t^*} e^{2\alpha\tau}d\tau$$

$$= \frac{I}{2\alpha} \cdot e^{-\alpha(t_1+t_2)}[\exp(2\alpha t^*) - 1]$$

$\kappa_{XX}(t,t) = \sigma_X^2 = ?$

For $t \leq t_0$, i.e. $t^* = t$

$\sigma_X^2 =$

For $t > t_0$, i.e. $t^* = t_0$

$\sigma_X^2 =$