Seoul National University
Dept. of Civil and Environmental Engineering

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457.212 Statistics for Civil & Environmental Engineers

In-Class Material: Class 17

Mathematical Expectations of Linear Functions (A&T: 4.3; Supplement #3)

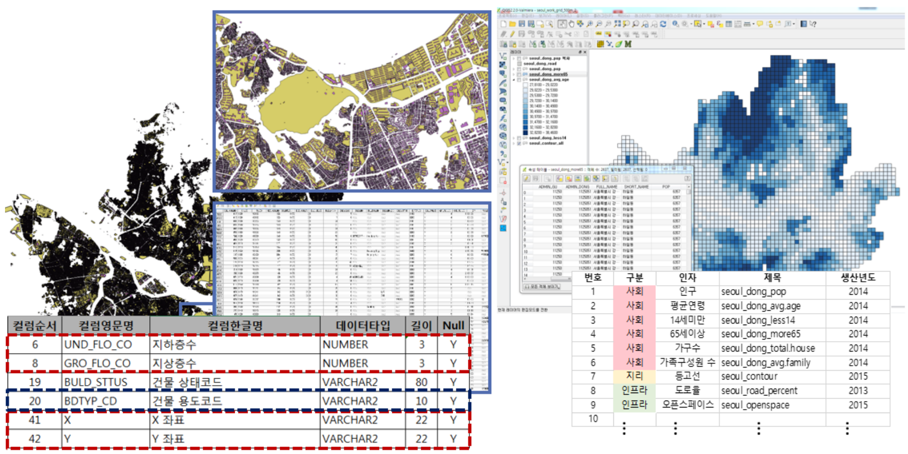
Linear Function: $Y_k = a_{k,0} + \sum_{i=1}^{n} a_{k,i} X_i \ , \ k = 1, \ldots, m$

Given: (   ), (   ) and (   )

Want to know: (   ), (   ) , (   ) and (   )

Useful when we are interested in the moments of the outputs only and/or the moments of the inputs are only available (not the type of the distribution).

Ex1) Seismic loss of buildings in a region: $L = \sum L_i$



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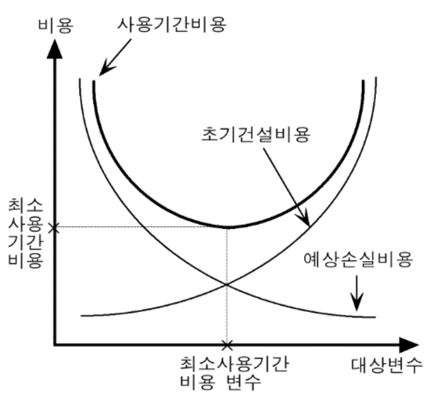
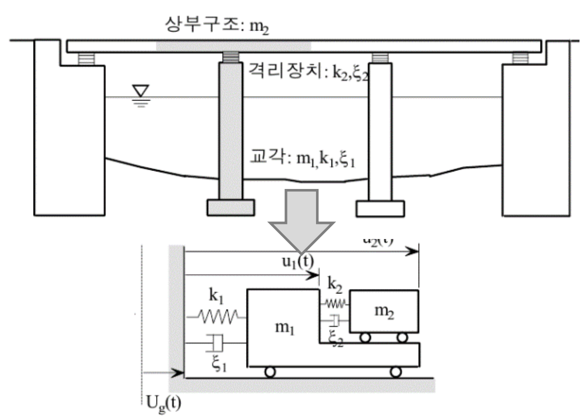
Ex2) Lifecycle cost of an infrastructure under hazards: $C_t = C_i + r \cdot \sum_{j=1}^{n} C_{d,j}$





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1. Linear Function of a Single Random Variable

$$Y = a_0 + a_1 \cdot X$$

(a) Mean of Y: $\mu_Y = a_0 + a_1\mu_X$

(b) Variance of Y: $\sigma_Y^2 = a_1^2\sigma_X^2$

(c) Standard deviation of Y: $\sigma_Y = |a| \cdot \sigma_X$

Example 1: Consider a building whose initial construction cost (c_i) is estimated as $100M and the discount rate (r) is 0.8. The uncertain damage cost (C_d) is a random variable whose mean and standard deviation are $40M and $10M, respectively. Find the mean and standard deviation of the lifecycle cost $C = c_i + r \cdot C_d$.

2. Linear Functions of Multiple Random Variables

$$Y_k = a_{k,0} + \sum_{i=1}^{n} a_{k,i} X_i \ , \ k = 1, \ldots, m$$

which means,

$$\begin{aligned}
Y_1 &= a_{1,0} + a_{1,1}X_1 + a_{1,2}X_2 + \cdots + a_{1,n}X_n \\
Y_2 &= a_{2,0} + a_{2,1}X_1 + a_{2,2}X_2 + \cdots + a_{2,n}X_n \\
&\vdots \\
Y_k &= a_{k,0} + a_{k,1}X_1 + a_{k,2}X_2 + \cdots + a_{k,n}X_n \\
&\vdots \\
Y_m &= a_{m,0} + a_{m,1}X_1 + a_{m,2}X_2 + \cdots + a_{m,n}X_n
\end{aligned}$$

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(a) Algebraic formula ($n \le 2$)

- Mean: $\mu_{Y_k} = a_{k,0} + \sum_{i=1}^{n} a_{k,i}\mu_{X_i}$
- Variance:

$$\sigma_{Y_k}^2 = \sum_{i=1}^{n} a_{k,i}^2\sigma_{X_i}^2 + \sum_{\substack{i=1 \\ i\neq j}}^{n}\sum_{j=1}^{n} a_{k,i}a_{k,j}\sigma_{X_i}\sigma_{X_j}\rho_{X_iX_j}$$

$$= \sum_{i=1}^{n} a_{k,i}^2\sigma_{X_i}^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_{k,i}a_{k,j}\sigma_{X_i}\sigma_{X_j}\rho_{X_iX_j}$$

- Covariance:

$$\mathrm{Cov}[Y_k, Y_l] = \sum_{i=1}^{n} a_{k,i}a_{l,i}\sigma_{X_i}^2 + \sum_{\substack{i=1 \\ i\neq j}}^{n}\sum_{j=1}^{n} a_{k,i}a_{l,j}\sigma_{X_i}\sigma_{X_j}\rho_{X_iX_j}$$

Example 2: Consider two functions

$$Y_1 = 2 + 3X_1 + 4X_2$$
$$Y_2 = 1 - X_1 + 2X_2$$

The means of X_1 and X_2 are 20 and 30, respectively. The coefficient of variation of each random variable is 10%. The correlation coefficient between X_1 and X_2 is 0.3.

(a) The means of Y_1 and Y_2 :

(b) The standard deviations of Y_1 and Y_2 :

(c) The covariance between Y_1 and Y_2 :

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(d) The correlation between Y_1 and Y_2:

(b) Matrix formula ($n \geq 3$)

The linear functions in (a) can be given in the following matrix form:

$$\mathbf{Y} = \mathbf{a}_0 + \mathbf{A}\mathbf{X}$$

in which

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{bmatrix}, \ \mathbf{a}_0 = \begin{bmatrix} a_{1,0} \\ a_{2,0} \\ \vdots \\ a_{m,0} \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & & & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \text{ and } \mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

Given: Mean vector and covariance matrix of $\mathbf{X}$

$$\mathbf{M}_\mathbf{X} = \begin{bmatrix} \mu_{X_1} \\ \mu_{X_2} \\ \vdots \\ \mu_{X_n} \end{bmatrix} \text{ and } \mathbf{\Sigma}_{\mathbf{XX}} = \begin{bmatrix} \mathrm{Cov}[X_1, X_1] & \mathrm{Cov}[X_1, X_2] & \cdots & \mathrm{Cov}[X_1, X_n] \\ \mathrm{Cov}[X_2, X_1] & \mathrm{Cov}[X_2, X_2] & \cdots & \mathrm{Cov}[X_2, X_n] \\ \vdots & & & \vdots \\ \mathrm{Cov}[X_n, X_1] & \mathrm{Cov}[X_n, X_2] & \cdots & \mathrm{Cov}[X_n, X_n] \end{bmatrix}$$

Then,

- Mean:

$$\mathbf{M}_\mathbf{Y} = \begin{bmatrix} \mu_{Y_1} \\ \mu_{Y_2} \\ \vdots \\ \mu_{Y_m} \end{bmatrix} = \mathbf{a}_0 + \mathbf{A}\mathbf{M}_\mathbf{X}$$

- Covariance matrix:

$$\mathbf{\Sigma}_{\mathbf{YY}} = \mathbf{A}\mathbf{\Sigma}_{\mathbf{XX}}\mathbf{A}^\mathrm{T}$$

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Example 2 (Contd.): Use the matrix formulas for the same problem:

$$Y_1 = 2 + 3X_1 + 4X_2$$
$$Y_2 = 1 - X_1 + 2X_2$$

The means of X_1 and X_2 are 20 and 30, respectively. The coefficient of variation of each random variable is 10%. The correlation coefficient between X_1 and X_2 is 0.3.

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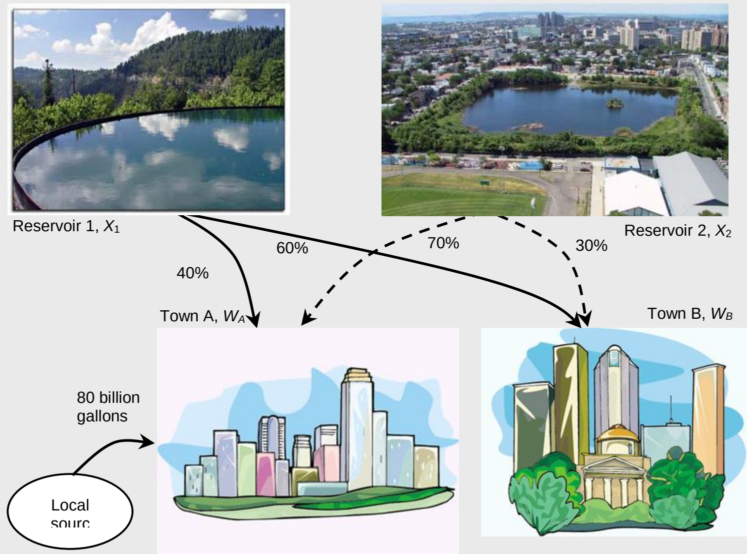
Example 3: Consider two towns (A and B) that get water supplies from Reservoir 1, Reservoir 2 and a Local Source. The means of the supplies from Reservoirs 1 and 2, X_1 and X_2, are 100 and 150 billion gallons, respectively. Their standard deviations are 35 and 30 billion gallons. The correlation coefficient is 0.30.



(a) Describe the water supplies to Town A and B, W_A and W_B in terms of X_1 and X_2:

(b) The means and standard deviations of W_A and W_B:

(c) The correlation coefficient between W_A and W_B:

(d) Confirm (b) and (c) by use of matrix formulas.

(e) When X_1 and X_2 are normal random variables, $P(W_A<150)$?