

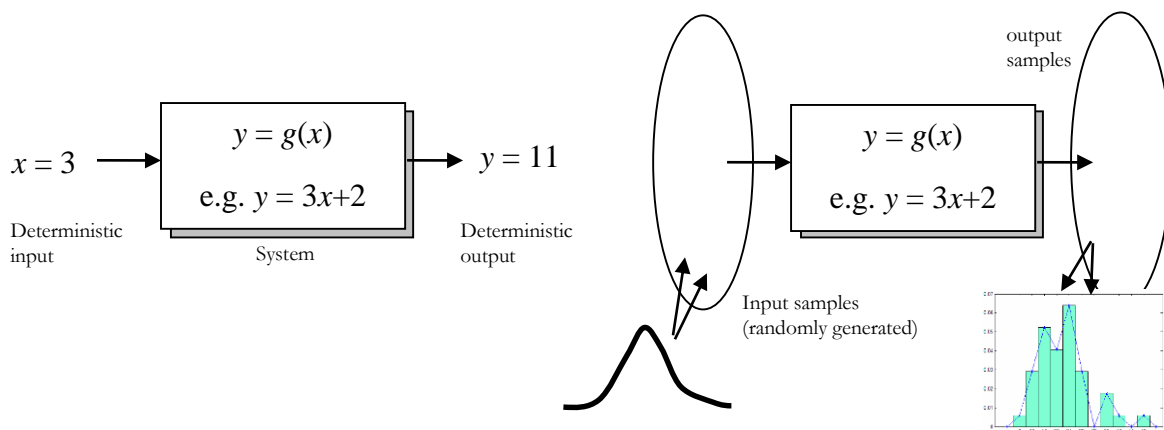
## 457.212 Statistics for Civil & Environmental Engineers

### In-Class Material: Class 19

### Monte Carlo Simulations; Propagation of Uncertainty; and Central Limit Theorem (A&T 5.2; Supplement #4-6)

#### 1. Monte Carlo Simulations

- Generate artificial samples randomly according to probability distribution models (e.g. PDF, CDF, etc.)
- Perform (deterministic) system analysis using each random input sample to obtain a set of artificial outputs



(a) How to generate random numbers – See **Supplement # 4** (using Matlab®)

- Generating single random variable: `shortnamernd(parameters)`  
 e.g. normal: `normrnd(mu, sigma, M, N)`  
 lognormal: `lognrnd(lambda, zeta, M, N)`
- Generating correlated (normal) random variables: `mvnrnd(mu, sigma, N)`

(b) Convergence of the estimate on the probability of an event by MCS

- Example:  $P(Y > 10)$  in the example above.
- For each sample  $x_i$ , the event of interest occurs with probability  $p$  (unknown).
- Generate  $N$  samples, i.e.  $\{x_1, \dots, x_N\}$  independently (i.e. s.i.)
- This is a B\_\_\_\_\_ trial. The number of occurrences  $\sim B(\quad, \quad)$
- Introduce  $Q_i$ , Bernoulli random variable

$$Q_i = \begin{cases} 1 & \text{the event occurs (prob., } p) \\ 0 & \text{the event does not occur (prob., } 1 - p) \end{cases}$$

- An estimate on  $p$  after  $N$  simulations:

$$\tilde{p} = \frac{\sum_{i=1}^N q_i}{N}$$

-  $\tilde{p}$  is \_\_\_\_\_ variable.

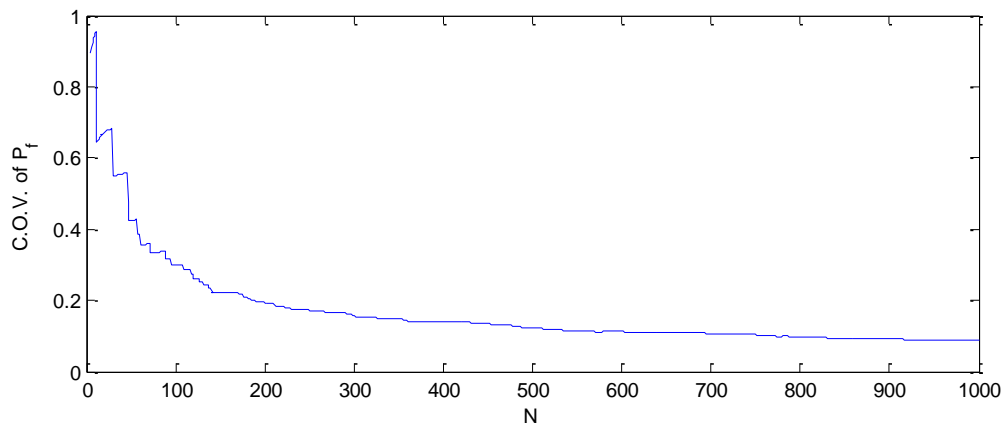
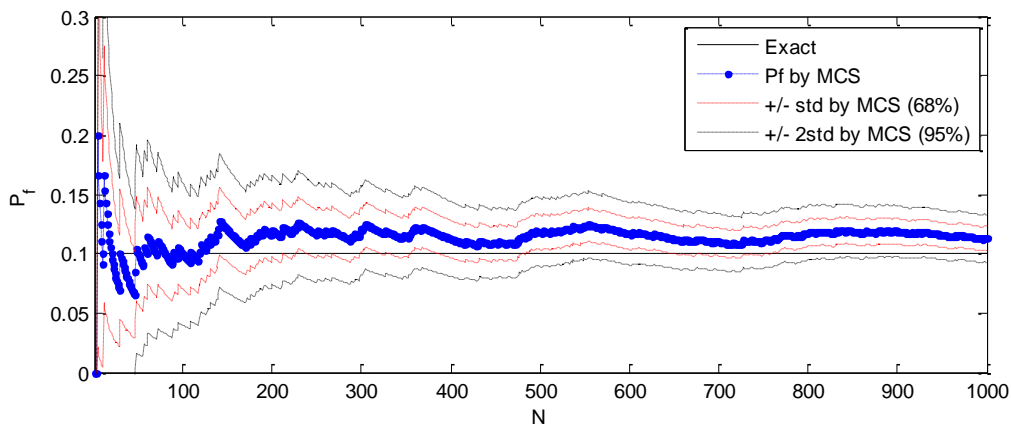
- It is known that

$$E[\tilde{p}] = p \quad (\text{Un_____ estimate})$$

$$\text{Var}[\tilde{p}] = \frac{p(1-p)}{N} \quad \text{The more samplings, the smaller _____}$$

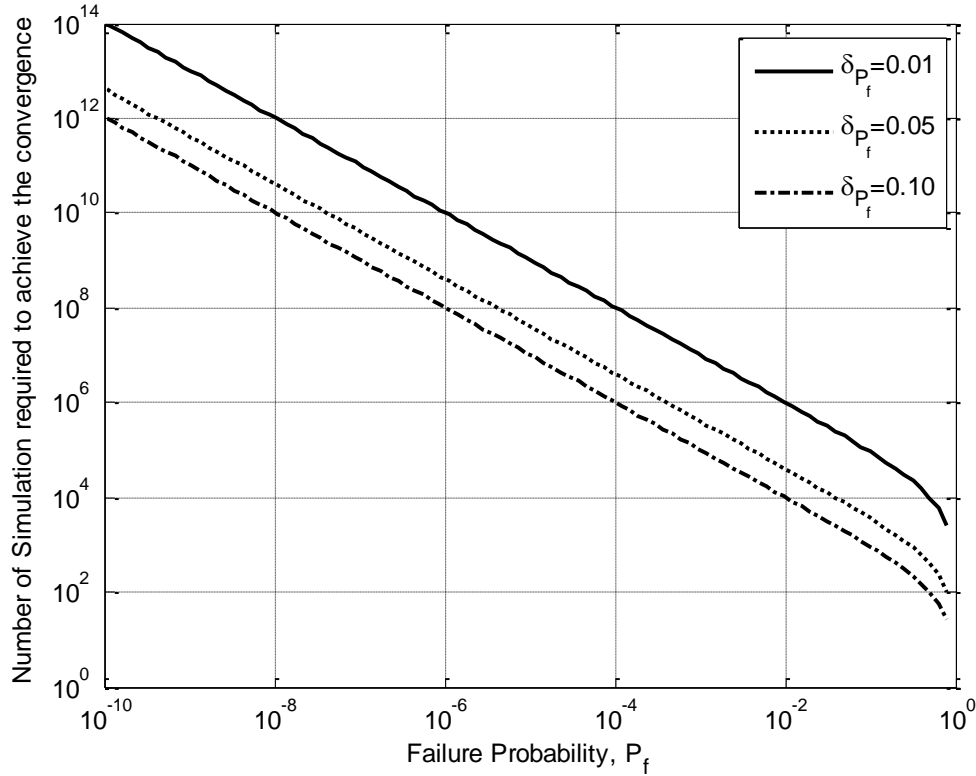
$$\delta_{\tilde{p}} = \frac{1}{\sqrt{N}} \sqrt{\frac{1-p}{p}} \quad \text{C.O.V. of the randomness in the estimate}$$

- Example:  $p = 0.01$



- Minimum number of MCS to achieve a target level of C.O.V.,  $\bar{\delta}$

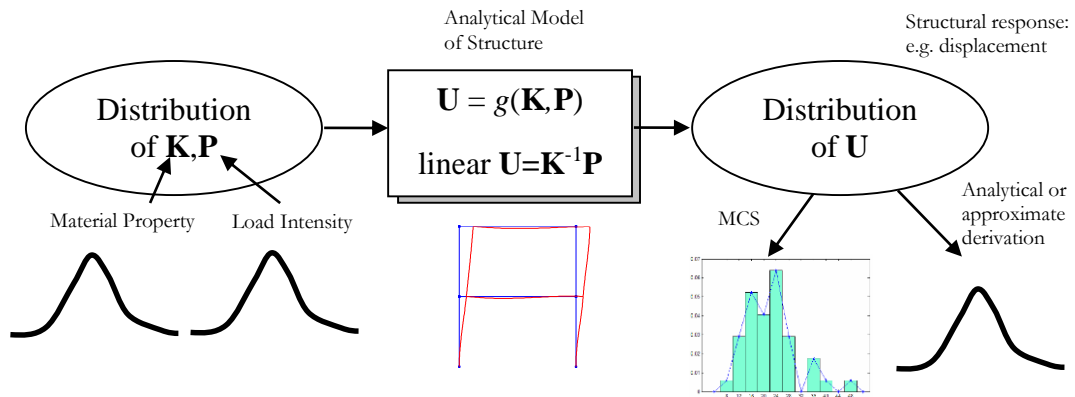
$$N_{\bar{\delta}} = \frac{1-p}{\bar{\delta}^2 p}$$



(e.g.  $p = 0.01$  and  $\bar{\delta} = 0.01$ ,  $N_{\bar{\delta}} \cong 10^6$ )

## 2. Propagation of Uncertainty

(See **Supplement #5** – Matlab® demonstration by FRAME93)



### 3. Central Limit Theorem

(See **Supplement # 6** – Matlab® demonstration by randomwalk.m)

(a) Sum of many random variables asymptotically converges to ( ) r.v.

$$Y = \sum_{i=1}^{\infty} X_i \rightarrow N(\mu_Y, \sigma_Y)$$

(b) Product of many (positive) random variables asymptotically converges to ( ) r.v.

$$Y = \prod_{i=1}^{\infty} X_i \rightarrow ?$$

$$\ln Y = \sum_{i=1}^{\infty} \ln X_i \rightarrow N(\mu_{\ln Y}, \sigma_{\ln Y}) \text{ (Central Limit Theorem)}$$

$$\text{Therefore, } Y = \exp(\ln Y) \rightarrow LN(\lambda = \mu_{\ln Y}, \zeta = \sigma_{\ln Y})$$

(c) Screen shots of randomwalk.m

