Instructor: Junho Song
junhosong@snu.ac.kr

457.643 Structural Random Vibrations
In-Class Material: Class 22

IV. Random Vibration Analysis of MDOF Systems (contd.)

◎ Spectral moments of MDOF generic response $y_p = \sum_{i=1}^{n} a_{pi} s_i(t)$

$$\lambda_m = \int_0^\infty \omega^m \cdot 2\Phi_{y_p y_p}(\omega) d\omega$$

Note

$$\Phi_{y_p y_p}(\omega) = \sum_i \sum_j a_{pi} a_{pj} \Phi_{s_i s_j}(\omega)$$

Thus,

$$\lambda_m = \sum_i \sum_j a_{pi} a_{pj} \int_0^\infty \omega^m \cdot 2\Phi_{s_i s_j}(\omega) d\omega$$

$$\lambda_m = \sum_i \sum_j a_{pi} a_{pj}$$

In words, the *m*-th order spectral moment of the generic response y_p can be obtained by the weighted sum of the *m*-th order (cross) spectral moments of the modal responses $s_i(t), i = 1, \ldots, n$

If WN approximation is made, the spectral moment is approximated as

$$\lambda_m \cong \sum_i \sum_j a_{pi} a_{pj}$$

Can use the closed-form formulas provided in the previous classnotes for $\lambda_{m,ij}^{WN}$

$$\lambda_{0,ij} = \mathrm{E}[s_i(t) \cdot s_j(t)] = \frac{4\pi\Phi_0(\omega_i\zeta_i + \omega_j\zeta_j)}{K_{ij}}, \quad \lambda_{2,ij} = \mathrm{E}[\dot{s}_i(t) \cdot \dot{s}_j(t)] = \frac{4\pi\Phi_0\omega_i\omega_j(\omega_i\zeta_j + \omega_j\zeta_i)}{K_{ij}}$$

$$\lambda_{1,ij} = \frac{2\Phi_0}{K_{ij}}\left\{\left[(\omega_i^2+\omega_j^2)\zeta_i + 2\omega_i\omega_j\zeta_j\right]\frac{1}{\sqrt{1-\zeta_i^2}}\tan^{-1}\left(\frac{\sqrt{1-\zeta_i^2}}{\zeta_i}\right) - (\omega_i^2-\omega_j^2)\ln\left(\frac{\omega_i}{\omega_j}\right) + \left[(\omega_i^2+\omega_j^2)\zeta_j + 2\omega_i\omega_j\zeta_i\right]\frac{1}{\sqrt{1-\zeta_j^2}}\tan^{-1}\left(\frac{\sqrt{1-\zeta_j^2}}{\zeta_j}\right)\right\}$$

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Example:

A frame structure with a light equipment attached (α 1) and small damping (ξ_1, ξ_2 1)

The ground acceleration process is assumed to be a white noise with the intensity Φ_0

Question: the mean square of the displacements $\mathrm{E}[x_1^2]$ and $\mathrm{E}[x_2^2]$

$$\mathbf{M} = \begin{bmatrix} & 0 \\ 0 & \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} \alpha k & \\ & (1+\alpha)k \end{bmatrix}$$

E.O.M.

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{MR}(-\ddot{x}_g)$$

Here $\mathbf{P} =$ and $\mathbf{F}(t) = F(t) =$

Eigenvalue analysis

$$|\mathbf{K} - \lambda\mathbf{M}| = 0$$

$$\begin{vmatrix} \alpha k - \lambda\alpha m & \\ & (1+\alpha)k - \lambda m \end{vmatrix} = 0$$

$$\lambda^2 - (2+\alpha)\frac{k}{m}\lambda + \frac{k^2}{m^2} = 0$$

$$\lambda = \left(1 + \frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 + 4\alpha}\right)\frac{k}{m}$$

For small $\alpha \ll 1$,

Instructor: Junho Song
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$$\omega_1 \cong \left(1 - \frac{\sqrt{\alpha}}{2}\right)\sqrt{\frac{k}{m}}$$

$$\omega_2 \cong \left(1 + \frac{\sqrt{\alpha}}{2}\right)\sqrt{\frac{k}{m}}$$

The corresponding modal vectors are

$$\boldsymbol{\phi}_1 = \begin{Bmatrix} \frac{1}{\sqrt{\alpha}} \\ 1 \end{Bmatrix}$$

$$\boldsymbol{\phi}_2 = \begin{Bmatrix} -\frac{1}{\sqrt{\alpha}} \\ 1 \end{Bmatrix}$$

Modal masses:

$$M_1 = \boldsymbol{\phi}_1^{\mathrm{T}}\mathbf{M}\boldsymbol{\phi}_1 = 2m$$

$$M_2 = \boldsymbol{\phi}_2^{\mathrm{T}}\mathbf{M}\boldsymbol{\phi}_2 = 2m$$

Modal participation factors:

$$\gamma_i = \frac{\boldsymbol{\phi}_i^{\mathrm{T}}\mathbf{P}}{M_i} = \frac{\boldsymbol{\phi}_i^{\mathrm{T}}}{M_i}$$

$$\gamma_1 = \frac{1+\sqrt{\alpha}}{2} \cong$$

$$\gamma_2 = \frac{1-\sqrt{\alpha}}{2} \cong$$

Effective modal participation factor:

$$\mathbf{A} = \mathbf{Q\Phi\Gamma} = \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix} = \begin{bmatrix} \frac{1}{2\sqrt{\alpha}} & -\frac{1}{2\sqrt{\alpha}} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

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Recall

$$\mathrm{E}[x_p^2] = \lambda_0^{(p)} = \sum_i \sum_j a_{pi} a_{pj} \lambda_{0,ij} = \sum_i \sum_j a_{pi} a_{pj} \rho_{0,ij} \sqrt{\lambda_{0,ii} \lambda_{0,jj}}$$

WN modal responses:

$$\lambda_{0,ii} = \frac{\pi \Phi_0}{2\xi_i \omega_i^3} \cong \frac{\pi \Phi_0}{2\xi \omega_0^3}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$

$$\rho_{0,12} = \frac{8\xi^2 r^{\frac{3}{2}}}{[(1-r)^2 + 4\xi^2 r](1+r)}$$

where $r = \omega_1/\omega_2$, and

$$1 - r = 1 - \frac{1-\frac{\sqrt{\alpha}}{2}}{1+\frac{\sqrt{\alpha}}{2}} = \frac{\sqrt{\alpha}}{1+\frac{\sqrt{\alpha}}{2}} \cong \sqrt{\alpha}, \;\; 1 + r = 1 + \frac{1-\frac{\sqrt{\alpha}}{2}}{1+\frac{\sqrt{\alpha}}{2}} = \frac{2}{1+\frac{\sqrt{\alpha}}{2}} \cong 2$$

Therefore,

$$\rho_{0,12} \cong \frac{4\xi^2}{4\xi^2 + \alpha}$$

Finally, from $\mathbf{A} = \begin{bmatrix} \frac{1}{2\sqrt{\alpha}} & -\frac{1}{2\sqrt{\alpha}} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and $\mathrm{E}[x_p^2] = \sum_i \sum_j a_{pi} a_{pj} \rho_{0,ij} \sqrt{\lambda_{0,ii}\lambda_{0,jj}}$, the mean square responses are derived as

$$\mathrm{E}[x_1^2] = \left(\frac{1}{2\sqrt{\alpha}}\right)^2 \lambda_{0,11} + 2\left(\frac{1}{2\sqrt{\alpha}}\right)\left(-\frac{1}{2\sqrt{\alpha}}\right) \rho_{0,12} \sqrt{\lambda_{0,11}\lambda_{0,22}} + \left(-\frac{1}{2\sqrt{\alpha}}\right)^2 \lambda_{0,22}$$

$$= \frac{\pi \Phi_0}{2\xi \omega_0^3} \cdot \frac{1}{2\alpha} \cdot \left(1 - \frac{4\xi^2}{4\xi^2 + \alpha}\right)$$

$$\mathrm{E}[x_2^2] = \frac{\pi \Phi_0}{2\xi \omega_0^3} \cdot \frac{1}{2} \cdot \left(1 + \frac{4\xi^2}{4\xi^2 + \alpha}\right)$$

For $\xi = 0.05$, the standard deviations of the displacements normalized by $\sqrt{\pi \Phi_0 / 2\xi \omega_0^3}$ are

Instructor: Junho Song
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	$\sigma_{x_1} / \sqrt{\frac{\pi\Phi_0}{2\xi\omega_0^3}}$			$\sigma_{x_2} / \sqrt{\frac{\pi\Phi_0}{2\xi\omega_0^3}}$		
	Exact	$\rho_{0,12}$ neglected	Error (%)	Exact	$\rho_{0,12}$ neglected	Error (%)
$\alpha = 0.01$	5	7.07	41	0.866	0.707	-18
$\alpha = 0.001$	6.71	22.4	233	0.975	0.707	-27

◎ Random vibration theory behind modal combination rules

Recall $y_r = \sum_{i=1}^{n} a_{ri} s_i$ and $(y_r^{max})^2 \cong (\sum a_{ri}\, s_i^{max})^2$ where

$y_r^{max} = \max_{0<t\le\tau} |y_r(t)|$ and $s_i^{max} = \max_{0<t\le\tau} |s_i(t)|$

1) Modal combination rules

SRSS (Rosenblueth 1951):

$$y_r^{max} \cong \left(\sum_{i=1}^{n} a_{ri}^2 (s_i^{max})^2 \right)^{1/2}$$

CQC (Der Kiureghian 1981, EESD)

$$y_r^{max} \cong \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ri} a_{rj} \rho_{0,ij}^{WN} s_i^{max} s_j^{max} \right)^{\frac{1}{2}}$$

$$= \left[\sum_{i=1}^{n} (a_{ri})^2 (s_i^{max})^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_{ri} a_{rj} \rho_{0,ij}^{WN} s_i^{max} s_j^{max} \right]^{1/2}$$

2) Random vibration theory

Recall

$$\lambda_m = \sum_i \sum_j a_{ri} a_{rj} \rho_{m,ij} \sqrt{\lambda_{m,ii} \lambda_{m,jj}}$$

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For example, consider $m = 0$

$$\lambda_0 = \mathrm{E}[Y_r^2] = \sigma_{Y_r}^2 = \sum_i \sum_j a_{ri} a_{rj} \rho_{0,ij} \sqrt{\lambda_{0,ii}\lambda_{0,jj}}$$

$$= \sum_i \sum_j a_{ri} a_{rj} \rho_{0,ij} \sigma_{s_i} \sigma_{s_j}$$

That is,

$$\sigma_{Y_r} = \left(\sum_i \sum_j a_{ri} a_{rj} \rho_{0,ij} \sigma_{s_i} \sigma_{s_j} \right)^{\frac{1}{2}}$$

Assume $\mathrm{E}[y_r^{max}] = p\sigma_{Y_r}$ and $\mathrm{E}[s_i^{max}] = p\sigma_{s_i}$, $\mathrm{E}\left[s_j^{max}\right] = p\sigma_{s_j}$

$$\frac{\mathrm{E}[y_r^{max}]}{p} = \left(\sum_i \sum_j a_{ri} a_{rj} \rho_{0,ij} \frac{\mathrm{E}[s_i^{max}]}{p} \frac{\mathrm{E}\left[s_j^{max}\right]}{p} \right)^{\frac{1}{2}}$$

Inspired by this, CQC rule is proposed as

$$y_r^{max} \cong \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ri} a_{rj} \rho_{0,ij}^{WN} s_i^{max} s_j^{max} \right)^{\frac{1}{2}}$$

This actually means

$$\mathrm{E}[y_r^{max}] \cong \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ri} a_{rj} \rho_{0,ij}^{WN} \mathrm{E}[s_i^{max}] \mathrm{E}\left[s_j^{max}\right] \right)^{\frac{1}{2}}$$

Herein $\mathrm{E}[s_i^{max}]$ and $\mathrm{E}\left[s_j^{max}\right]$ are obtained from ___________ spectrum.

3) SRSS works well when modal frequencies are well-separated, say $r = \frac{\omega_i}{\omega_j} < \frac{0.2}{0.2+\xi_i+\xi_j}$ ($\omega_j > \omega_i$) because $\rho_{0,ij} \cong 0.10$ for the range.

4) Approximation introduced in CQC for practicality

$$\rho_{0,ij} \cong \rho_{0,ij}^{WN}$$

Instructor: Junho Song
junhosong@snu.ac.kr

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④ CQC's approximation for practicality

$\rho_{0,ij} \cong \rho_{0,ij}^{WN}$ (good FO approx. available, indep. of Φ_0 see plot ↓)

general "~~~" (depends on $\Phi_{FF}(\omega)$)

Error is reasonably small
(see comparison w/ $\rho_{0,ij}$ to Kanai-Tajimi spectrum) e.g.
i.e. filtered white noise
(ω_g : dominant freq, ζ_g : bandwidth) of ground motion

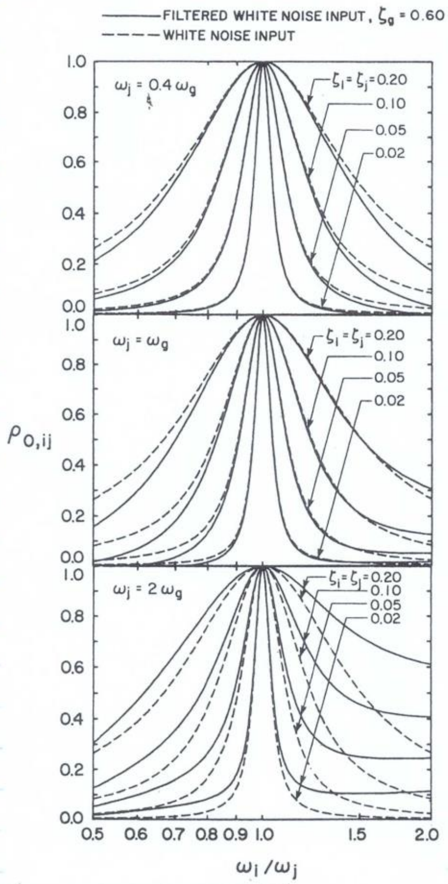

Figure 2. Comparison of correlation coefficients for responses to white-noise and filtered white-noise inputs

$$\rho_{0,ij} = \frac{2\sqrt{(\zeta_i\zeta_j)}\,[(\omega_i+\omega_j)^2(\zeta_i+\zeta_j)+(\omega_i^2-\omega_j^2)(\zeta_i-\zeta_j)]}{4(\omega_i-\omega_j)^2+(\omega_i+\omega_j)^2(\zeta_i+\zeta_j)^2}$$

$$\rho_{1,ij} = \frac{2\sqrt{(\zeta_i\zeta_j)}\,[(\omega_i+\omega_j)^2(\zeta_i+\zeta_j)-4(\omega_i-\omega_j)^2/\pi]}{4(\omega_i-\omega_j)^2+(\omega_i+\omega_j)^2(\zeta_i+\zeta_j)^2}$$

$$\rho_{2,ij} = \frac{2\sqrt{(\zeta_i\zeta_j)}\,[(\omega_i+\omega_j)^2(\zeta_i+\zeta_j)-(\omega_i^2-\omega_j^2)(\zeta_i-\zeta_j)]}{4(\omega_i-\omega_j)^2+(\omega_i+\omega_j)^2(\zeta_i+\zeta_j)^2}$$

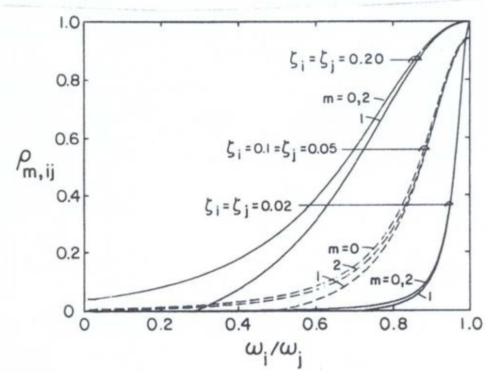

Figure 3. Coefficients $\rho_{m,ij}$ for response to white-noise

From
A. Der Kiureghian (1981)
"A Response Spectrum Method for Random Vibration Analysis of MDF Systems" Earthquake Engineering & Str. Dynamics