Seoul National University
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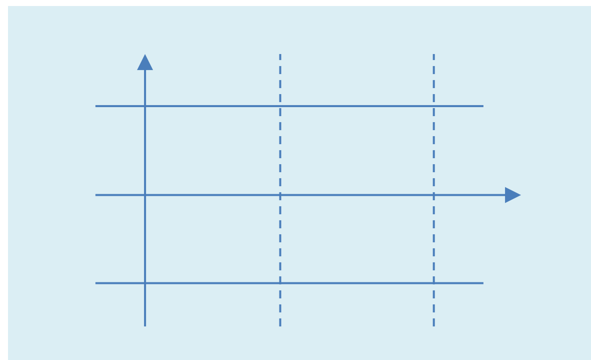
457.643 Structural Random Vibrations
In-Class Material: Class 23

V. Crossings & Failure Analysis

◎ Failure probabilities

1) Instantaneous failure probability

$P(|X(t)| > a)$ or $P(X(t) > a)$



e.g. Gaussian with $\mu_X(t)$ and $\sigma_X(t)$

$X(t) \sim N(\mu_X(t), \sigma_X^2(t))$

$$P(X(t) > a) = 1 - F_{X(t)}(a)$$
$$= 1 - \Phi\left(\underline{\qquad\qquad}\right)$$

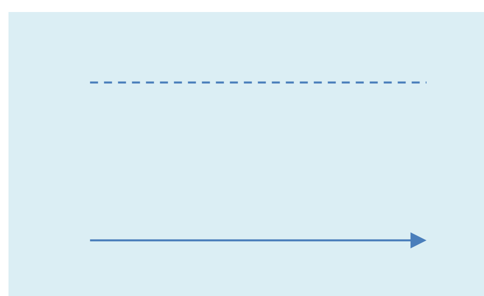
2) First-passage failure probability

$$P\left(\max_{0<t\leq\tau} |X(t)| > a\right) = P(at\ least \qquad crossing\ in\ (0, \tau])$$

can be estimated by checking the probability distribution of _______ values, or

by deriving from __________ rates and other characteristics

3) Accumulated damage



e.g. Fatigue damage index
(L&S 11.8~11.11, 12.9)

$D(t)$: damage measure (e.g. counts)

◎ Crossing statistics

1) $N^+(a;t)$: Number of upcrossings of level a in $(0, t)$

$p^+(a;t)$: Probability of an upcrossing of level a in $(t, t+dt]$

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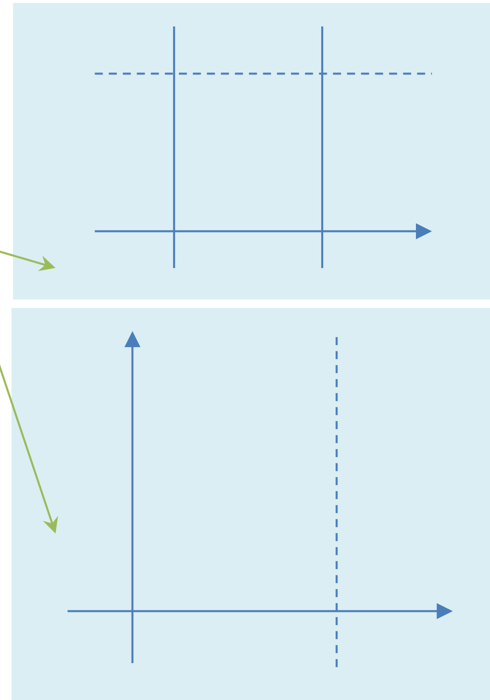
Upcrossing event at $(t, t+dt]$

Conditions:

- $X(t) \quad a$
- $\dot{X}(t) \quad 0$
- $X(t+dt) \cong X(t) + \dot{X}(t)dt \qquad a$



Therefore,

$$p^+(a;t) = P\big[\{ \qquad < X(t) < \quad \} \cap \{\dot{X}(t) \quad 0\}\big]$$

$$= \int_0^\infty \int f_{X\dot{X}}(x,\dot{x};t)\,dx\,d\dot{x}$$

$$= \int_0^\infty f_{X\dot{X}}(a,\dot{x};t)\dot{x}\,dt\,d\dot{x}$$

$$= dt \int_0^\infty \dot{x} f_{X\dot{X}}(a,\dot{x};t)\,d\dot{x}$$

For the bottom figure, the third condition is interpreted as $a - \dot{x}dt < x$ and thus $\dot{x} > -\frac{1}{dt}x + \frac{1}{dt}a$

2) $dN^+(a;t)\left(= \frac{\partial N^+(a;t)}{\partial t}dt\right)$: Number of crossings in $(t, t+dt]$

$$\mathrm{E}[dN^+(a;t)] = 0 \times P(0 \text{ crossings}) + 1 \times P(1 \text{ crossing}) + 2 \times P(2 \text{ crossings}) + \cdots$$

$$\cong P(1 \text{ crossing in } (t, t+dt])$$

$$= p^+(a;t)$$

$$= dt \int_0^\infty \dot{x} f_{X\dot{X}}(a,\dot{x};t)\,d\dot{x}$$

3) Average number of upcrossings in $(t, t+dt]$, i.e. "mean upcrossing rate"

$$\boxed{\nu^+(a;t) = \mathrm{E}\left[\frac{dN^+(a;t)}{dt}\right] = \int_0^\infty \dot{x} f_{X\dot{X}}(a,\dot{x};t)\,d\dot{x}}$$

Stephen O. Rice (1907-1986) → "Rice formula" (1944, 1945)

Downcrossing rate?

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$$\nu^-(a;t) = -\int_{-\infty}^{0} \dot{x} f_{X\dot{X}}(a,\dot{x};t)d\dot{x} = \int_{-\infty}^{0} |\dot{x}| f_{X\dot{X}}(a,\dot{x};t)d\dot{x}$$

All crossings?

$$\begin{aligned}\nu(a;t) &= \nu^+(a;t) + \nu^-(a;t) \\ &= \int_{-\infty}^{\infty} |\dot{x}| f_{X\dot{X}}(a,\dot{x};t)d\dot{x}\end{aligned}$$

- More rigorous derivation available in L&S (p. 265)

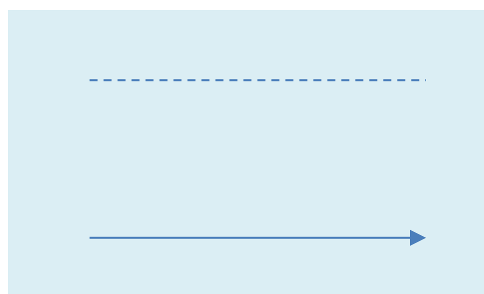
4) Mean number of crossing in $(t_1, t_2]$

$$\mathrm{E}[N(a;t_2) - N(a;t_1)] = \int_{t_1}^{t_2} \nu(a;t)dt$$

5) If $X(t)$ is stationary,

- $f_{X\dot{X}}(x,\dot{x};t) \rightarrow f_{X\dot{X}}(x,\dot{x})$ (if zero-mean Gaussian, $f_{X\dot{X}}(x,\dot{x}) = f_X(x) \cdot f_{\dot{X}}(\dot{x})$)
- $\nu(a;t) \rightarrow \nu(a)$
- $\mathrm{E}[N(a;t_2) - N(a;t_1)] \rightarrow \nu(a) \cdot (t_2 - t_1)$

6) Relationship between crossing rate and peak distribution (approximation for narrow-band processes)



If $X(t)$ is stationary <u>narrow-band</u> process, almost every upcrossings over μ is associated with one and only one peak, then…

$$P(\text{a randomly selected peak} > a) \cong \frac{\nu^+(a)}{\nu^+(\mu)} \cong 1 - F_p(a)$$

where $F_p(\cdot)$ is the CDF of a local peak

The PDF of a local peak is approximated as

$$f_p(a) \cong -\frac{1}{\nu^+(\mu)} \cdot \frac{d\nu^+(a)}{da}$$

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Example: A stationary Gaussian process with zero-mean

$$f_{X\dot{X}}(x,\dot{x}) = f_X(x)\cdot f_{\dot{X}}(\dot{x})$$
$$= \frac{1}{2\pi\sigma_X\sigma_{\dot{X}}}\exp\left\{-\frac{1}{2}\left[\left(\frac{x}{\sigma_X^2}\right)^2+\left(\frac{\dot{x}}{\sigma_{\dot{X}}^2}\right)^2\right]\right\}$$

$$\nu^+(a) = \int_0^\infty \dot{x} f_{X\dot{X}}(a,\dot{x})d\dot{x}$$
$$= \int_0^\infty \dot{x}\frac{1}{2\pi\sigma_X\sigma_{\dot{X}}}\exp\left\{-\frac{1}{2}\left[\left(\frac{a}{\sigma_X^2}\right)^2+\left(\frac{\dot{x}}{\sigma_{\dot{X}}^2}\right)^2\right]\right\}d\dot{x}$$
$$= \frac{1}{2\pi\sigma_X\sigma_{\dot{X}}}\exp\left(-\frac{a^2}{2\sigma_X^2}\right)\int_0^\infty \dot{x}\exp\left(-\frac{\dot{x}^2}{\sigma_{\dot{X}}^2}\right)d\dot{x}$$

One can show that $\int_0^\infty \dot{x}\exp\left(-\frac{\dot{x}^2}{\sigma_{\dot{X}}^2}\right)d\dot{x} = \sigma_{\dot{X}}^2$ (hint: change variable $\dot{x}^2 \to t$)

Therefore,

$$\boxed{\begin{aligned}\nu^+(a) &= \frac{1}{2\pi}\frac{\sigma_{\dot{X}}}{\sigma_X}\exp\left(-\frac{a^2}{2\sigma_X^2}\right)\\ &= \frac{1}{2\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}\exp\left(-\frac{a^2}{2\sigma_X^2}\right)\end{aligned}}$$

Some notable results:

- $\nu^-(a) =$
- $\nu(a) =$
- $\nu^+(0) = \frac{1}{2\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}$
- $\sqrt{\frac{\lambda_2}{\lambda_0}} = 2\pi\nu^+(0)$: circular apparent frequency

  e.g. WN response: $\lambda_2 = \frac{\pi\Phi_0}{2\xi\omega_0}$ and $\lambda_0 = \frac{\pi\Phi_0}{2\xi\omega_0^3}$ → $\sqrt{\frac{\lambda_2}{\lambda_0}} = \omega_0$
- NB approximation for local peak distribution: $f_p(a) \cong -\frac{1}{\nu^+(0)}\cdot\frac{d\nu^+(a)}{da} = \frac{a}{\lambda_0}\exp\left(-\frac{a^2}{2\lambda_0}\right)$

  → "Rayleigh" distribution

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◎ Distribution of local peaks (NOT NB approximation; L&S pp. 488-490)

$$F_p(a;t) = \frac{\int_{-\infty}^{0}\int_{-\infty}^{a}|\ddot{x}|f_{X\dot{X}\ddot{X}}(x,0,\ddot{x};t)dxd\ddot{x}}{\int_{-\infty}^{0}|\ddot{x}|f_{\dot{X}\ddot{X}}(0,\ddot{x};t)d\ddot{x}}, \quad f_p(a;t) = \frac{dF_p(a;t)}{da} = \frac{\int_{-\infty}^{0}|\ddot{x}|f_{X\dot{X}\ddot{X}}(a,0,\ddot{x};t)d\ddot{x}}{\int_{-\infty}^{0}|\ddot{x}|f_{\dot{X}\ddot{X}}(0,\ddot{x};t)d\ddot{x}}$$

Example: The PDF and CDF of the local peaks of a stationary Gaussian process $X(t)$:
("Rice" distribution; Ex 11.1 in L&S)

$$f_P(a) = \frac{\sqrt{1-\alpha^2}}{\sqrt{2\pi}\sigma_X}\exp\left[-\frac{(a-\mu_X)^2}{2(1-\alpha^2)\sigma_X^2}\right] + \frac{\alpha(a-\mu_X)}{\sigma_X^2}\exp\left[-\frac{(a-\mu_X)^2}{2\sigma_X^2}\right]\Phi\left[\frac{\alpha(a-\mu_X)}{\sqrt{1-\alpha^2}\sigma_X}\right]$$

$$F_P(a) = \Phi\left(\frac{a-\mu_X}{\sqrt{1-\alpha^2}\sigma_X}\right) - \alpha\exp\left[-\frac{(a-\mu_X)^2}{2\sigma_X^2}\right]\Phi\left[\frac{\alpha(a-\mu_X)}{\sqrt{1-\alpha^2}\sigma_X}\right]$$

※ How was it derived?

- $f_{X\dot{X}\ddot{X}}(x,\dot{x},\ddot{x}) = f_{X\ddot{X}}(x,\ddot{x})\cdot f_{\dot{X}}(\dot{x})$ ($\because$ stationary and Gaussian)

- $\rho_{X\ddot{X}}$? Note $\mathrm{COV}[X,\ddot{X}] = -\lambda_2$

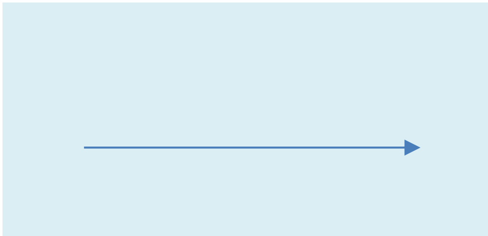
  $\because \Phi_{X\ddot{X}}(\omega) = (-\mathrm{i}\omega)^2\Phi_{XX}(\omega) = -\omega^2\Phi_{XX}(\omega) = -\Phi_{\dot{X}\dot{X}}(\omega)$

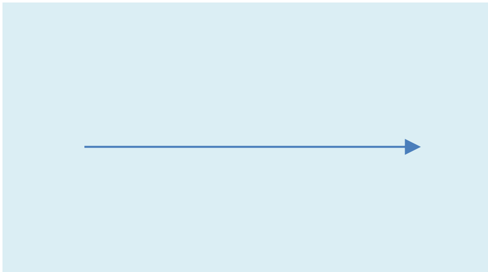
  $\therefore \rho_{X\ddot{X}} = -\dfrac{\lambda_2}{\sqrt{\lambda_0\lambda_4}} = -\alpha$

- $\alpha = \dfrac{\lambda_2}{\sqrt{\lambda_0\lambda_4}} = \dfrac{\nu_X^+(0)}{\nu_{\dot{X}}^+(0)}\left(\because = \dfrac{\frac{1}{2\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}}{\frac{1}{2\pi}\sqrt{\frac{\lambda_4}{\lambda_2}}}\right)$

Note: α is another measure of the bandwidth (cf. $0 < s < \infty$ and $0 < \delta < 1$)

- $0 < \alpha < 1$
- $\alpha \cong 0$: $\nu_{\dot{X}}^+(0) \gg \nu_X^+(0)$ wide-band process
- $\alpha \cong 1$: $\nu_{\dot{X}}^+(0) \cong \nu_X^+(0)$ narrow-band process





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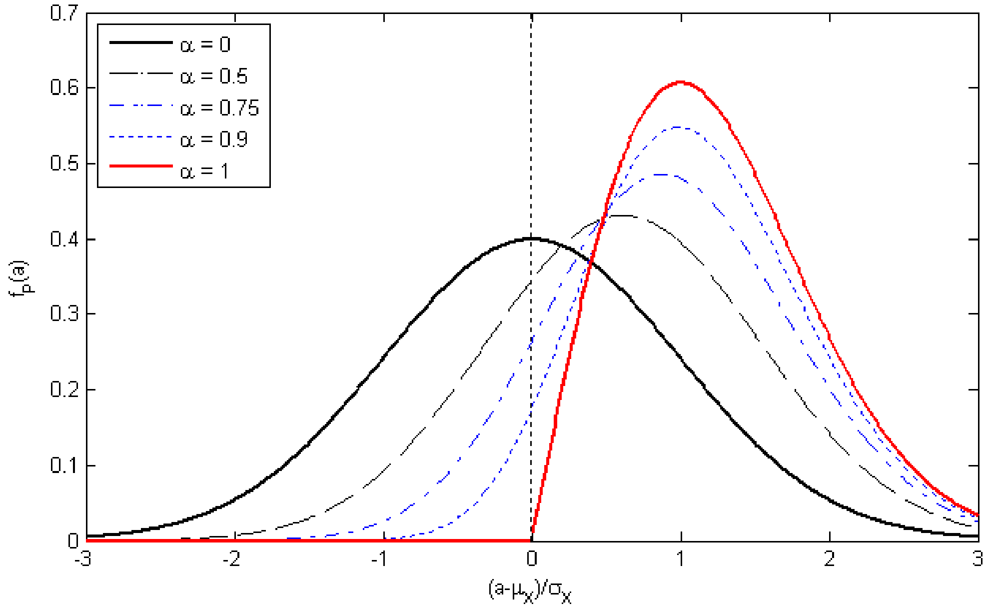

Note

(1) $\alpha = 0$: wide-band $\quad f_P(a) = \dfrac{1}{\sqrt{2\pi}\sigma_X}\exp\left[-\dfrac{(a-\mu_X)^2}{2\sigma_X^2}\right]$ (Gaussian)

(2) $\alpha = 1$: narrow-band $\quad f_P(a) = \dfrac{(a-\mu_X)}{\sigma_X^2}\exp\left[-\dfrac{(a-\mu_X)^2}{2\sigma_X^2}\right]$ (Rayleigh)

(3) The average fraction of local peaks below the mean value.

$$F_P(\mu_X) = \frac{1-\alpha}{2}$$

0.5 for $\alpha = 0$ (Gaussian) and 0 for $\alpha = 1$ (Rayleigh)