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457.643 Structural Random Vibrations

In-Class Material: Class 25

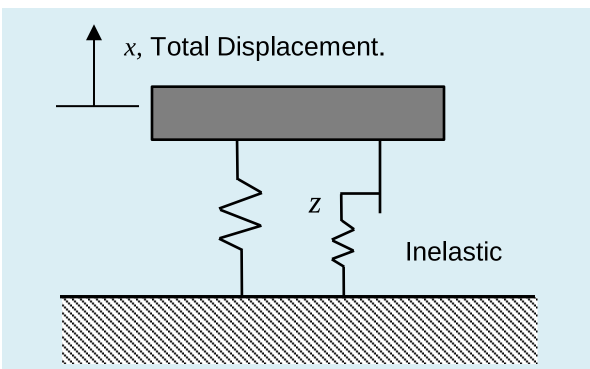
VI. Nonlinear Random Vibration Analysis

◎ (Differential equation based) hysteretic constitutive models in structural dynamics

"Hysteresis"

- Origin: ferromagnetic materials
- Memory-based multi-valued relation between an input signal & output (generally referring to "rate-independent" relationship only; ~~viscous materials~~)

Mechanical model for differential equation based hysteresis model



z: Auxiliary variable representing inelastic behavior ("internal variable" – Capecchi & de Felice 2001, ASCE JEM) ~ displacement of inelastic spring

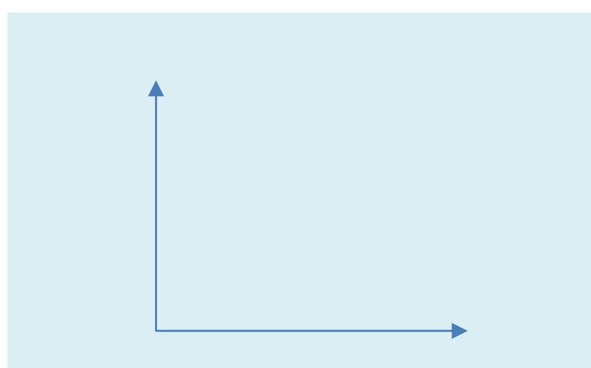
- $z = x$: no slide
- $z = 0$: slide

(nonlinearity determined by difference between z and x)

Resisting force:

$$\boxed{f_s(x,z) = \alpha k_0 x + (1-\alpha)k_0 z}$$

- α: post-to-pre-yield stiffness ratio
  - ✓ $\alpha = 0$: perfectly plastic
  - ✓ $\alpha = 1$: linear elastic
- k_0: initial stiffness



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Evolution of z follows a nonlinear differential equation

$$\boxed{\dot{z} = \dot{x} \cdot h(x, \dot{x}, z)}$$

Meaning of the nonlinear function $h(\cdot)$?

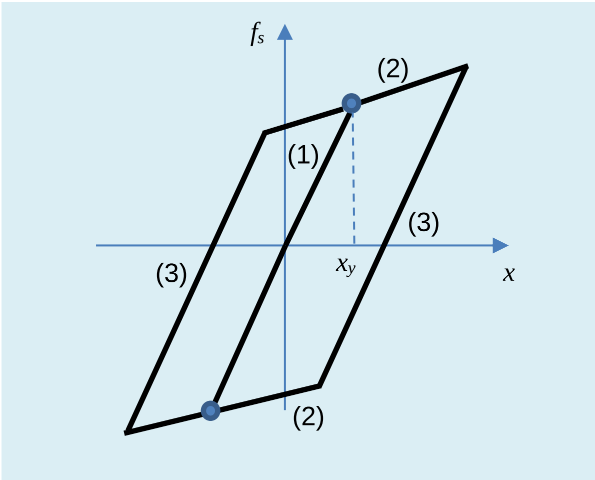
Song, J., and A. Der Kiureghian (2006). Generalized Bouc-Wen model for highly asymmetric hysteresis. *Journal of Engineering Mechanics. ASCE*, 132(6), 610-618.

$$\frac{dz}{dt} = \frac{dx}{dt} \cdot h(\cdot)$$

Therefore, $h(\cdot)\left(= \frac{dz}{dx}\right)$ determines the slope of z with respect to x at a given time.

◎ Bilinear model (Kaul & Penzien 1974 JEMD; Asano & Iwan 1984 EESD)

Main idea: describe inelastic spring in the mechanical model by a Coulomb slider (i.e. no slide until it reaches the yield displacement)



(1) $-x_y < z < x_y$:

the Coulomb slider does not slide, i.e.

$z = x$ and $\dot{z} = \dot{x}$

$f_s(x, z) = \alpha k_0 x + (1 - \alpha) k_0 x = k_0 x$ (linear)

(2) $z > x_y, \dot{x} > 0$ or $z < -x_y, \dot{x} < 0$:

Coulomb slider slides (i.e. $\dot{z} = 0$)

(3) $z > x_y, \dot{x} < 0$ or $z < -x_y, \dot{x} > 0$:

Coulomb slider stops sliding $\dot{z} = \dot{x}$

Differential-equation model by Kaul & Penzien (1974):

$$\dot{z} = \dot{x} \cdot \{U(z + x_y) - U(z - x_y) + U(z - x_y) \cdot U(-\dot{x}) + U(-z - x_y) \cdot U(\dot{x})\}$$

where $U(\cdot)$ denotes the unit step function.

How to solve the nonlinear system differential equation, i.e.

E.O.M. with $f_s = \alpha k_0 x + (1 - \alpha) k_0 z$ plus $\dot{z} = \dot{x} \cdot h(x, \dot{x}, z)$

e.g. Runge-Kutta method (after transforming to state-space formulation $\dot{\boldsymbol{y}} = \boldsymbol{g}(\boldsymbol{y}) + \boldsymbol{f}$)

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◎ Bouc-Wen class model

Bouc (1967) first proposed, and Wen (1976) later modified to the form

$$\boxed{\dot{z} = \dot{x} \cdot [A - |z|^{n} \psi(x, \dot{x}, z)]}$$

where

- A: scale of hysteresis loop
- n: smoothness of transition from pre-yielding to post-yielding
- $\psi(x, \dot{x}, z)$: "shape-control" function

Reviews are available in Song & ADK (2006, JEM), and Ismail et al. (2009, Archi. Comp. Meth. Engrg.)

1) Bouc (1967, 1971)
    - $n = 1$
    - $\psi(x, \dot{x}, z) = \gamma + \beta \mathrm{sgn}(\dot{x} z)$
2) Wen (1976)
    - n: generalized
    - $\psi(x, \dot{x}, z) = \gamma + \beta \mathrm{sgn}(\dot{x} z)$

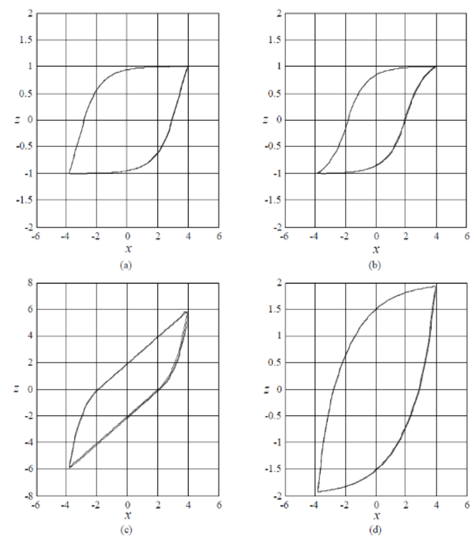

Figure 3.3 Hysteresis loops by Bouc-Wen model ($A = 1, n = 1$) (a) $\gamma = 0.5$, $\beta = 0.5$, (b) $\gamma = 0.1$, $\beta = 0.9$, (c) $\gamma = 0.5$, $\beta = -0.5$ and (d) $\gamma = 0.75$, $\beta = -0.25$

The parameters γ and β in the "shape-control" function determine the shapes of the hysteresis loops (Song and ADK 2006)

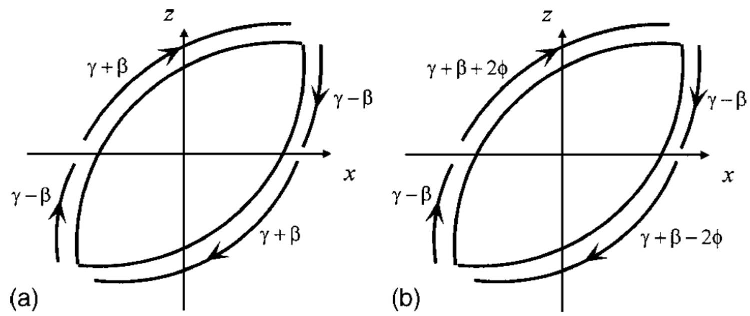

Fig. 2. Values of shape-control function for: (a) original Bouc–Wen model; and (b) model by Wang and Wen

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3) Baber & Wen (1981): Considered the degradation effect by making the model parameters functions of ϵ, "the dissipated energy"

4) Baber & Noori (1984): Introduced additional parameters to describe "pinching" effect

5) Wang & Wen (1998): Described "asymmetric" shape by adding additional terms

$$\psi(x,\dot{x},z) = \gamma + \beta\text{sgn}(\dot{x}z) + \phi[\text{sgn}(\dot{x}) + \text{sgn}(z)]$$

➔ Added more DOFs (see the figure above)

6) Generalized Bouc-Wen (Song & ADK, 2006)

Generalize the "shape-control" function to describe highly asymmetric behavior

$$\psi(x,\dot{x},z) = \beta_1\text{sgn}(\dot{x}z) + \beta_2\text{sgn}(x\dot{x}) + \beta_3\text{sgn}(xz) + \beta_4\text{sgn}(\dot{x}) + \beta_5\text{sgn}(z) + \beta_6\text{sgn}(x)$$

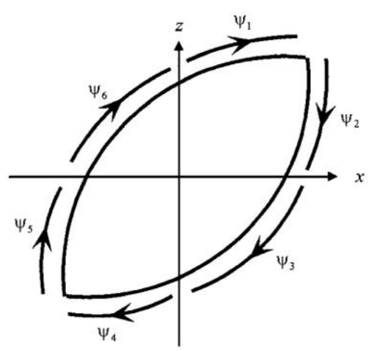

Fig. 3. Values of shape-control function for generalized Bouc–Wen model

Six phases can now have all different values, and the values are determined as

$$\begin{bmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\\\psi_5\\\psi_6\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1\\-1 & -1 & 1 & -1 & 1 & 1\\1 & -1 & -1 & -1 & -1 & 1\\1 & 1 & 1 & -1 & -1 & -1\\-1 & -1 & 1 & 1 & -1 & -1\\1 & -1 & -1 & 1 & 1 & -1\end{bmatrix}\begin{bmatrix}\beta_1\\\beta_2\\\beta_3\\\beta_4\\\beta_5\\\beta_6\end{bmatrix}$$

The model parameters $\beta_i, i = 1, \ldots, 6$ can be fitted by use of

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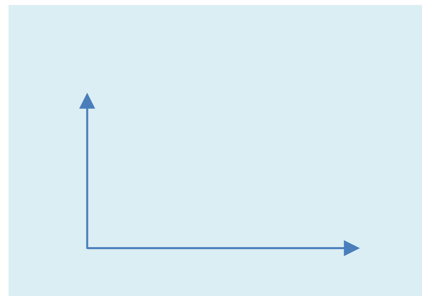
$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 1 & 0 & 1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{bmatrix}$$

※ Weakness of Bouc-Wen class models:

- can violate the requirement of classical plasticity theory ("Drucker's postulate"; Bažant 1978); can create negative dissipative energy when "loading-unloading" occurs without load reversal
- But this problem is not critical if $\mathrm{E}[f_s] \cong 0$ (Wen 1989, Hurtado & Barbat 1996)



※ Bouc-Wen class models are widely-used in structural dynamics and earthquake engineering because

1) Can describe a wide-class of phenomena (pinching, degradation, etc.)

2) Facilitates efficient time history analysis (no IF or THEN)

3) Facilitates efficient random vibration analysis

   e.g. Nonlinear random vibration analysis for Bouc-Wen model by Equivalent Linearization Method (Wen 1980)

◎ Nonlinear time-history analysis of structural system with Bouc-Wen class models

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{R}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{z}) = -\mathbf{M}\mathbf{1}\ddot{x}_g$$

where $\mathbf{R}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{z})$ uses $f_s = \alpha k_0 x + (1-\alpha) k_0 z$ to describe the resistant force of each B-W element. The auxiliary variable follows the nonlinear differential equation $\dot{z} = \dot{x} \cdot h(x, \dot{x}, z)$.

Transformed to state-space formulation, i.e. $\mathbf{y} = \{x_1, \dot{x}_1, x_2, \dot{x}_2, \ldots, z_1, \ldots, z_m\}$

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Example: Two connected equipment items in an electrical substation (Song, 2004)

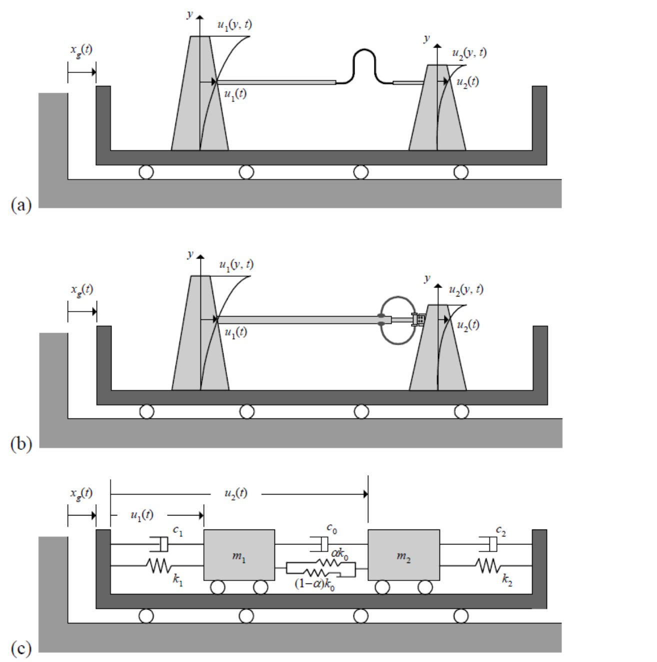

Figure 2.1 Mechanical models of equipment items connected by rigid bus connectors: (a) RB-FSC-connected system, (b) Bus-slider-connected system, and (c) idealized system with SDOF equipment models

$$\dot{\boldsymbol{y}} = \boldsymbol{g}(\boldsymbol{y}) + \boldsymbol{f}$$

where

$$\boldsymbol{y} = \{u_1, \dot{u}_1, u_2, \dot{u}_2, z\}^{\mathrm{T}}$$

$$\boldsymbol{g}(\boldsymbol{y}) = \left\{ \begin{array}{c} \dot{u}_1 \\ -\left(\dfrac{k_1 + \alpha k_0}{m_1}\right) u_1 - \left(\dfrac{c_1 + c_0}{m_1}\right) \dot{u}_1 + \dfrac{\alpha k_0}{m_1} u_2 + \dfrac{c_0}{m_1} \dot{u}_2 + \dfrac{(1-\alpha) k_0}{m_1} z \\ \dot{u}_2 \\ \dfrac{\alpha k_0}{m_2} u_1 + \dfrac{c_0}{m_2} \dot{u}_1 - \left(\dfrac{k_2 + \alpha k_0}{m_2}\right) u_2 - \left(\dfrac{c_2 + c_0}{m_2}\right) \dot{u}_2 - \dfrac{(1-\alpha) k_0}{m_2} z \\ \Delta \dot{u} \cdot h(\Delta u, \Delta \dot{u}, z) \end{array} \right\}$$

$$\boldsymbol{f} = \left\{ 0 \quad -\frac{l_1}{m_1} \ddot{x}_g \quad 0 \quad -\frac{l_2}{m_2} \ddot{x}_g \quad 0 \right\}^{\mathrm{T}}$$

Can solve the differential equation by a numerical method such as the fourth and fifth order Runge-Kutta-Fehlberg (RKF) algorithm.

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◎ Equivalent Linearization Method (ELM; a.k.a. stochastic linearization method)

Among various methods such as Fokker-Planck equation, stochastic averaging, moment closure, perturbation (Lutes and Sarkani 2004), ELM is considered as a nonlinear random vibration approach with the highest potential for practical use (Pradlwarter & Schuëller 1991)

- Applicable to both stationary and nonstationary processes
- Applicable to a wide class of nonlinear behavior
- Can handle MDOF systems and FE models
- Takes significantly less efforts than Monte Carlo simulations (especially for low-probability events)

Consider an original nonlinear system: $\dot{\mathbf{y}} = \boldsymbol{g}(\boldsymbol{y}) + \boldsymbol{f}$:

One can find an "equivalent linear" system: $\dot{\mathbf{y}}_e = \mathbf{A} \cdot \mathbf{y}_\mathrm{e} + \boldsymbol{f}$ such that the mean-square error (caused by linearization) $\mathrm{E}[(\boldsymbol{g}(\boldsymbol{y}) - \mathbf{A}\boldsymbol{y})^\mathrm{T}(\boldsymbol{g}(\boldsymbol{y}) - \mathbf{A}\boldsymbol{y})]$ is minimized.

Note: ELM based on the error definition above is considered "standard" ELM while the error measure $\mathrm{E}[(\boldsymbol{g}(\boldsymbol{y}_e) - \mathbf{A}\boldsymbol{y}_e)^\mathrm{T}(\boldsymbol{g}(\boldsymbol{y}_e) - \mathbf{A}\boldsymbol{y}_e)]$ is called "SPEC-alternative" ELM (Crandall 2001).

Crandall, S.H. (2001) Is stochastic equivalent linearization a subtly flawed procedure? *Probabilistic Engineering Mechanics*, 16:169-176

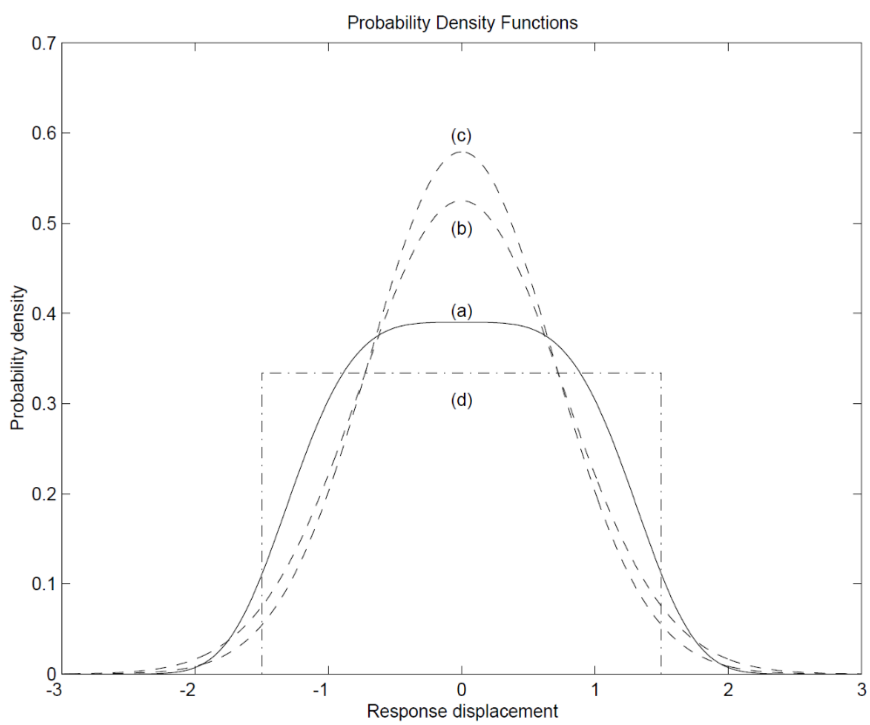

Fig. 2. Probability density functions for cubic oscillator.