Ch 14. Complex Integration

서울대학교
조선해양공학과
서유택
2017.11

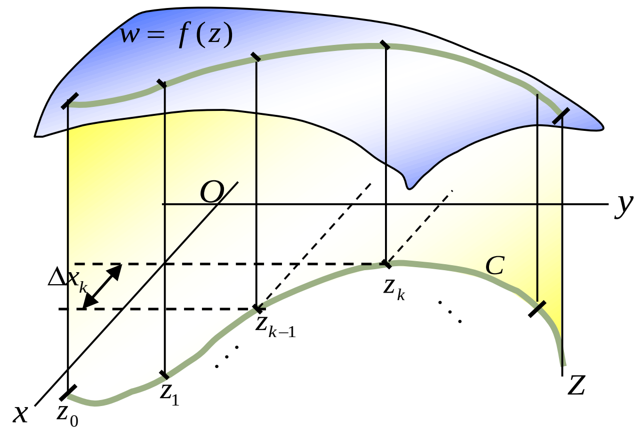
※ 본 강의 자료는 이규열, 장범선, 노명일 교수님께서 만드신 자료를 바탕으로 일부 편집한 것입니다

14.1 Line Integral in the Complex Plane

☑ Complex Line Integral (복소 선적분)

: Integrated over a given curve C (in the complex plane) or a portion of it.

$$\int_C f(z)dz$$



Complex line integral

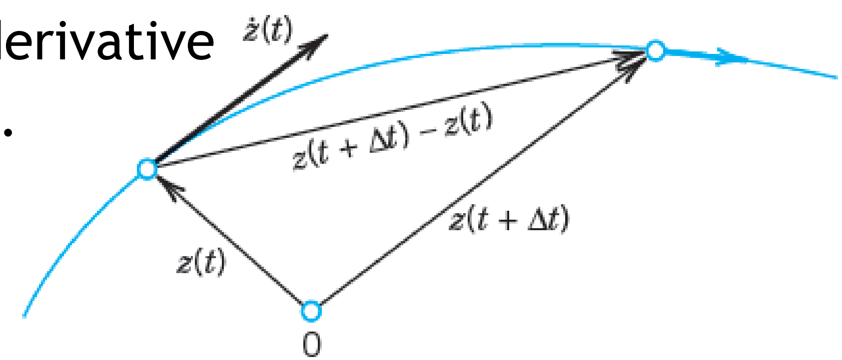
- Path of Integration: $C : z(t) = x(t) + iy(t) \quad (a \le t \le b)$
- Positive Sense (양의 방향): The sense of increasing t

  Ex) $C : z(t) = t + 3it \quad (0 \le t \le 1) \Rightarrow$ the line segment $y = 3x$
- C is Smooth curve

  : C has a continuous and nonzero derivative

  $$\dot{z}(t) = \frac{dz}{dt} = \dot{x}(t) + i\dot{y}(t)$$ at each point.

  $$\dot{z}(t) = \lim_{\Delta t \to 0} \frac{z(t+\Delta t) - z(t)}{\Delta z}$$



Tangent vector $\dot{z}(t)$ of a curve C in the complex plane given by $z(t)$.

14.1 Line Integral in the Complex Plane

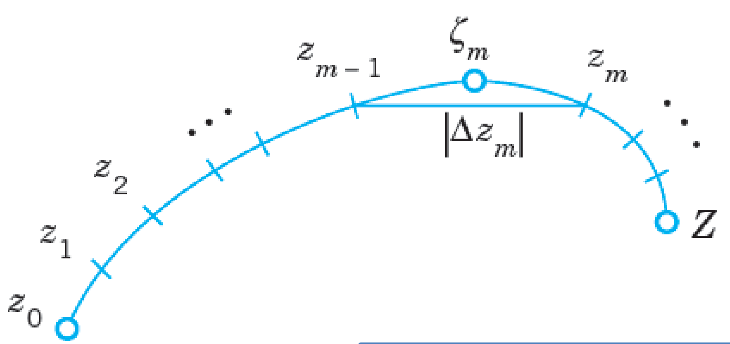
ζ : zeta
ξ : xi (크사이)

☑ Definition of the Complex Line Integral

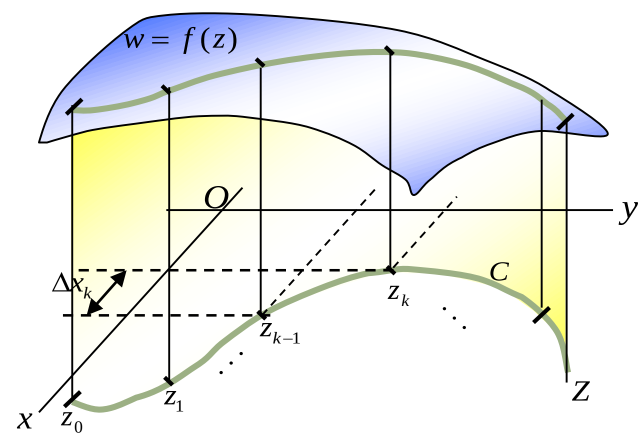
: Integrated over a given curve C (in the complex plane) or a portion of it.

$$t_0(=a),\ t_1,\ \dots,\ t_{n\text{-}1}, t_n(=b) \qquad z_0,\ z_1,\ \dots,\ z_{n\text{-}1}, z_n(=Z)$$

- ζ_1 between z_0 and z_1, and ζ_m between $z_{m\text{-}1}$ and z_m
- $S_n = \sum_{m=1}^{n} f(\varsigma_m)\Delta z_m$ where $\Delta z_m = z_m - z_{m-1}$
- $n \to \infty \ \Rightarrow |\Delta t_m| \to 0 \ \Rightarrow \ |\Delta z_m| \to 0$ ⇨ $\lim_{n\to\infty} S_n = \int_c f(z)dz$



Complex line integral



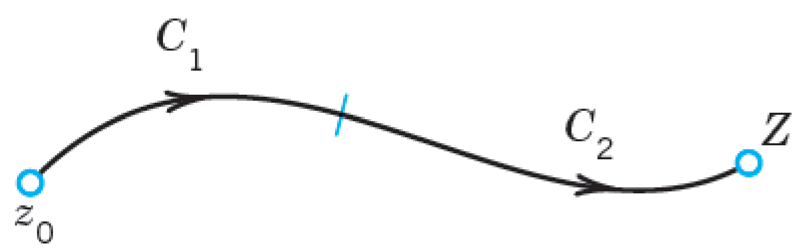
14.1 Line Integral in the Complex Plane

☑ Basic Properties Directly Implied by the Definition

1. Linearity: $\int_C [k_1 f_1(z) + k_2 f_2(z)] dz = k_1 \int_C f_1(z) dz + k_2 \int_C f_2(z) dz$

2. Sense reversal (방향 뒤바뀜): $\int_{z_0}^{Z} f(z) dz = -\int_{Z}^{z_0} f(z) dz$

3. Partitioning of Path: $\int_C f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz$



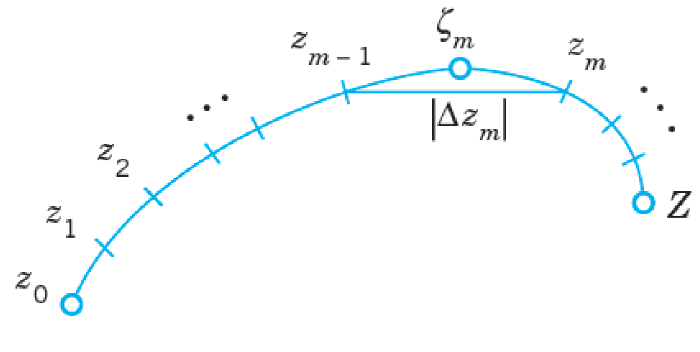
Partitioning of Path

14.1 Line Integral in the Complex Plane

☑ Existence of the Complex Line Integral

ζ : zeta
ξ : xi (크사이)

- f(z) is continuous and C is piecewise smooth imply the existence of the line integral.



$$f(z) = u(x,y) + iv(x,y)$$

$$\zeta_m = \xi_m + i\eta_m \quad \text{and} \quad \Delta z_m = \Delta x_m + i\Delta y_m$$

$$S_n = \sum_{m=1}^{n} f(\zeta_m)\Delta z_m \ \text{ where } \Delta z_m = z_m - z_{m-1}$$

$$\Rightarrow \ S_n = \sum_{m=1}^{n}(u+iv)(\Delta x_m + i\Delta y_m) \ \ \textit{where } u = u(\xi_m, \eta_m),\ v = v(\xi_m, \eta_m)$$

$$S_n = \sum u\Delta x_m - \sum v\Delta y_m + i\left[\sum u\Delta y_m - \sum v\Delta x_m\right]$$

$$\text{As } n \to \infty,\ \Delta x_m \ \&\ \Delta y_m \to 0$$

$$\lim_{n\to\infty} S_n = \int_c f(z)dz = \int_c u dx - \int_c v dy + i\left[\int_c u dy - \int_c v dx\right]$$

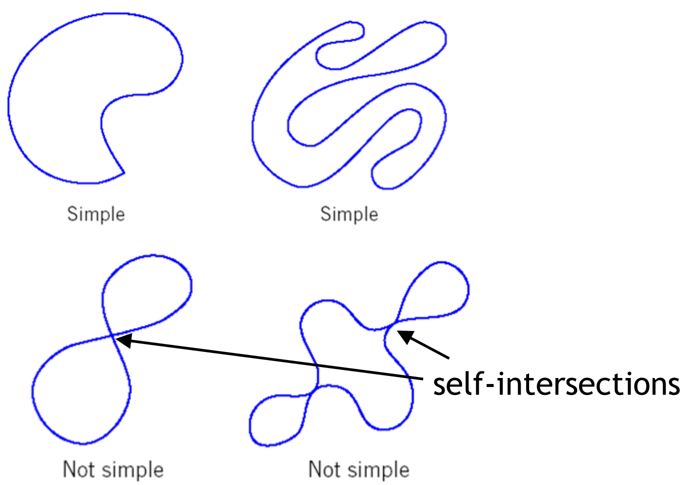
14.1 Line Integral in the Complex Plane

☑ First Evaluation Method: Indefinite Integration (부정적분) and Substitution of Limits (상하한의 대입)

- Simple Closed Curve: Closed curve without self-intersections
- D is Simply connected: Every simple closed curve encloses only points of D.
- Ex. A circular disk (원판) is simply connected, whereas an annulus (환형) is not simply connected.

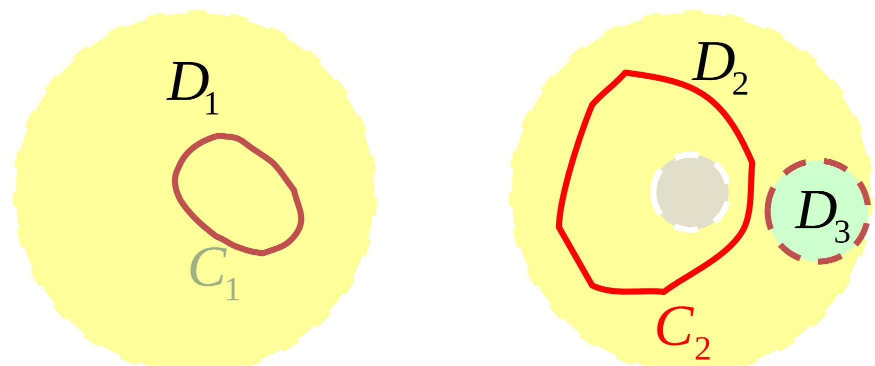
• Simple closed (단순 닫힌)

A simple closed path is a closed path that does not intersect or touch itself



• Simply connected (단순 연결)

A domain *D* is called simply connected if every simple closed curve encloses only points of *D*.



- **Simply connected**
  : D_1, D_3
- **Not simply connected**
  : D_2

14.1 Line Integral in the Complex Plane

☑ Theorem 1 Indefinite Integration of Analytic Functions

- Let $f(z)$ be analytic in a simply connected domain D.
- There exists an indefinite integral of $f(z)$ in the domain D.
- That is, an analytic function $F(z)$ such that $F'(z) = f(z)$ in D.
- For all paths in D joining two points z_0 and z_1 in D we have

$$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0), \quad F'(z) = f(z)$$

- A function $f(z)$ that is analytic for all z is called an entire function.
- If $f(z)$ is entire, we can take for D the complex plane which is certainly simply connected.

Ex.)

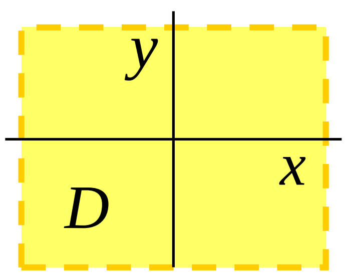
$$\int_0^{1+i} z^2 dz = ?$$

$$\int_0^{1+i} z^2 dz = \frac{1}{3} z^3 \Big|_0^{1+i} = \frac{1}{3}(1+i)^3 = -\frac{2}{3} + \frac{2}{3} i$$

$$\int_{-\pi i}^{\pi i} \cos z dz = ?$$

$\sin z = \sin x \cosh y + i \cos x \sinh y$

$$\int_{-\pi i}^{\pi i} \cos z dz = \sin z \Big|_{-\pi i}^{\pi i} = 2 \sin \pi i = 2(\sin 0 \cosh \pi + i \cos 0 \sinh \pi) = 2i \sinh \pi = 23.097 i$$



***D*: simple connected domain**

☑ Theorem 2 Integration by the Use of the Path

Let C be a piecewise smooth path, represented by $z = z(t)$, where $a \le t \le b$. Let $f(z)$ be a continuous function on C. Then

$$\int_C f(z)dz = \int_a^b f\left[z(t)\right]\dot{z}(t)\,dt \qquad \left(\dot{z} = \frac{dz}{dt}\right)$$

Proof)

$$\int_C f(z)dz = \int_C u\,dx - \int_C v\,dy + i\left[\int_C u\,dy + \int_C v\,dx\right]$$

$$\int_a^b f[z(t)]\dot{z}(t)dt$$

$$= \int_a^b (u+iv)(\dot{x}+i\dot{y})dt$$

$$= \int_C [u\dot{x} - v\dot{y} + i(u\dot{y}+v\dot{x})]dt = \int_C [u\,dx - v\,dy + i(u\,dy + v\,dx)]$$

$$= \int_C u\,dx - \int_C v\,dy + i\left[\int_C u\,dy + \int_C v\,dx\right] = \int_C f(z)dz$$

c.f.* $\quad F(x,y) = F_1(x,y)\mathbf{i} + F_2(x,y)\mathbf{j}$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P(x,y)dx + Q(x,y)dy$$

$$z(t) = x(t) + iy(t), \quad \dot{z}(t) = \dot{x}(t) + i\dot{y}(t)$$

$$\frac{dx}{dt} = \dot{x}, \quad \frac{dy}{dt} = \dot{y}, \qquad dx = \dot{x}dt, \quad dy = \dot{y}dt$$

$$f(z) = u[x(t), y(t)] + iv[x(t), y(t)]$$

14.1 Line Integral in the Complex Plane

☑ Theorem 2 Integration by the Use of the Path

Let C be a piecewise smooth path, represented by $z = z(t)$, where $a \le t \le b$. Let $f(z)$ be a continuous function on C. Then

$$\int_C f(z)dz = \int_a^b f[z(t)]\dot{z}(t)dt \qquad \left(\dot{z} = \frac{dz}{dt}\right)$$

☑ Steps in Applying Theorem 2

A. Represent the path C in the form $z(t)$, $a \le t \le b$

B. Calculate the derivative $\dot{z}(t) = \dfrac{dz}{dt}$

C. Substitute $z(t)$ for every z in $f(z)$ (hence $x(t)$ for x and $y(t)$ for y)

D. Integrate $f[z(t)]\dot{z}(t)$ over t from a to b.

☑ Ex. 5 A Basic Result: Integral of 1/z Around the Unit Circle

Show that by integrating $1/z$ counterclockwise around the unit we obtain

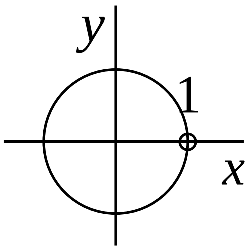
$$\oint_C \frac{dz}{z} = 2\pi i \quad (C \text{ the unit circle, counterclockwise})$$

$$\int_C f(z)dz = \int_a^b f[z(t)]\dot{z}(t)dt$$

Sol)

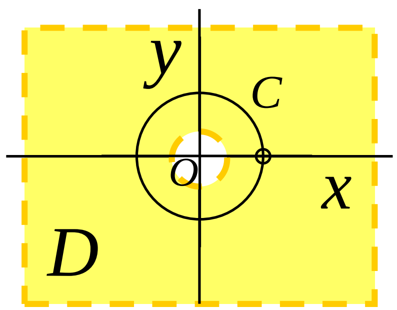
A. Represent the unit circle C by $z(t) = \cos t + i\sin t = e^{it}$

$$(0 \le t \le 2\pi)$$



B. Differentiation gives $\dot{z}(t) = ie^{it}$ (chain rule)

C. By substitution, $f(z(t)) = \dfrac{1}{z(t)} = e^{-it}$



D. Result

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} e^{-it} ie^{it} dt = i\int_0^{2\pi} dt = 2\pi i$$

Q?

The function is not analytic at 0.

$$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0)$$

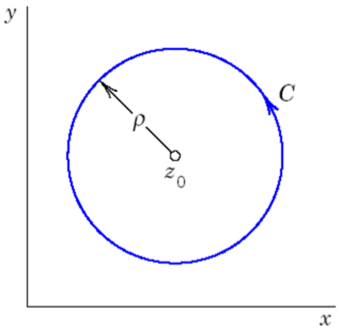
unit circle must contain $z = 0$,
not simply connected
➡ Theorem 1 can't be used for this problem.

☑ Ex. 6 Integral of $1/z^m$ with Integer Power m

Let $f(z) = (z - z_0)^m$ where m is the integer and z_0 a constant. Integrate counterclockwise around the circle C of radius ρ with center at z_0.

Sol)

$$\int_C f(z)dz = \int_a^b f[z(t)]\dot{z}(t)dt$$



$$z(t) = z_0 + \rho(\cos t + i\sin t) = z_0 + \rho e^{it} \quad (0 \le t \le 2\pi)$$

$$(z - z_0)^m = \rho^m e^{imt}, \quad dz = i\rho e^{it}dt$$

$$\oint_C (z - z_0)^m dz = i\rho^{m+1}\int_0^{2\pi} e^{i(m+1)t}dt$$

$$= i\rho^{m+1}\left[\int_0^{2\pi}\cos(m+1)t\,dt + i\int_0^{2\pi}\sin(m+1)t\,dt\right]$$

$$\text{if} \quad m = -1 \quad \oint_C (z - z_0)^m dz = i\int_0^{2\pi} 1dt \; = 2\pi i$$

Not simply connected
($f(z) = 1/(z - z_0)$)

$$\text{if} \quad m \ne -1 \text{ and integer } = \frac{i\rho^{m+1}}{m+1}\left[\sin(m+1)t - i\cos(m+1)t\right]_0^{2\pi} = 0$$

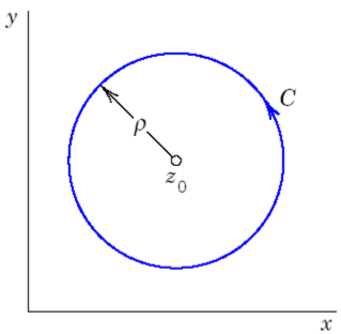
14.1 Line Integral in the Complex Plane

☑ Ex. 6 Integral of $1/z^m$ with Integer Power m

Let $f(z) = (z - z_0)^m$ where m is the integer and z_0 a constant. Integrate counterclockwise around the circle C of radius ρ with center at z_0.

Sol)

$$\int_C f(z)dz = \int_a^b f[z(t)]\dot{z}(t)dt$$



$$z(t) = z_0 + \rho(\cos t + i\sin t) = z_0 + \rho e^{it} \quad (0 \le t \le 2\pi)$$

$$(z - z_0)^m = \rho^m e^{imt}, \quad dz = i\rho e^{it} dt$$

$$\oint_C (z - z_0)^m dz = \begin{cases} 2\pi i & (m = -1), \\ 0 & (m \neq -1 \text{ and integer}) \end{cases}$$

Simple connectedness and analytic function is quite essential (충분조건) in Theorem 1.

Simple connectedness and analytic function ⇨ $\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0)$

14.1 Line Integral in the Complex Plane

If analytic and simply connected

$$\int_C f(z)dz = \int_a^b f[z(t)]\dot{z}(t)dt$$

☑ Dependence on path

A complex line integral depends not only on the endpoints of the path but in general also on the path itself.

☑ Ex. 7 Integral of a Nonanalytic Function. Dependence on Path

Integrate $f(z) = \text{Re}\, z = x$ from 0 to $1+2i$ (a) along C* (b) along C consisting of C_1 and C_2 (Not satisfied: $u_x = v_y$, $u_y = -v_x$ ➡ Nonanalytic)

Sol) (a) C^* : $z(t) = t + 2it \quad (0 \le t \le 1)$

$$\dot{z}(t) = 1 + 2i, \quad f[z(t)] = x(t) = t$$

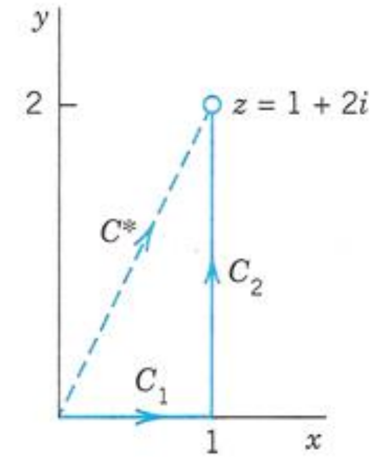
$$\therefore \int_{C^*} \text{Re}\, z dz = \int_0^1 t(1+2i)dt = \frac{1}{2}(1+2i) = \boxed{\frac{1}{2} + i}$$

(b) $C_1 : z(t) = t \quad (0 \le t \le 1) \quad \Rightarrow \quad \dot{z}(t) = 1, \quad f(z(t)) = x(t) = t$

$C_2 : z(t) = 1 + it \quad (0 \le t \le 2) \quad \Rightarrow \quad \dot{z}(t) = i, \quad f(z(t)) = x(t) = 1$

$$\therefore \int_C \text{Re}\, z dz = \int_{C_1} \text{Re}\, z dz + \int_{C_2} \text{Re}\, z dz = \int_0^1 t dt + \int_0^2 1 \cdot i dt = \boxed{\frac{1}{2} + 2i}$$

$\neq$



Paths

14.1 Line Integral in the Complex Plane

☑ Bounds for Integrals (적분한계값). ML-Inequality

Basic formula: $\left|\int_C f(z)\,dz\right| \le ML$ $\quad\quad (\textbf{ML-inequality})$

L is the length of C and M a constant such that $|f(z)| \le M$ everywhere on C.

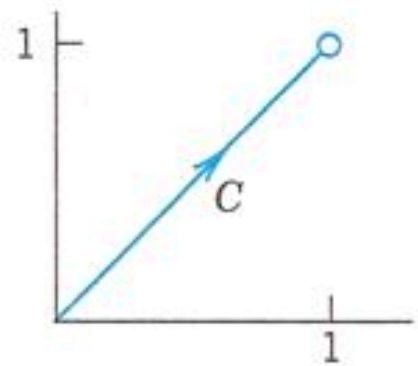
☑ Ex. 8 Estimation of an Integral

Find an upper bound for the absolute value of the integral

$$\int_C z^2\,dz, \qquad C \text{ the straight-line segment from 0 to } 1+i$$

Sol)

$$L = \sqrt{2}, \quad |f(z)| = |z^2| \le 2 \;\Rightarrow\; \left|\int_C z^2\,dz\right| \le 2\sqrt{2} = 2.8284$$



Paths

14.1 Line Integral in the Complex Plane

$$\int_C f(z)dz = \int_a^b f\left[z(t)\right]\dot{z}(t)\,dt$$

☑ Ex. 9) Integration

$$\int_C (z+z^{-1})dz,\ C \text{ the unit circle}$$

Sol)

$$\int_C z\,dz + \int_C z^{-1}dz = 0 + 2\pi i$$

☑ Ex. 10) Integration

$$\int_C \left(\frac{5}{z-2i} - \frac{6}{(z-2i)^2}\right)dz,\ C \text{ the unit circle } |z-2i| = 4 \text{ clockwise}$$

Sol)

$$z(t) = 2i + 4e^{-ti}\ \ (-\pi \le t \le \pi),\ dz = -4e^{-ti}$$

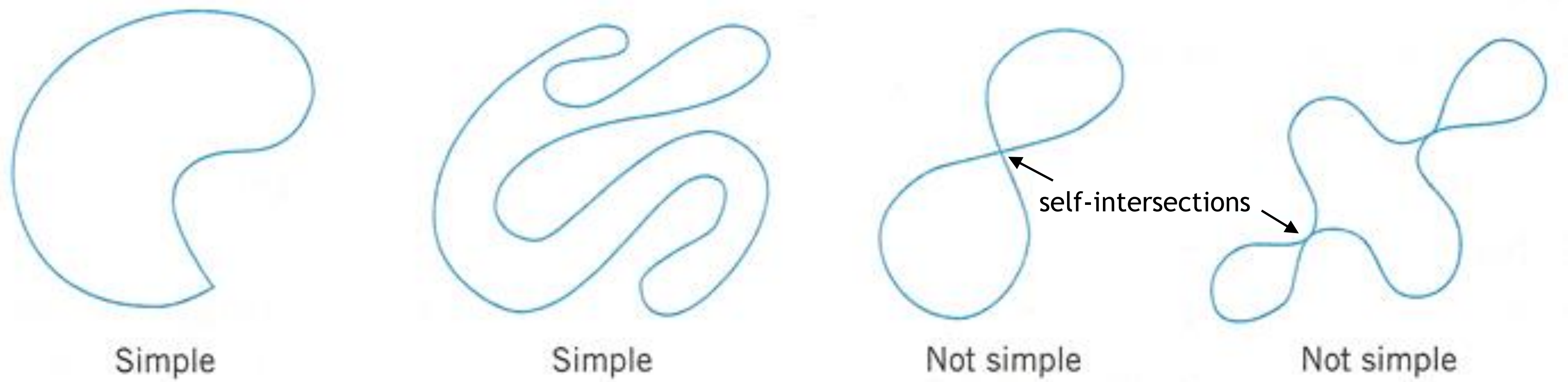
$$\int_C \left(\frac{5}{z-2i} - \frac{6}{(z-2i)^2}\right)dz\ = -4\int_{-\pi}^{\pi}\left(\frac{5}{4}e^{ti} - \frac{3}{8}e^{2ti}\right)e^{-ti}dt$$

$$= -4\int_{-\pi}^{\pi}\left(\frac{5}{4} - \frac{3}{8}e^{ti}\right)dt = -10\pi i + \frac{3}{2}[\sin t - i\cos t]_{-\pi}^{\pi} = -10\pi i$$

14.2 Cauchy's Integral Theorem

☑ **Simple Closed Path (단순 닫힌 경로): A closed path that does not intersect or touch itself.**

- Ex. A circle is simple, but a curve shaped like an 8 is not simple



Closed paths

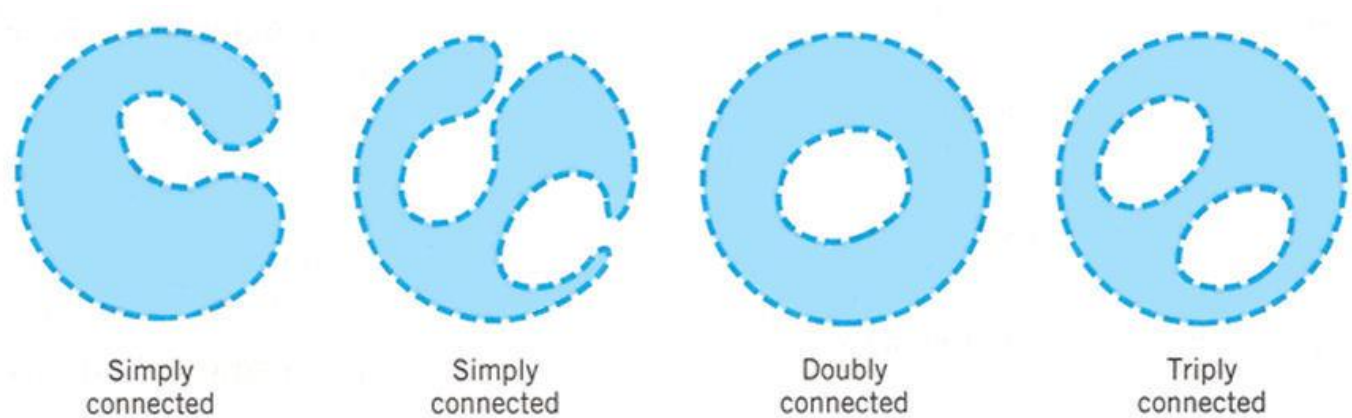
14.2 Cauchy's Integral Theorem

☑ Definition of simple connectedness

- Simply Connected Domain D (단순연결영역): A domain such that every simple closed path in D encloses only points of D

  Ex.) The interior of a circle, ellipse, or any simple closed curve
- Multiply Connected (다중연결): A domain that is not simply connected.

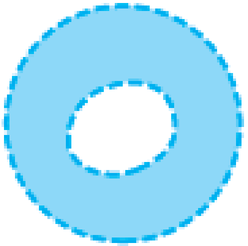
  Ex.) A domain that is not a disk without the center $0 < |z| < 1$



Simply and multiply connected domains

14.2 Cauchy's Integral Theorem

- Bounded Domain (유계영역): A domain that lies entirely in some circle about the origin
- D is p-fold Connected (p중 연결): Its boundary consists of p closed connected sets (curves, segments, or single points) without common points
- D has p-1 holes
- Ex. An annulus is doubly connected (p = 2)

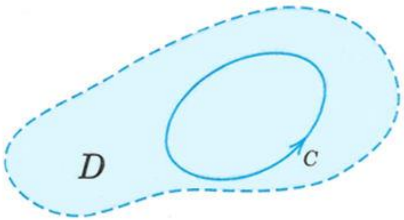


Doubly connected

☑ Theorem 1 Cauchy's Integral Theorem

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D

$$\oint_C f(z)\,dz = 0$$

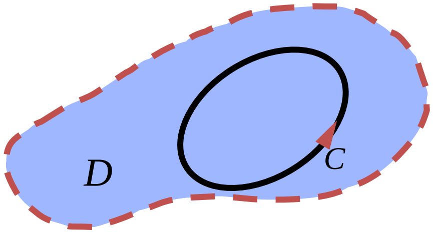


14.2 Cauchy's Integral Theorem

☑ Theorem 1 Cauchy's Integral Theorem

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D

$$\oint_C f(z)\,dz = 0$$



$$\text{Sec. 14.1.}\quad \int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0), \qquad F'(z) = f(z)$$

$$\text{if} \quad z_1 = z_0, \quad \text{then} \int_{z_0}^{z_1} f(z)dz = 0$$

$$\int_{z_0}^{z_1} f(z)dz = 0 \quad (\because \text{closed path}, \quad z_1 = z_0)$$

14.2 Cauchy's Integral Theorem

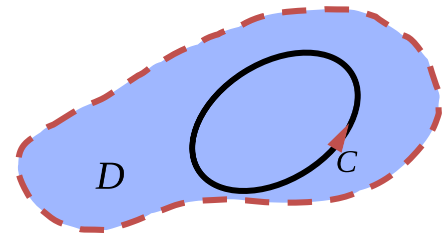
Cauchy - Riemann equation

(1) $u_x = v_y, \quad u_y = -v_x$

☑ Theorem 1 Cauchy's Integral Theorem

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D

$$\oint_C f(z)\,dz = 0$$



☑ Ex. 1 No Singularities (특이점) (Entire Functions, 완전함수)

$$\oint_C e^z\,dz = 0,$$

$$\oint_C \cos z\,dz = 0,$$

$$\oint_C z^n\,dz = 0 \quad (n = 0, 1, \cdots)$$

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y)$$

$$u = e^x \cos y, \; v = e^x \sin y$$

$$u_x = e^x \cos y = v_y$$

$$u_y = -e^x \sin y = -v_x$$

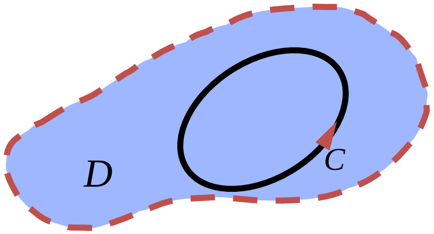
Integrals are zero for any closed path, since these functions are entire (analytic for all z).

14.2 Cauchy's Integral Theorem

☑ Theorem 1 Cauchy's Integral Theorem

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D

$$\oint_C f(z)\,dz = 0$$



☑ Ex. 1 No Singularities (Entire Functions)

$$\oint_C e^z dz = 0,$$

$$\oint_C \cos z\,dz = 0,$$

$$\oint_C z^n dz = 0 \quad (n = 0, 1, \cdots)$$

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

$$u = \cos x \cosh y, \; v = -\sin x \sinh y$$

$$u_x = -\sin x \cosh y = v_y$$

$$u_y = \cos x \sinh y = -v_x$$

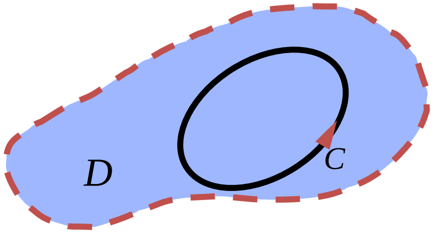
Integrals are zero for any closed path, since these functions are entire (analytic for all z).

14.2 Cauchy's Integral Theorem

☑ Theorem 1 Cauchy's Integral Theorem

If $f(z)$ is <u>analytic</u> in a <u>simply connected</u> domain D, then for every simple closed path C in D

$$\oint_C f(z)\,dz = 0$$



☑ Ex. 2 Singularities outside the contour

$\cos z = \cos x \cosh y - i \sin x \sinh y$

$$\oint_C \sec z\,dz = 0$$

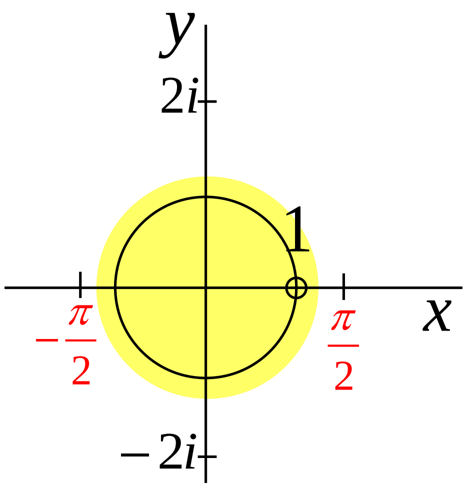
$$\oint_C \frac{dz}{z^2+4} = 0$$

$\sec z = \dfrac{1}{\cos z}$ is not analytic

at $z = \pm\pi/2,\ \pm 3\pi/2, \cdots$

$(\because \cos z = 0)$,

but all these points lie outside *C*; non lies on *C* or inside *C*.



$(C:$ unit circle$)$

14.2 Cauchy's Integral Theorem

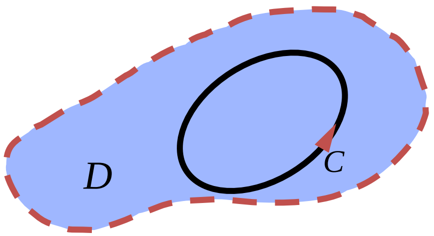
Cauchy - Riemann equation

(1) $u_x = v_y, \quad u_y = -v_x$

☑ Theorem 1 Cauchy's Integral Theorem

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D

$$\oint_C f(z)\,dz = 0$$

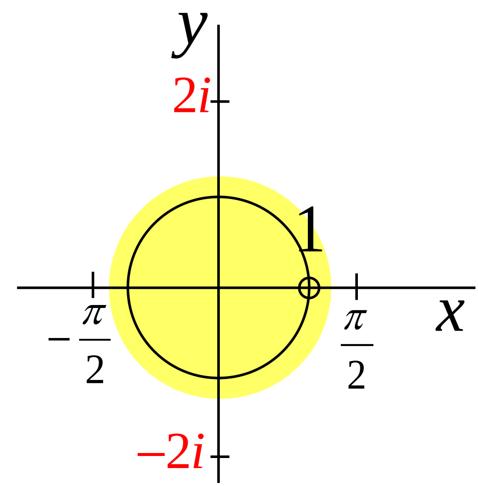


☑ Ex. 2 Singularities outside the contour

$$\oint_C \sec z\,dz = 0$$

$$\oint_C \frac{dz}{z^2+4} = 0$$

$\dfrac{1}{z^2+4}$ is not analytic at $z = \pm 2i$ outside C.



$(C:\ \text{unit circle})$

14.2 Cauchy's Integral Theorem

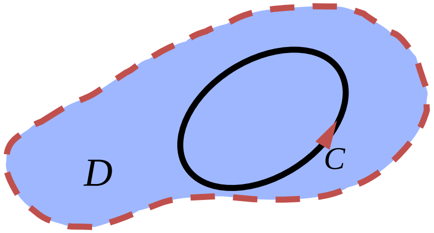
Cauchy - Riemann equation

(1) $u_x = v_y, \quad u_y = -v_x$

☑ Theorem 1 Cauchy's Integral Theorem

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D

$$\oint_C f(z)\,dz = 0$$



☑ Ex. 3 Not analytic function

$$\int_C f(z)dz = \int_a^b f[z(t)]\dot{z}(t)dt$$

$$\oint_C \bar{z}dz = ?$$

$$\oint_C \frac{dz}{z^2} = ?$$

$\left(C : z(t) = e^{it} \text{ is the unit circle}\right)$

$$z(t) = e^{it} \quad (0 \le t \le 2\pi)$$

$$\dot{z}(t) = ie^{it} \qquad f[z(t)] = \bar{z}(t) = e^{-it}$$

$$\oint_C \bar{z}dz = \int_0^{2\pi} e^{-it} \cdot ie^{it}dt = i\int_0^{2\pi} dt = 2\pi i$$

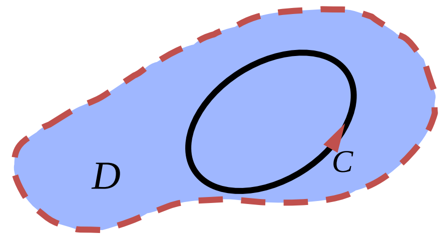
This does not contradict Cauchy's integral theorem because it is not analytic.

14.2 Cauchy's Integral Theorem

☑ Theorem 1 Cauchy's Integral Theorem

If $f\,(z)$ is <u>analytic</u> in a <u>simply connected</u> domain D, then for every simple closed path C in D

$$\oint_C f(z)\,dz = 0$$



☑ Ex. 3 Not analytic function $\left(C: z(t) = e^{it} \text{ is the unit circle}\right)$

$$\oint_C \bar{z}\,dz = ?$$

$$\oint_C \frac{dz}{z^2} = ?$$

$$\int_C f(z)dz = \int_a^b f[z(t)]\dot{z}(t)dt$$

$$z(t) = e^{it} \quad (0 \le t \le 2\pi)$$

$$\dot{z}(t) = ie^{it} \qquad f[z(t)] = 1/z^2(t) = e^{-i2t}$$

$$\oint_C \frac{dz}{z^2} = \int_0^{2\pi} e^{-i2t} \cdot ie^{it}dt = i\int_0^{2\pi} e^{-it}dt = i\int_0^{2\pi} (\cos t - i\sin t)dt$$

$$= i[\sin t + i\cos t]_0^{2\pi} = 0$$

This result does <u>not follow from Cauchy's integral theorem</u>, because $f\,(z)$ is not analytic at $z = 0$. Hence the condition that f be analytic in D is sufficient (충분조건) rather than necessary for Cauchy's theorem to be true.

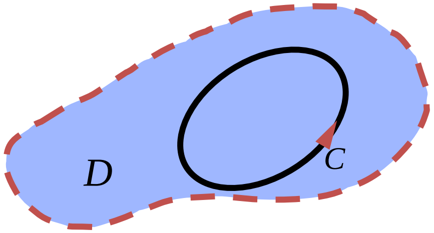
Analytic and simply connected $\underset{\not\Longleftarrow}{\Longrightarrow}$ $\oint_C f(z)\,dz = 0$

14.2 Cauchy's Integral Theorem

☑ Theorem 1 Cauchy's Integral Theorem

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D

$$\oint_C f(z)\,dz = 0$$



☑ Ex. 5 Simple Connectedness Essential

Unit circle C lies in the annulus $\frac{1}{2} < |z| < \frac{3}{2}$ where $\frac{1}{z}$ is analytic, but this domain is not simply connected → Cauchy' theorem cannot be applied.

- The condition that the domain D be simply connected is essential.

"by Cauchy's theorem, if $f(z)$ is analytic on a simple closed path C and everywhere inside C, with no exception, not even a single point, then $\oint_C f(z)\,dz = 0$ holds."

14.2 Cauchy's Integral Theorem

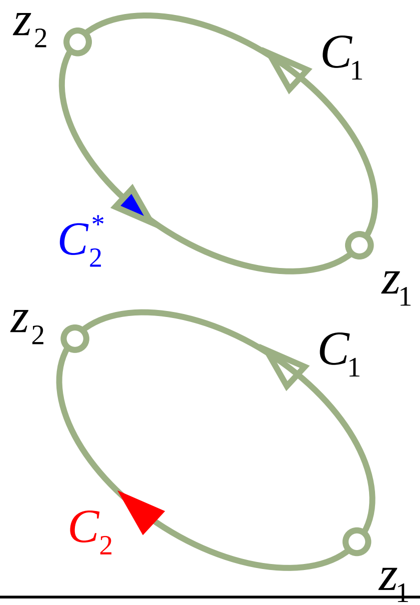
☑ Independence of Path

Integral of $f(z)$ is independent of path in a domain D

$\Leftrightarrow$ for every z_1, z_2 in D, its value depends only on the initial point z_1 and the terminal point z_2, but not on the choice of the path C in D.

☑ Theorem 2 Independence of Path

If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D.



$$\oint_C f(z)dz = \int_{C_1} fdz + \int_{C_2^*} fdz = 0$$

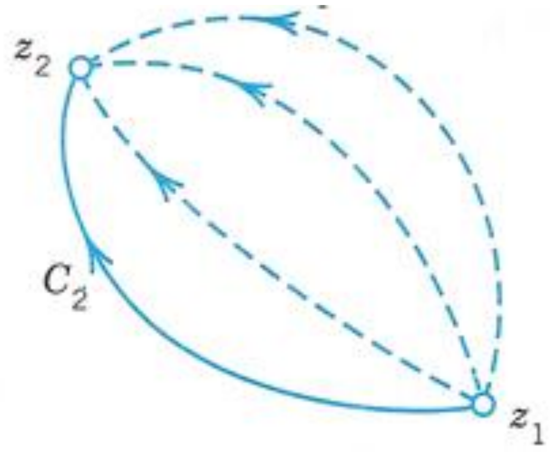
$$\therefore \int_{C_1} fdz = -\int_{C_2^*} fdz$$

$$\int_{C_1} fdz = \int_{C_2} fdz$$

14.2 Cauchy's Integral Theorem

☑ Principle of Deformation of Path (경로변형의 원리)

As long as our deforming path (a continuous deformation of the path of an integral, keeping the end fixed) always contains only points at which $f(z)$ is analytic, the integral retains the same value.



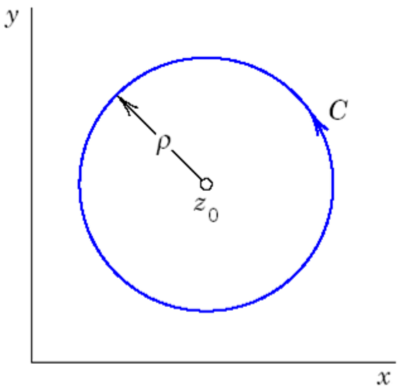
Continuous deformation of path

14.2 Cauchy's Integral Theorem

☑ Theorem 2 Independence of Path

If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D.

☑ Ex. 6 Basic Result: Integral of Integer Powers



$$f(z) = (z - z_0)^m$$

m : integer,
z_0 : constant

if $m < 0$,

$$f(z) = \frac{1}{(z - z_0)^p}$$

is not analytic at z_0.

if $m \geq 0$,

$$f(z) = (z - z_0)^m$$

is analytic for all z.

$$\oint_C (z - z_0)^m \, dz = \begin{cases} 2\pi i & (m = -1), \\ 0 & (m \neq -1 \text{ and integer}) \end{cases}$$

The integral is independent of the radius ρ.

$$\oint_C \frac{1}{z^2} \, dz = 0 \text{ (even if Not Analytic at } z = 0\text{)}$$

14.2 Cauchy's Integral Theorem

☑ Theorem 3 Existence of Indefinite Integral (부정적분)

- If $f(z)$ is analytic in a simply connected domain D, then there exists an indefinite integral $F(z)$ of $f(z)$ in D, which is analytic in D,
- and for all paths in D joining any two points z_0 and z_1 in D, the integral of $f(z)$ from z_0 to z_1 can be evaluated by formula

$$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0)$$

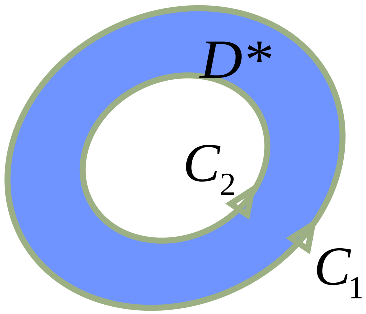
14.2 Cauchy's Integral Theorem

☑ Multiply Connected Domains (다중연결영역)

➢ Doubly connected domain D

- If a function $f(z)$ is analytic in any domain D^* that contains D and its boundary curves, we claim that

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$



Proof)

+: Counterclockwise
-: Clockwise

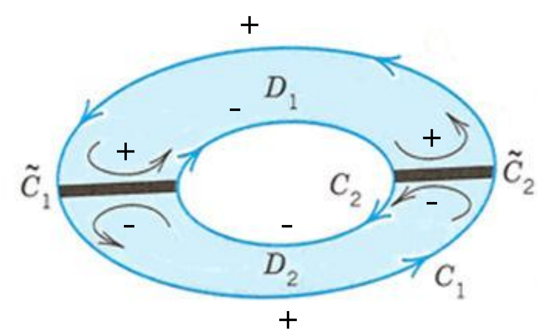
$$D_1 \;:\; \int_{C_{1up}} f(z)dz + \cancel{\int_{\tilde{C}_1} f(z)dz} - \int_{C_{2up}} f(z)dz + \cancel{\int_{\tilde{C}_2} f(z)dz} = 0$$

$$D_2 \;:\; \int_{C_{1down}} f(z)dz - \cancel{\int_{\tilde{C}_2} f(z)dz} - \int_{C_{2down}} f(z)dz - \cancel{\int_{\tilde{C}_1} f(z)dz} = 0$$

$\tilde{C}_1, \tilde{C}_2$: two cuts

$$D_1 + D_2 \;:\; \oint_{C_1} f(z)dz - \oint_{C_2} f(z)dz = 0$$

$$\therefore \oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$



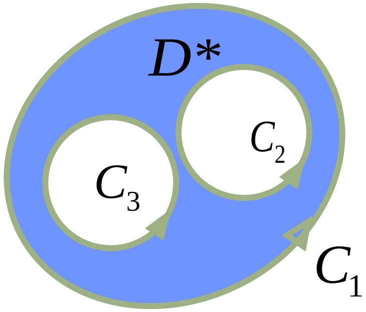
Doubly connected domain

14.2 Cauchy's Integral Theorem

☑ Multiply Connected Domains

- Triply connected domain
  - If a function $f(z)$ is analytic in any domain D^* that contains D and its boundary curves, we claim that

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz$$

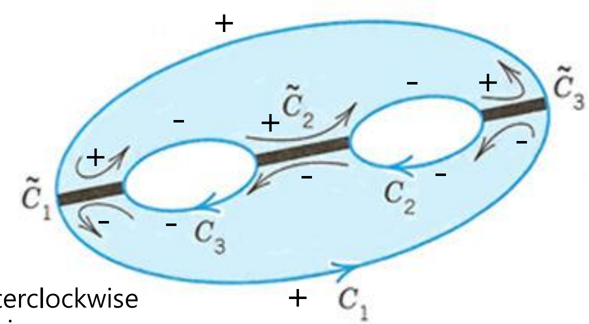


Proof) $D_1 : \int_{C_{1up}} f(z)dz + \cancel{\int_{\tilde{C}_1} f(z)dz} - \int_{C_{3up}} f(z)dz + \cancel{\int_{\tilde{C}_2} f(z)dz} - \int_{C_{2up}} f(z)dz + \cancel{\int_{\tilde{C}_3} f(z)dz} = 0$

$D_2 : \int_{C_{1down}} f(z)dz - \cancel{\int_{\tilde{C}_3} f(z)dz} - \int_{C_{2down}} f(z)dz - \cancel{\int_{\tilde{C}_2} f(z)dz} - \int_{C_{3down}} f(z)dz - \cancel{\int_{\tilde{C}_1} f(z)dz} = 0$

$D_1 + D_2 : \int_{C_1} f(z)dz - \int_{C_3} f(z)dz - \int_{C_2} f(z)dz = 0$

$$\therefore \oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz$$



+: Counterclockwise
-: Clockwise

Triply connected domain

14.3 Cauchy's Integral Formula

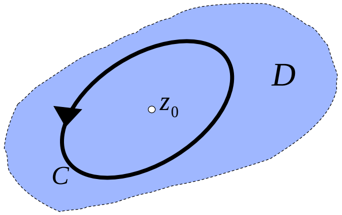
☑ Theorem 1 Cauchy's Integral Formula

If $f(z)$ is analytic in a simply connected domain D. Then for any point z_0 in D and any simply closed path C in D that encloses z_0.

$$\oint_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0) \qquad (\textbf{Cauchy's integral formula})$$

the integration being taken counterclockwise. Alternatively

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz \qquad (\textbf{Cauchy's integral formula})$$



Cauchy's integral formula

14.3 Cauchy's Integral Formula

☑ **Proof)** $f(z) = f(z_0) + [f(z) - f(z_0)]$

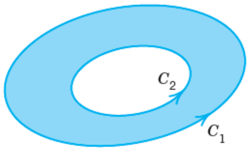
$$\oint_C (z-z_0)^m dz = \begin{cases} 2\pi i & (m=-1), \\ 0 & (m \neq -1 \text{ and integer}) \end{cases}$$

$$\oint_C \frac{f(z)}{z-z_0} dz = \boxed{f(z_0) \oint_C \frac{dz}{z-z_0}} + \oint_C \frac{f(z)-f(z_0)}{z-z_0} dz$$

$$= f(z_0) \cdot 2\pi i$$

- The integrand of the second integral is analytic, except at z_0.
- By (6) in Sec. 14.2

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

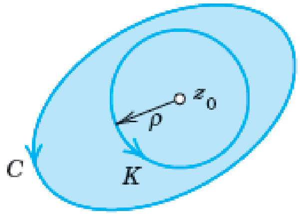


- If an $\varepsilon > 0$ being given

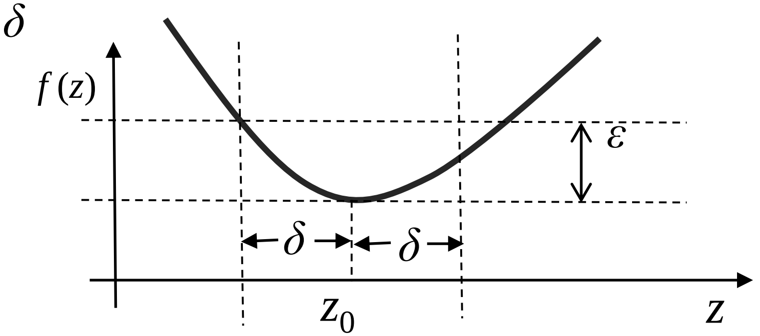
We can find $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ for all z in the disk $|z - z_0| < \delta$

- Choosing the radius of ρ of K smaller than δ

For z on circle on $|z - z_0| = \delta$



$$\left| \frac{f(z)-f(z_0)}{z-z_0} \right| < \frac{\varepsilon}{\delta} < \frac{\varepsilon}{\rho}$$



14.3 Cauchy's Integral Formula

☑ **Proof)** $f(z) = f(z_0) + [f(z) - f(z_0)]$

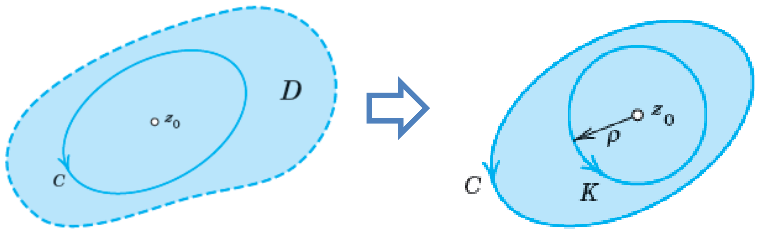
$$\oint_C \frac{f(z)}{z-z_0}dz = f(z_0)\oint_C \frac{dz}{z-z_0} + \boxed{\oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz}$$

$$\left|\int_C f(z)\,dz\right| \le ML \quad (\textbf{ML-inequality})$$

L is the length of C and M a constant such that $|f(z)| \le M$ everywhere on C.

- We can replace C by a small circle K of radius ρ and center z_0

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| < \underbrace{\boxed{\frac{\varepsilon}{\rho}}}_{M}$$



Length of $K = 2\pi\rho$ by $ML-inequality$

$$\left|\oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| < \frac{\varepsilon}{\rho}2\pi\rho = 2\pi\varepsilon \qquad \varepsilon \to 0,\ \left|\oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \to 0$$

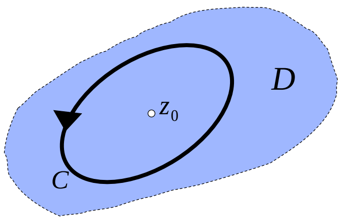
$$\therefore \oint_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$$

14.3 Cauchy's Integral Formula

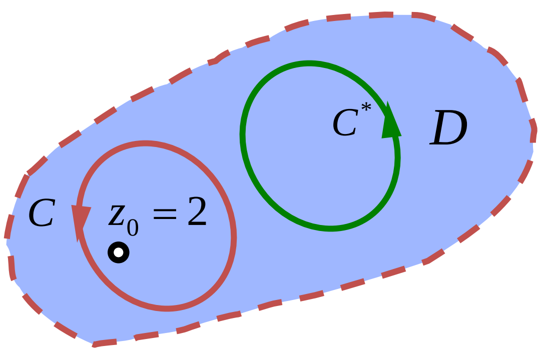
☑ Theorem 1 Cauchy's Integral Formula

If $f(z)$ is analytic in a simply connected domain D. Then for any point z_0 in D and any simply closed path C in D that encloses z_0.

$$\oint_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0) \qquad f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz$$



☑ Ex. 1 Cauchy's Integral Formula



For any contour C enclosing $z_0 = 2$ $\quad \oint_C \frac{e^z}{z-2}dz = ?$

Sol)

$$\oint_C \frac{e^z}{z-2}dz = 2\pi i\, f(z_0) = 2\pi i e^{z_0} = 2\pi i e^2$$
$$= 46.4268\, i$$

For any contour C^* for which $z_0 = 2$ lies outside

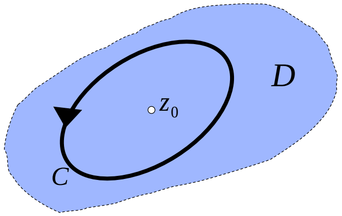
$$\oint_{C^*} \frac{e^z}{z-2}dz = 0$$, since e^z is entire.

14.3 Cauchy's Integral Formula

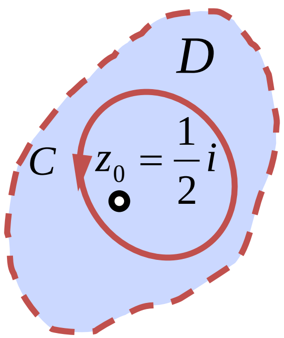
☑ Theorem 1 Cauchy's Integral Formula

If $f(z)$ is analytic in a simply connected domain D. Then for any point z_0 in D and any simply closed path C in D that encloses z_0.

$$\oint_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0) \qquad f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz$$



☑ Ex. 3 Cauchy's Integral Formula



For any contour C enclosing $z_0 = 1/2\mathrm{i}$ $\quad \oint_C \frac{z^3-6}{2z-i}dz = ?$

Sol)

$$\oint_C \frac{z^3-6}{2z-i}dz = \oint_C \frac{\frac{1}{2}z^3-3}{z-\frac{1}{2}i}dz = 2\pi i(\tfrac{1}{2}z^3-3)\Big|_{z=\frac{1}{2}i}$$

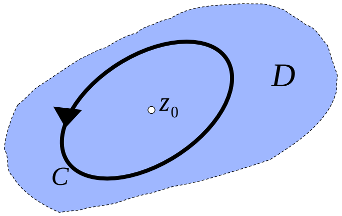
$$= 2\pi i(\tfrac{1}{2}(\tfrac{1}{2}i)^3-3) = \frac{\pi}{8}-6\pi i$$

14.3 Cauchy's Integral Formula

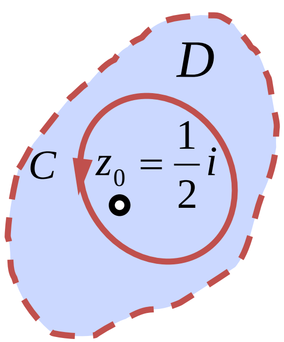
☑ Theorem 1 Cauchy's Integral Formula

If $f(z)$ is analytic in a simply connected domain D. Then for any point z_0 in D and any simply closed path C in D that encloses z_0.

$$\oint_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0) \qquad f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz$$



☑ Ex.) Cauchy's Integral Formula



For the unit circle $\oint_C \frac{z^3}{(2z-1)}dz$

Sol)

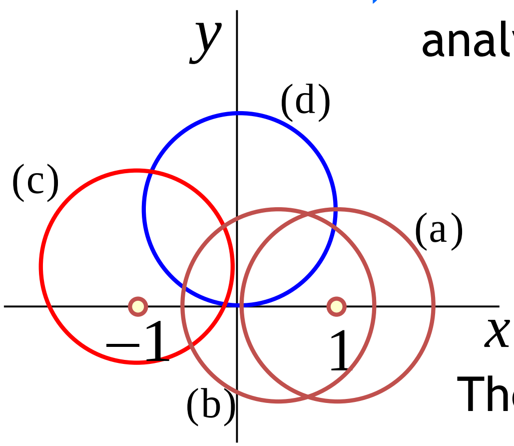
$$\frac{1}{2}\oint_C \frac{z^3}{(z-\frac{1}{2})}dz = \pi i z^3\Big|_{z=i/2} = \frac{\pi}{8}$$

$$\oint_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$$

☑ Ex. 3 Integration Around Different Contours

Integrate $g(z) = \dfrac{z^2+1}{z^2-1} = \dfrac{z^2+1}{(z+1)(z-1)}$ **counterclockwise around each of the four circles.**

Sol) The circles (a) and (b) enclose the point $z_0 = 1$ where $g(z)$ is not analytic.



$$g(z) = \frac{z^2+1}{z^2-1} = \boxed{\frac{z^2+1}{z+1}} \cdot \frac{1}{z-1} \qquad \therefore f(z) = \frac{z^2+1}{z+1}$$

$$\therefore \int_C \frac{z^2+1}{z^2-1}dz = 2\pi i\, f(1) = 2\pi i.$$

The circles (c) enclose the point $z_0 = -1$ where $g(z)$ is not analytic.

$$g(z) = \frac{z^2+1}{z^2-1} = \boxed{\frac{z^2+1}{z-1}} \cdot \frac{1}{z+1} \qquad \therefore f(z) = \frac{z^2+1}{z-1}$$

$$\therefore \int_C \frac{z^2+1}{z^2-1}dz = 2\pi i\, f(-1) = -2\pi i.$$

Singular points : (-1,0), (1,0)

Circles (a), (b), (c) enclose a singular point.

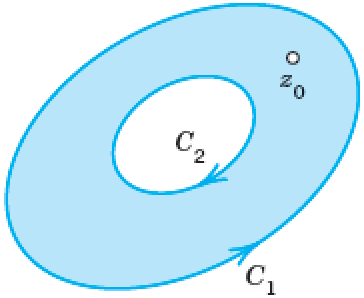
14.3 Cauchy's Integral Formula

☑ Multiply Connected Domains

$f(z)$ is analytic on C_1 and C_2 and in the ring-shaped domain bounded by C_1 and C_2 and z_0 is any point in that domain

$$f(z_0) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z - z_0}dz + \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z - z_0}dz$$

where the outer integral is taken counterclockwise and the inner clockwise.



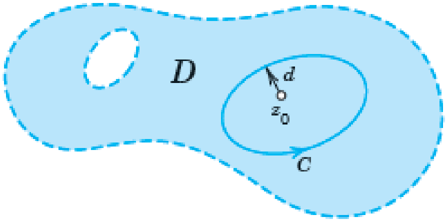
14.4 Derivatives of Analytic Functions

☑ Theorem 1 Derivatives of an Analytic Function

If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D, which are then also analytic functions in D. The values of these derivatives at a point z_0 in D are given by the formulas

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz$$

$$f''(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}\,dz$$



and in general

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz, \quad (n = 1, \ 2, \ \cdots)$$

here C is any simply closed path in D that enclose z_0 and whose full interior belongs to D; and we integrate counterclockwise around C.

14.4 Derivatives of Analytic Functions

$$f(z_0)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz$$

☑ **Proof)** $f'(z_0)=\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$

by Cauchy's integral formula

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=\frac{1}{2\pi i\Delta z}\left[\oint_C \frac{f(z)}{z-(z_0+\Delta z)}dz-\oint_C \frac{f(z)}{z-z_0}dz\right]$$

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=\frac{\cancel{\Delta z}}{2\pi i\cancel{\Delta z}}\oint_C \frac{f(z)}{(z-z_0+\Delta z)(z-z_0)}dz \Rightarrow \Delta z\to 0$$

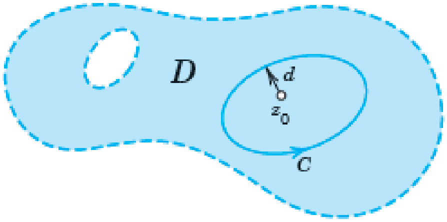
$$\Rightarrow f'(z_0)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz \ ?$$

The above right side can be proved by the following becomes 0 as $\Delta z\to 0$

$$\oint_C \frac{f(z)}{(z-z_0+\Delta z)(z-z_0)}dz-\oint_C \frac{f(z)}{(z-z_0)^2}dz=\oint_C \frac{f(z)\Delta z}{(z-z_0+\Delta z)(z-z_0)^2}dz$$

we show that by the *ML*-inequality,

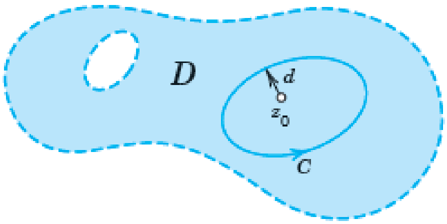
$|f(z)|\le K$ for all z on C, $(z-z_0)^2\ge d^2$ $\quad \frac{1}{(z-z_0)^2}\le\frac{1}{d^2}$



14.4 Derivatives of Analytic Functions

☑ **Proof)** We will show the following by the ML-inequality.

$$\oint_C \frac{f(z)\Delta z}{(z-z_0+\Delta z)(z-z_0)^2}dz \to 0 \quad as \quad \Delta z \to 0$$



$$(z-z_0)^2 \geq d^2 \qquad \boxed{\frac{1}{(z-z_0)^2} \leq \frac{1}{d^2}}$$

$$d \leq |z-z_0| \leq |z-z_0-\Delta z+\Delta z| \leq |z-z_0-\Delta z| + |\Delta z| \quad \Rightarrow \quad \boxed{d-|\Delta z| \leq |z-z_0-\Delta z|}$$

$$\text{let } |\Delta z| \leq \frac{1}{2}d \Rightarrow -\frac{1}{2}d \leq -|\Delta z| \quad \Rightarrow \frac{1}{2}d \leq d-|\Delta z| \leq |z-z_0-\Delta z| \quad \Rightarrow \quad \boxed{\frac{1}{|z-z_0-\Delta z|} \leq \frac{2}{d}}$$

Let *L* be the length of *C*. If $|\Delta z| \leq d/2$

$$\left|\int_C f(z)dz\right| \leq ML \quad (\textbf{ML-inequality})$$

$$\left|\oint_C \frac{f(z)\Delta z}{(z-z_0+\Delta z)(z-z_0)^2}dz\right| \leq KL|\Delta z|\frac{2}{d}\frac{1}{d^2} \qquad \boxed{|f(z)| \leq K}$$

This approaches zero as $\Delta z \to 0$ $\quad \therefore f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz$

14.4 Derivatives of Analytic Functions

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} dz$$

$$f''(z_0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^3} dz$$

☑ Ex. 1 Evaluation of Line Integrals

$$\oint_C \frac{\cos z}{(z - \pi i)^2} dz = 2\pi i (\cos z)' \Big|_{z=\pi i}$$

$$= -2\pi i \sin \pi i = 2\pi \sinh \pi \qquad \text{(counterclockwise)}$$

for any contour enclosing the point πi

☑ Ex. 2 Evaluation of Line Integrals

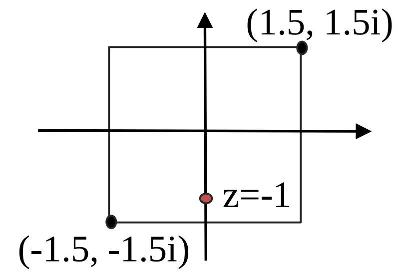
$$\oint_C \frac{z^4 - 3z^2 + 6}{(z + i)^3} dz = \pi i \left(z^4 - 3z^2 + 6\right)'' \Big|_{z=-i}$$

$$= \pi i \left[12z^2 - 6\right]_{z=-i} = -18\pi i \quad \text{(counterclockwise)}$$

for any contour enclosing the point $-i$.

14.4 Derivatives of Analytic Functions

☑ Ex. 3 Evaluation of Line Integrals



$$\oint_C \frac{\mathrm{Ln}(z+3)}{(z-2)(z+1)^2}\,dz$$

C : the boundary of the square with vertices $\pm 1.5,\ \pm 1.5i$, counterclockwise

$$f(z)=\frac{\mathrm{Ln}(z+3)}{(z-2)}$$

$$f'(z)=\frac{1}{(z+3)(z-2)}-\frac{Ln(z+3)}{(z-2)^2}$$

$$\oint_C \frac{\mathrm{Ln}(z+3)}{(z-2)(z+1)^2}\,dz=\oint_C \frac{\mathrm{Ln}(z+3)/(z-2)}{(z+1)^2}\,dz$$

$$=2\pi i\left[\frac{1}{(z+3)(z-2)}-\frac{Ln(z+3)}{(z-2)^2}\right]\Bigg|_{z=-1}$$

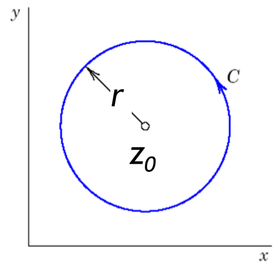
$$=\frac{\pi i}{9}(-3-2Ln2)$$

$$f'(z_0)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz$$

$$f''(z_0)=\frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}\,dz$$

☑ Cauchy's Inequality. Liouville's and Morera's Theorems

- Cauchy's Inequality: $\left|f^{(n)}(z_0)\right| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right| \le \frac{n!}{2\pi} M \frac{1}{r^{n+1}} 2\pi r$ $\quad \left|f(z)\right| \le M$



$$\Rightarrow \left|f^{(n)}(z_0)\right| \le \frac{n!M}{r^n}$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

☑ Theorem 2 Liouville's Theorems

If an entire function is bounded in absolute value in the whole complex, then this function must be a constant.

Proof) $|f(z)| < K$ for all z. $\left|f^{(n)}(z_0)\right| \le \frac{n!M}{r^n} \Rightarrow \left|f'(z_0)\right| \le \frac{K}{r}$

Since $f(z)$ is entire, this holds for every r. For large r

$$f'(z_0) = 0 \Rightarrow f'(z_0) = u_x + iv_x = 0 \Rightarrow u_x = v_x = 0$$

$$\therefore \quad f(z) = u + iv = \text{const for all } z$$

14.4 Derivatives of Analytic Functions

☑ **Theorem 3 Morera's Theorems (Converse (역) of Cauchy's Integral Theorem)**

If $f(z)$ is continuous in a simply connected domain D and if

$$\oint_C f(z)dz = 0$$

for every closed path in D, then $f(z)$ is analytic in D.

$$\oint_C \frac{dz}{z^2} = 0 \quad C: \text{unit circle with its center} = 0$$

This was proved in Sec. 14.2. Even if the integral is zero, because it is not continuous at $z = z_0$, it doesn't follow Morera's theorem. Thus, it is not analytic in D.