Chapter 6 Virtual Work Principles

Principle of virtual Displacement = stiffness method = Displacement method

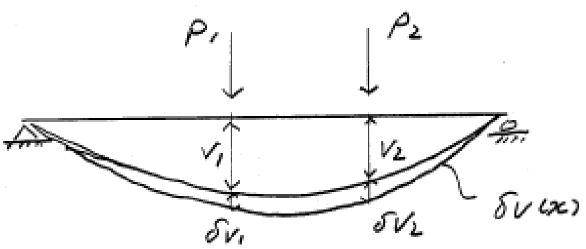
Principle of virtual force = flexibility method = force method

The principle of virtual displacements finds its most powerful application in the development of approximate solutions.

Advantages:

Without the force-equilibrium, the governing equation $(\underline{K}\underline{u} = \underline{P})$ can be obtained, assuming a displacement function.

Even a displacement function not satisfying the equilibrium can be used to obtain an approximate solution.



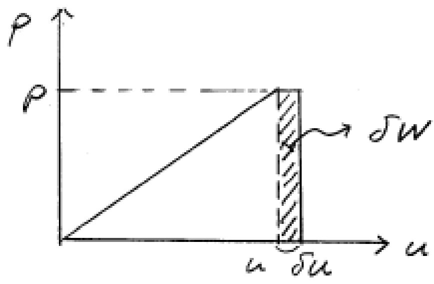
Principle of Virtual Displacement

$\delta W = \delta U \quad (or\ \delta W_{ext} = \delta W_{\text{int}})$

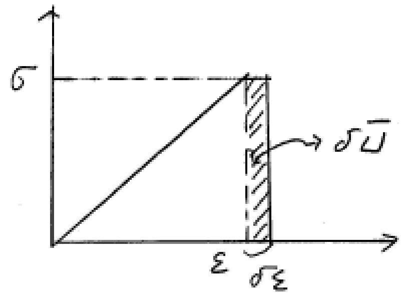
$\delta W =$ virtual external work

$\delta U =$ virtual internal work (virtual strain energy)

$$\delta W = \delta \underline{u} \cdot \underline{P}$$



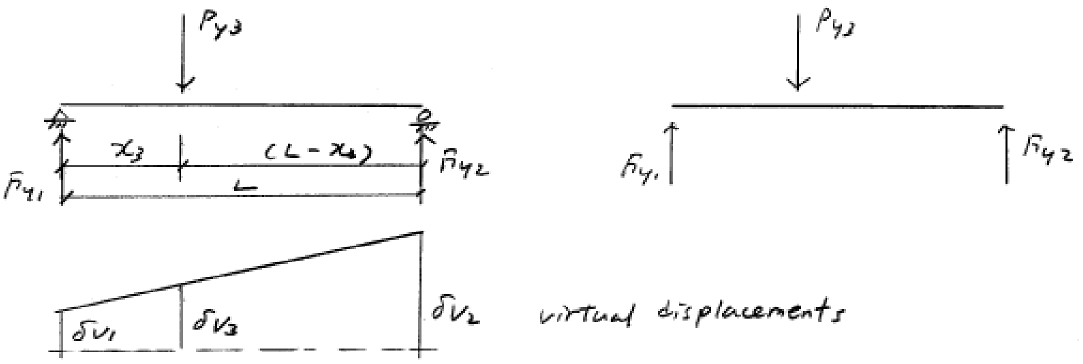
$$\delta U = \int \delta \bar{U} dv \qquad \delta \bar{U} = \delta \underline{\varepsilon} \cdot \underline{\sigma}$$
$$= \int \delta \underline{\varepsilon} \cdot \underline{\sigma} dv$$



6.1 Principle of virtual Displacements – Rigid Bodies

$$\delta W = \delta U = 0$$

For a particle subjected to a system of force in equilibrium, the work due to a virtual displacement is zero.



δv_1 and δv_2 does not induce internal work because of the rigid body motion.

$$\delta W = F_{y1}\delta v_1 + F_{y2}\delta v_2 - P_{y3}\delta v_3$$

$$\delta v = \left(1 - \frac{x}{L}\right)\delta v_1 + \frac{x}{L}\delta v_2 \qquad \text{(satisfy the compatibility for rigid body)}$$

$$\delta v_3 = \left(1 - \frac{x_3}{L}\right)\delta v_1 + \left(\frac{x_3}{L}\right)\delta v_2$$

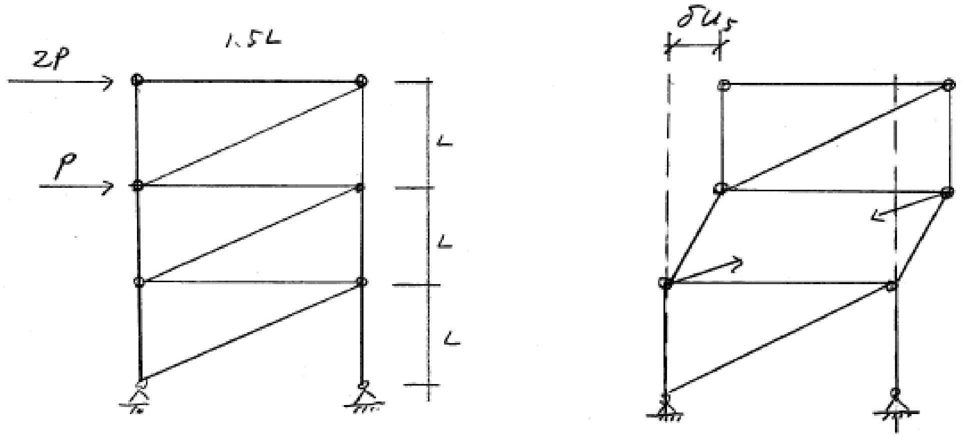
$$\delta W = \underset{\overset{//}{0}}{\underline{\left[F_{y1} - P_{y3}\left(1 - \frac{x_3}{L}\right)\right]}}\delta v_1 + \underset{\overset{//}{0}}{\underline{\left[F_{y2} - P_{y3}\left(\frac{x_3}{L}\right)\right]}}\delta v_2 = 0$$

A particle is in equilibrium under the action of a system of forces if the virtual work is zero for every independent virtual displacement.

$$F_{y1} = P_{y3}\left(1 - \frac{x_3}{L}\right)$$

$$F_{y2} = \frac{P_{y3}x_3}{L}$$

Example 6.1



After releasing member force F_{3-6}, lateral displacement δu_5 cause a rigid body motion which does not cause internal deformation and energy.

$$\delta W = 0 \qquad \Rightarrow \qquad \text{solve } F_{3-6}$$

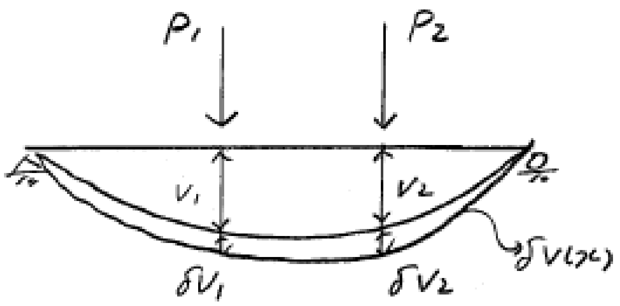
$$\delta W = 2P(\delta u_5) + P(\delta u_5) - F_{3-6}\left[\frac{1.5}{\sqrt{3.25}}(\delta u_5)\right] = 0$$

$$F_{3-6} = 2\sqrt{3.25}\ \mathrm{P}$$

6.2 Principle of virtual Displacements – Deformable bodies

Principle of virtual displacement for deformable bodies

$\delta W = \delta U$



For a deformable structure in equilibrium under the action of a system of applied force, the external virtual work due to an admissible virtual displaced state is equal to the internal virtual work due to the same virtual displacements.

$\delta W = \delta U \qquad \Rightarrow \qquad \underset{\sim}{K}\underset{\sim}{u} = \underset{\sim}{P}$

General Mechanics		Energy Principle
force – equilibrium		principle of virtual displacement
Displacement – compatibility	$\Rightarrow$	Displacement - compatibility
force – Displ. Relationship		force – Displ. Relationship

Since energy is expressed as displacement x force, in order to satisfy $\delta W = \delta U$, at least internal force ≈ external force in average sense if the displacement-compatibility is satisfied.

Internal force is defined as the function of the assumed displacement.

6.3 Virtual Displacement analysis procedure and Detailed Expressions

6.3.1 General procedure

The principle of virtual displacements finds its most powerful application in the development of approximate solutions.

Only displacement functions, which satisfy b/c, are required while force-equilibrium is assumed to be satisfied by using the condition of energy conservation.

6.3.2 Internal virtual work

1) Axial force member

$$\delta\bar{U}\left(\text{virtual energy density}\right)$$
$$=\delta\varepsilon_x\sigma_x$$

Internal work is defined as the function of displacement.

$\delta U\left(\text{virtual energy}\right)$

$$=\int \delta\bar{U}dV$$

$$=\int \delta\varepsilon_x \sigma_x dV$$

$$=\int_o^L \delta\varepsilon_x \sigma_x \mathrm{dx}$$

$$=\int_o^L \delta\varepsilon_x EA\varepsilon_x dx \ =\int_o^L (\frac{d\delta u}{dx})EA(\frac{du}{dx})dx$$

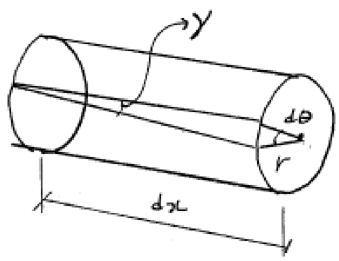
2) Torsional Member (pure torsion)

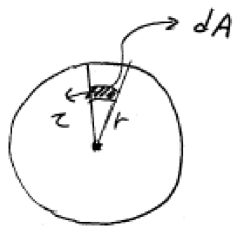
$$\delta\bar{U} = \delta\underset{\sim}{\varepsilon} \cdot \underset{\sim}{\sigma} = \delta\gamma \cdot \tau$$

$$\gamma = r\frac{d\theta}{dx} = r \cdot \beta$$

$$\tau = G\gamma$$

$$M_x = \int \tau \cdot r dA = G\beta \int r^2 dA = GJ\beta$$





$$\delta U = \int \delta\gamma \cdot \tau dV$$

$$=\int_o^L \int_o^A \delta\gamma \cdot \tau c$$

$$=\int\int \delta\beta G r^2 \beta$$

$$=\int \delta\beta GJ\beta dx$$

$$=\int (\frac{d\delta\theta_x}{dx})GJ(\frac{d\theta_x}{dx})dx$$

$$=\int \delta\beta \cdot M_x dx \qquad (M_x = \mathrm{GJ}\frac{d\theta_x}{dx})$$

3) flexural member

$$\delta\bar{U} = \delta\underset{\sim}{\varepsilon} \cdot \underset{\sim}{\sigma} = \delta\varepsilon_x\sigma_x$$

$$\delta\varepsilon_x = -y\frac{d^2\delta v}{dx^2} = -y\delta\phi$$

$$\sigma_x = E\varepsilon_x = -Ey\phi$$

$$\begin{aligned}
\delta U &= \int \delta\varepsilon_x\sigma_x \ dV \\
&= \int_L \int_A \delta\varepsilon_x\sigma_x dA \ dx \\
&= \int \int_A \delta\phi E y^2 \phi dA \ dx \\
&= \int \delta\phi E I_z \phi \ dx \qquad (\int y^2 dA = I_z) \\
&= \int_o^L \int \left(\frac{d^2\delta v}{dx^2}\right) \mathrm{EI}_z \left(\frac{d^2 v}{dx^2}\right) dx \\
&= \int_o^L \delta\phi M_z \ dx \qquad (M_z = EI_z\phi)
\end{aligned}$$

6.3.3 External virtual work

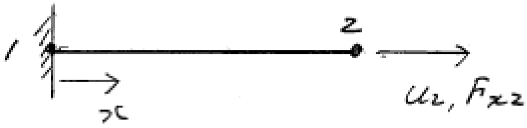
$$\delta W = \sum \delta u_i P_i + \int \delta\underset{\sim}{u} \cdot \underset{\sim}{b} \ dV$$

$$(\text{or} \int \delta\underset{\sim}{u} \cdot \underset{\sim}{b} \ dx)$$

Example 6.2

6.4 Construction of Analytical Solutions by the principle of virtual displacements

6.4.1 Exact solutions



select $u = \frac{x}{L} u_2$ which satisfy the B/C's: at $x = 0 \quad u = 0$
at $x = L \quad u = u_2$

$$u = \frac{x}{L} u_2 \qquad \varepsilon_x = \frac{du}{dx} = \frac{u_2}{L}$$

$$\delta u = \frac{x}{L} \delta u_2 \qquad \delta \varepsilon_x = \frac{1}{L} \delta u_2$$

$$\begin{aligned}\delta U &= \int \delta \underline{\varepsilon} \cdot \underline{\sigma} \; dV \\ &= \int_0^L \delta \varepsilon_x EA \varepsilon_x \; dx \\ &= \int_0^L \frac{\delta u_2}{L} EA \frac{u_2}{L} \; dx \\ &= \frac{EA}{L} (\delta u_2) u_2\end{aligned}$$

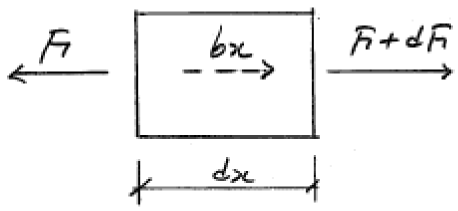
$$\delta W = \delta u_2 F_{x2}$$

$$\delta U = \delta W \qquad \Rightarrow \qquad F_{x2} = \frac{EA}{L} u_2$$

$$\underline{d} = \frac{L}{EA}$$

Why is $u = \frac{x}{L} u_2$ exact displacement function?

The exact displacement function is the one that satisfies the force-equilibrium.



By equilibrium, $dF + b_x dx = 0$

$$b_x = -\frac{dF}{dx}$$

When $b_x = 0,\quad \frac{dF}{dx} = 0$

$$F = \sigma \cdot A = EA\varepsilon = EA\frac{du}{dx}$$

$$\Rightarrow \frac{d}{dx}(EA\frac{du}{dx}) = 0$$

When E and A are constant,

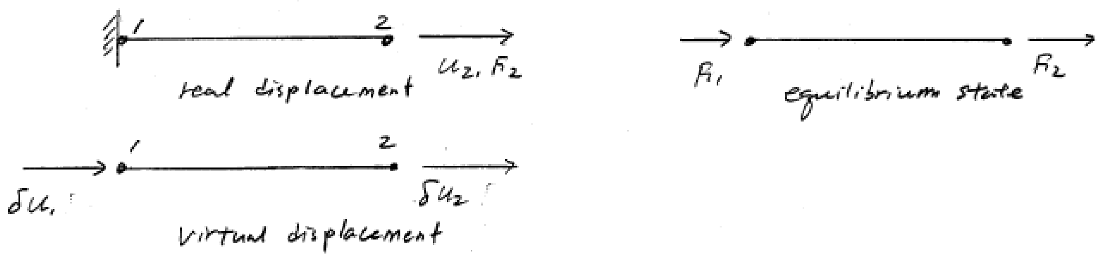
$$\frac{d^2u}{dx^2} = 0 \qquad \Rightarrow \qquad \text{the condition for the exact displacement function}$$

$$\text{u=}\frac{x}{L}u_2 \qquad \text{satisfies } \left(\frac{d^2u}{dx^2} = 0\right)$$

Virtual displacement with different $\left[\begin{array}{l}\textbf{B/C}\\ \textbf{displacement shape}\end{array}\right.$

yields the same equilibrium equation

$$\delta U = \int \delta \underline{\varepsilon} \underline{\sigma} \; dV$$



For virtual displacement $\delta u_v = (1 - \frac{x}{L})\delta u_1 + \frac{x}{L}\delta u_2$ Eq. 1

- Virtual displacement is applied to the equilibrium system and is not related to the actual displ. B/C. Thus δu_v in Eq. 1 is valid.

- $F_1 = -F_2 = -\frac{EA}{L}u_2 \;\Rightarrow\;$ Reaction can be calculated

- $\delta u_v = \frac{x}{L}\delta u_2$ is a special case of Eq. 1

$$\delta U = \int \frac{d\delta u_v}{dx} EA \frac{du_r}{dx} dx \qquad\qquad u_r = \frac{x}{L} u_2$$

$$= \int (-\frac{\delta u_1}{L} + \frac{\delta u_2}{L}) EA \frac{du_r}{dx} dx$$

$$= -\delta u_1 \frac{EA}{L} u_2 + \delta u_2 \frac{EA}{L} u_2$$

$$\delta W = \delta u_1 F_1 + \delta u_2 F_2$$

$$\delta U = \delta W \quad \Rightarrow \quad F_1 = -\frac{EA}{L} u_2$$

$$F_2 = \frac{EA}{L} u_2$$

If we use for both real displacement and virtual displacement,

$$u_r = (1 - \frac{x}{L}) u_1 + \frac{x}{L} u_2$$

$$\delta u_v = (1 - \frac{x}{L}) \delta u_1 + \frac{x}{L} u_2$$

$$\delta U = \int \frac{d\delta u_v}{dx} EA \frac{du_r}{dx} \ dx$$

$$= \int (-\frac{\delta u_1}{L} + \frac{\delta u_2}{L}) EA (-\frac{u_1}{L} + \frac{u_2}{L}) \ dx$$

$$\delta W = \delta u_1 F_1 + \delta u_2 F_2$$

$$\delta U = \delta W \qquad F_1 = \frac{EA}{L}(u_1 - u_2)$$

$$F_2 = \frac{EA}{L}(u_2 - u_1)$$

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Different displacement shape for virtual displacement

$$\begin{bmatrix} \delta u_v = (\frac{x}{L})^2 \delta u_2 \\ \delta u_v = \sin\frac{\pi x}{2L}\delta u_2 \end{bmatrix} \rightarrow$$ satisfy displacement boundary condition.

but not satisfy the force-equilibrium.

Real displacement $u_r = \frac{x}{L} u_2$

$$\delta U = \int \frac{d\delta u_v}{dx} EA \frac{du_r}{dx}\ dx$$

Integration by part

$$\delta U = \int \frac{d\delta u_v}{dx} EA \frac{du_r}{dx}\ dx$$

$$\left(\frac{d\delta u_v}{dx} = a' \quad \frac{du_r}{dx} = b, \qquad \int a'b = |ab| - \int ab' \right)$$

$$= EA \left| \delta u_v \frac{du_r}{dx} \right|_0^L - EA \int \delta u_v \frac{d^2 u_r}{dx^2} dx \qquad \left(\frac{d^2 u_r}{dx^2} = 0 \quad \because \frac{dF}{dx} = 0 \right)$$

$$= EA \left| \delta u_v \frac{du_r}{dx} \right|_0^L$$

This result indicates that δU is affected by the values at the starting and last points, regardless of the shape of the u_v

Thus if virtual displacement δu_v satisfies the displacement B/C, any form of virtual displacement can be used.

(Here, $x = 0 \quad \delta u_v = 0$, $x = L \quad \delta u_v = \delta u_2$)

However, the real displacement $\mathrm{u}_r \ (= \frac{x}{L} u_2)$ should satisfy

$\frac{d^2 u_r}{dx^2} = 0$ (when $b_x = 0$), which is the force-equilibrium condition.

$$\begin{cases} \delta u_v = \left(\frac{x}{L} \right)^2 \delta u_2 \\ u_r = \left(\frac{x}{L} \right)^2 u_2 \end{cases} \qquad \begin{cases} \delta u_v = \sin(\frac{\pi x}{2L}) \delta u_2 \\ u_r = \sin(\frac{\pi x}{2L}) u_2 \end{cases}$$

$\Rightarrow$ cannot get the exact solution, because $\frac{d^2 u_r}{dx^2} \neq 0$

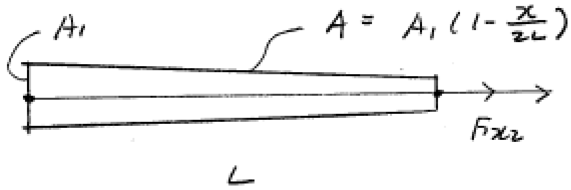
If the chosen read displacements corresponds to stresses that satisfy identically the conditions of equilibrium, any form of admissible virtual displacement will suffice to produce the exact solution.

6.4.2 Approximate solutions and the significance of the chosen virtual displacements

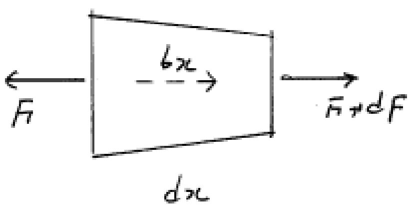
Principle of virtual work is applicable to seeking approximate solutions for

- Frame work Analysis: tapered section, nonlinearity, instability, dynamics
- All finite element analysis

For example, tapered truss element.



Equilibrium condition



$$\frac{dF}{dx} = -b_x$$

If $b_x = 0$, $\underline{\frac{dF}{dx} = 0}$

If $u = \frac{x}{L}u_2$ is used,

$$\sigma_x = E\varepsilon = E\frac{du}{dx} = E\frac{u_2}{L}$$

But, $\dfrac{dF_x}{dx} = d\left[\sigma_x A_1(1-\dfrac{x}{2L})\right]/dx \;= d\left[\mathrm{E}\dfrac{u_2}{L}A_1(1-\dfrac{x}{2L})\right]/dx \neq 0$

⇒ violate the force-equilibrium condition

Nevertheless, we can use the approximate displacement function

$u = \dfrac{x}{L}u_2$ which is exact only for prismatic elements.

$$\delta U = \int_o^L \frac{d\delta u}{dx} EA \frac{du}{dx} dx$$
$$= \int_o^L \left(\frac{\delta u_2}{L}\right)\left(\frac{u_2}{L}\right) EA_1(1-\frac{x}{2L})dx$$
$$= \delta u_2 u_2 \frac{3EA_1}{4L}$$

$$\delta W = \delta u_2 \cdot F_{x2}$$

$$\delta U = \delta W \quad \Rightarrow \quad F_{x2} = \frac{3EA_1}{4L}u_2$$

$$\text{for} \quad \left(\begin{array}{l} u = \dfrac{x}{L}u_2 \\ \delta u = \left(\dfrac{x}{L}\right)^2 \delta u_2 \end{array}\right) \quad \Rightarrow \quad F_{x2} = \frac{2EA_1}{3L}u_2$$

$$\text{for} \quad \left(\begin{array}{l} u = \dfrac{x}{L}u_2 \\ \delta u = \sin\left(\dfrac{\pi x}{2L}\right)\delta u_2 \end{array}\right) \quad \Rightarrow \quad F_{x2} = \frac{0.6817EA_1}{L}u_2$$

Exact solution = $0.721\dfrac{EA_1}{L}u_2$

Discussions

1) Although u is not exact displacement function,

$\delta W = \delta U$ (energy conservation) force to provide a basis for the calculation of the undetermined parameter u_2

2) Although the approximate real displacement cannot satisfy the equilibrium conditions at every locations, the enforcement of the condition $\delta W = \delta U$ results in average satisfaction of the equilibrium throughout the structure.

3) The standard procedure for choosing the form of virtual displacement is to adopt the same form as the real displacement for convenience and to make the stiffness matrix symmetric.

Requirement of displacement function (real displacement)

1) Displacement Boundary condition should be satisfied.

$$\begin{aligned} u &= f(\text{nodal displacements}) \\ &= f(u_1, u_2) \end{aligned}$$

In case of 2-node truss element, there are two nodes. Thus, only two terms can be used when a polynomial equation is used.

For example $u = a_0 + a_1 x$

$$x = 0,\ u = u_1, \quad x = L,\ u = u_2 \quad \Rightarrow \quad a_0 = u_1 \quad a_1 = \frac{u_2 - u_1}{L}$$

2) Rigid body motion (no strain) and constant strain should be described.

$$u = a_0 + a_1 x \quad \text{(O.K)}$$

$$u = a_0 + a_1 \sin\frac{\pi}{2L} x \quad \text{(Not OK)}$$

3) Force-equilibrium should be satisfied.

For axial force member

$$\frac{dF}{dx} = b_x \quad \Rightarrow \quad \frac{d}{dx}(EA\frac{du}{dx}) = b_x$$

1) is the essential condition to get at least an approximate solution

2) is the convergence condition to get a reasonably accurate solution by increasing the number of elements.

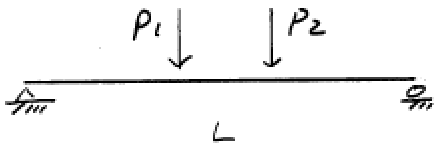
3) is the condition to obtain the exact solution.

Generally, virtual displacement function is the same as the real displacement Function.

$\Rightarrow$ symmetric matrix

Example 6.5

Rayleigh-Ritz method



$$\upsilon = a_1 \sin\frac{\pi x}{L} + a_2 \sin\frac{2\pi x}{L}$$

$a_1, a_2 =$ generalized displacements

$\delta W = \delta U \Rightarrow$ solve approximate solution

6.5 Principle of Virtual Force

Principle of Virtual Force

≡ flexibility method

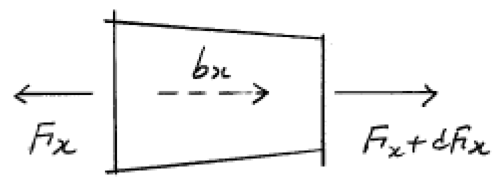
≡ Force method

6.5.1 Equations of Equilibrium

The fundamental requirement on virtual force systems is that they meet the relevant conditions of equilibrium.

For axial force member,

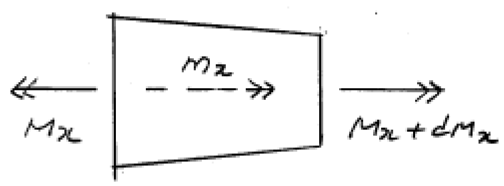
$$\frac{dF}{dx} = -b_x$$
$$\sigma_x = F_A / A$$



For torsional member,

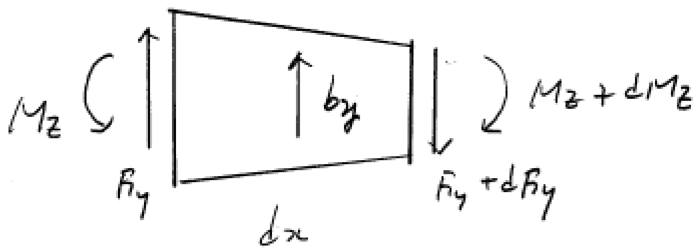
$$\frac{dM_x}{dx} = -m_x$$

$$\tau = \frac{M_x r}{J}$$



For flexural member,

$$\sum F_y = -dF_y + b_y dx = 0$$
$$\Rightarrow \frac{dF_y}{dx} = b_y$$
$$\sum M_z = dM_z - F_y \cdot dx = 0$$
$$\Rightarrow \quad \frac{dM_z}{dx} = F_y$$
$$\frac{d^2 M_z}{dx^2} = b_y$$



6.5.2 Characteristics of virtual force systems

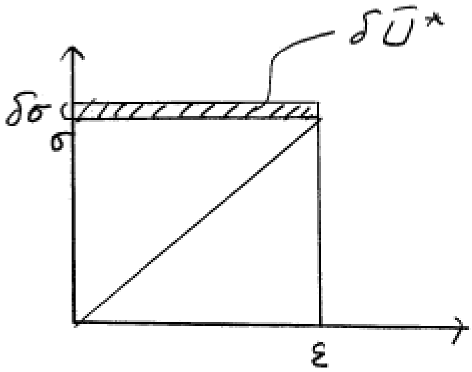
External Equilibrium Equation. $\sum F = 0$

Internal Equilibrium Equation.

$$\begin{cases} \dfrac{dF_x}{dx} = -b_x \\ \dfrac{dM_x}{dx} = -m_x \\ \dfrac{dF_y}{dx} = b_y \quad \text{and} \quad \dfrac{dM_z}{dx} = F_y \end{cases}$$

Virtual complementary Strain energy

$$\delta U^* = \int \delta \overline{U}^* dv$$
$$= \int \delta \underline{\sigma} \cdot \underset{\sim}{\varepsilon}\ dv$$
$$= \int \delta \underline{\sigma} \underline{\underline{E}}^{-1} \underline{\sigma}\ dv$$



$\delta W^* = \delta U^*$ gives the conditions of compatibility.

The strains and displacements in a deformable system are compatible and consistent with the constraints if and only if the external complementary virtual work is equal to the internal complementary virtual work for every system of virtual force and stresses that satisfy the conditions of equilibrium.

Even if the real force state does not correspond to a deformational state that exactly satisfies compatibility, $\delta W^* = \delta U^*$ can be used to enforce an approximate satisfaction of the conditions of compatibility.

For axial force member,

$$\delta U^* = \int \delta \underline{\sigma}\, \underline{E}^{-1} \underline{\sigma}\ dV$$

$$= \int \delta\sigma \frac{1}{E} \sigma\ dV$$

$$= \int \delta\sigma \frac{A}{E} \sigma dx$$

$$(\sigma = \frac{F_x}{E} \qquad \delta\sigma = \frac{\delta F_x}{E})$$

$$= \int \frac{1}{EA} \delta F_x\ F_x\ \mathrm{d}x$$

For torsional member

$$\delta U^* = \int \delta \underset{\sim}{\sigma}\, \underset{\sim}{E}^{-1} \underset{\sim}{\sigma}\ dV$$

$$= \int \delta\tau \frac{1}{G} \tau\ dV \qquad \tau = \frac{M_x r}{J}$$

$$= \int \frac{1}{GJ^2} \int \delta M_x \cdot r^2 M_x dA\ dx$$

$$= \int \frac{1}{GJ} \delta M_x M_x\ \mathrm{d}x$$

For flexural member

$$\delta U^* = \int \delta \underset{\sim}{\sigma}\, \underset{\sim}{E}^{-1} \underset{\sim}{\sigma}\ dV$$

$$= \int \delta\sigma \frac{1}{E} \sigma\ dV \qquad \sigma = -\frac{My}{I}$$

$$= \int \frac{1}{EI} \delta M \cdot M\ \ dx$$

6.5.4 Construction of Analytical Solutions by virtual force principle

Axial force member

$$\delta U^* = \int \delta F_x \cdot \frac{F_x}{EA} dx \qquad F_x = F_{x2}$$

$$= \delta F_{x2} \frac{L}{EA} F_{x2}$$

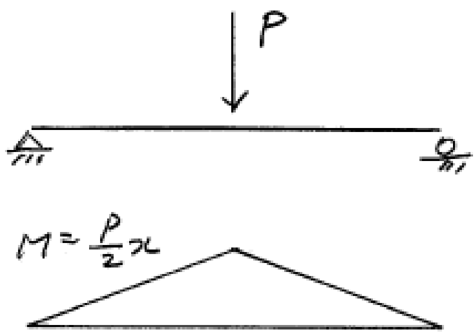
$$\delta W^* = \delta F_{x2} \cdot u_2$$

$$\delta W^* = \delta U^* \quad \Rightarrow \quad U_2 = \frac{L}{EA} F_{x2}$$

Flexural member

$$M = \frac{P}{2} x$$

$$\delta M = \delta P \cdot \frac{x}{2}$$



$$\delta U^* = \int \delta M \frac{M}{EI} dx$$

$$= \frac{2}{EI} \int_0^{\ell/2} \delta P \cdot P \cdot \frac{x^2}{4} dx$$

$$= \frac{\ell^3}{48EI} \delta P \cdot P$$

By using equilibrium equation with low orders, the solution can be found conveniently.

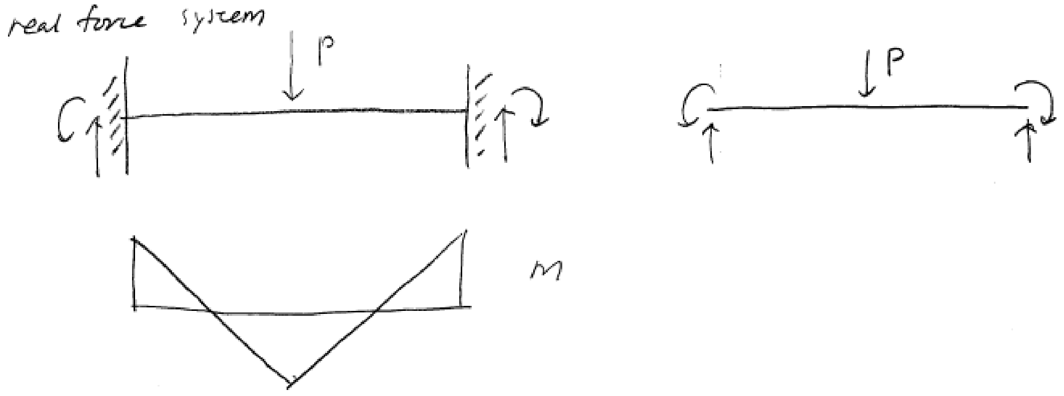
$$\delta W^* = \delta P \cdot v$$

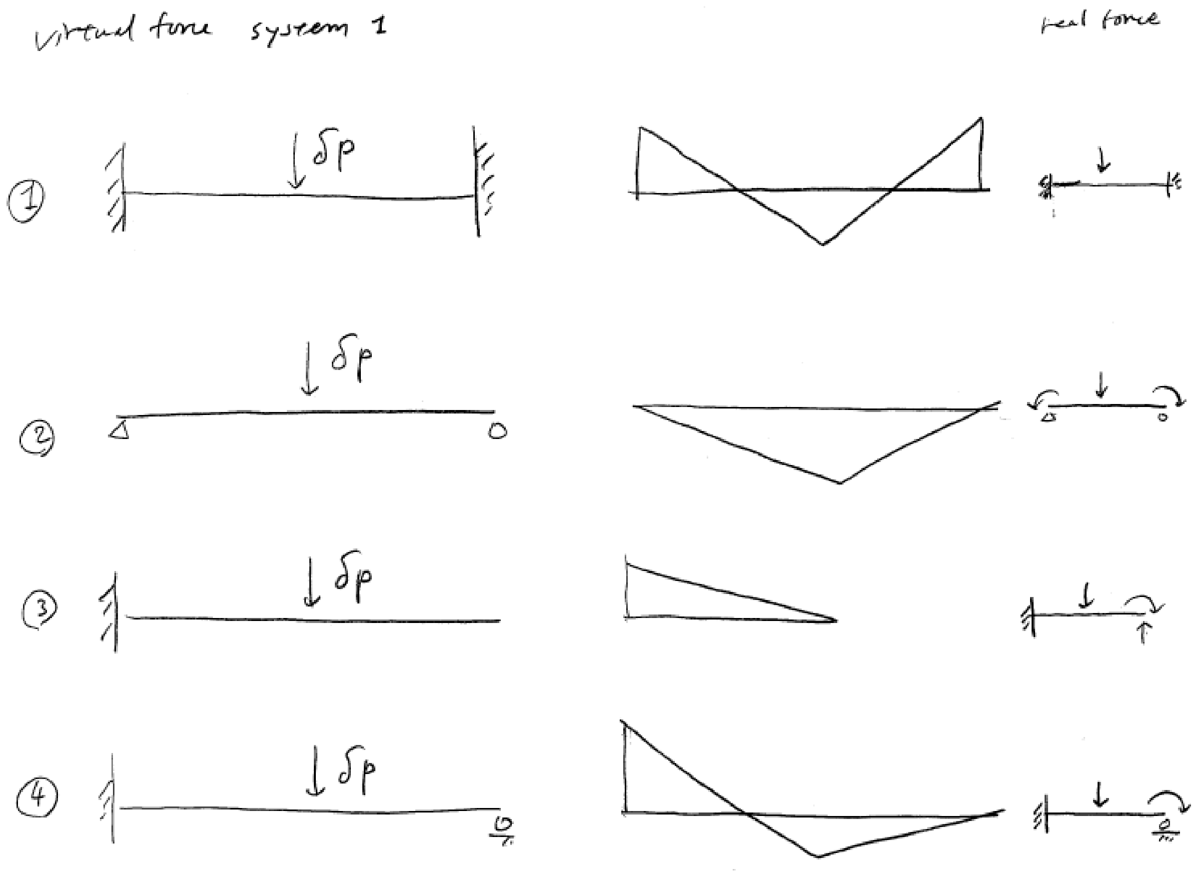
$$\delta W^* = \delta U^* \qquad \Rightarrow \qquad v = \frac{P\ell^3}{48EI}$$

If we set $\delta P = 1$,

The principle of virtual force ≡ unit load method.

Discussion on virtual force system





$\delta U_{①}^{\ *} = \delta U_{②}^{\ *} = \delta U_{③}^{\ *} = \delta U_{④}^{\ *}$

The force-equilibrium cause a relative displacement (deformation).

On the other hand, with different support conditions, the relative displacement could be the same when the force-equilibrium is the same.

Discussion on stiffness method vs flexibility method

For principle of virtual displacement

$$\delta W = \delta U$$

$$\delta W = \int \delta\varepsilon\sigma \; dV$$

$$= \int_{\ell} \delta v EIv \; dx \qquad (\frac{d^4 v}{dx^4} = 0)$$

v ⇒ 3rd order equation to satisfy the force-equilibrium

If the cross-section is variable along the length, v function becomes more complicated.

On the other hand, for principle of virtual force,

$$\delta W^* = \delta U^*$$

$$\delta U^* = \int \delta\sigma \cdot \varepsilon \; dV$$

$$= \int \delta M \frac{M}{EI} \; dV$$

M ⇒ 1st order equation for the determinate system.

As the element properties become more complicate, the flexibility method is easier in the derivation of the flexibility matrix and stiffness matrix.