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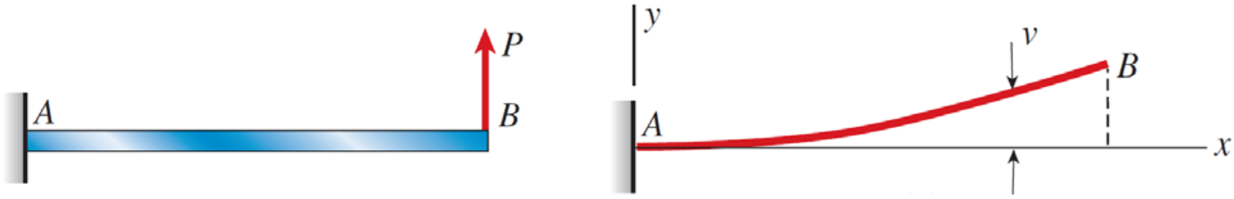
Chapter 5 Stresses in Beams (Basic Topics)

5.1 Introduction; 5.2 Pure Bending and Nonuniform Bending

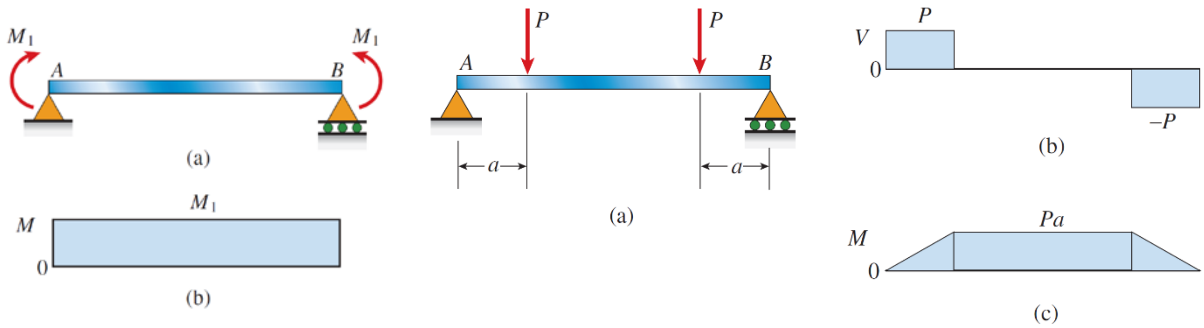
⊙ Coordinate Axes and Deflection Curve

1. Coordinate axes: right-handed coordinate system (as shown in the figure below)
2. Deflection: displacement from the original position, measured in y direction



⊙ Pure Bending and Nonuniform Bending

1. Pure bending: flexure of a beam under a constant ____________________
2. (Left) Pure bending in the entire beam (=0); and (Right) partially pure bending where   = 0

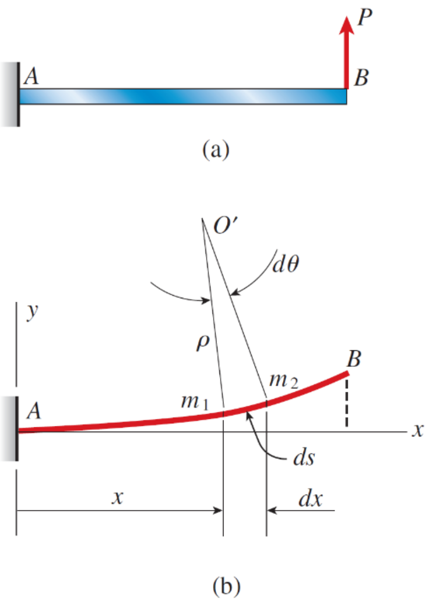


5.3 Curvature of a Beam

⊙ Definition of Curvature and Sign Convention

1. Curvature: a measure of how sharply a beam is bent
2. O': Center of curvature
3. Distance m_1O': Radius of curvature
4. Curvature

$$\kappa = \frac{1}{\rho}$$



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5. From geometry, $\rho d\theta = ds$, thus

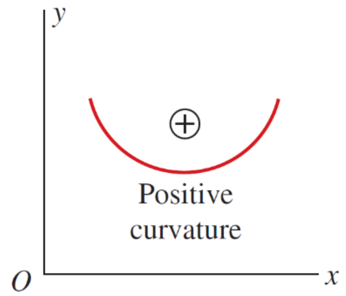
$$\kappa = \frac{1}{\rho} = \frac{d\theta}{ds}$$

6. For a small deflection, $ds \cong dx$, thus

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{dx}$$

7. For a prismatic bar with homogeneous material under pure bending, the curvature is constant. If under nonuniform bending, the curvature varies.

8. Sign conventions for curvature (→)



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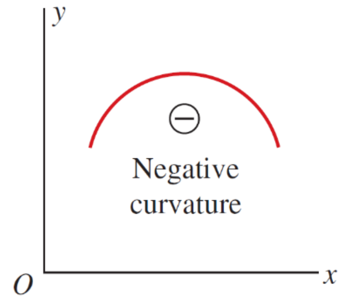
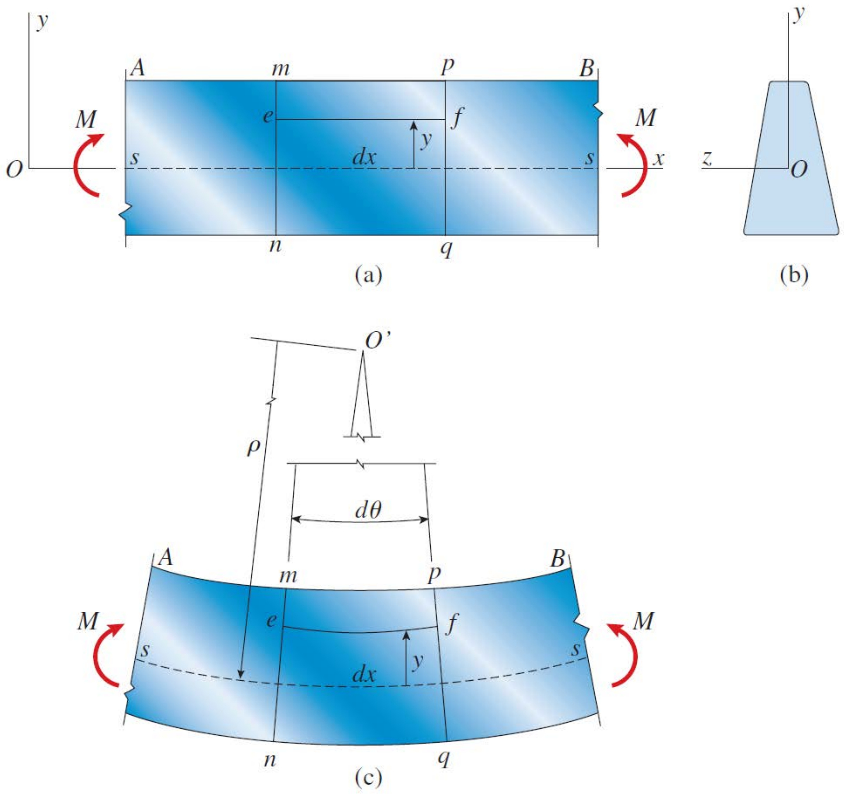


(b)

5.4 Longitudinal Strains in Beams

⦿ Conditions Used for Derivations and Terminologies

1. Beam under **pure bending** with bending moment M
2. Theorem: **cross-sections of a beam remain p________** during pure bending (Proof: all elements of the beam must deform in an identical manner, which is possible only if …)



3. **Neutral surface**: a surface in which longitudinal lines do not change in l_________
4. **Neutral axis**: intersection of the neutral surface with any cross-sectional plane (- axis in the figure)

⦿ Longitudinal Strain

1. Note $\rho d\theta =$
2. Length of ef (distance y from the neutral axis) after deformation:

$$L_1 = (\rho - y)d\theta = dx - \frac{y}{\rho}dx$$

3. Longitudinal strain at ef (distance y from the neutral axis)

$$\boldsymbol{\varepsilon_x} = \frac{L_1 - }{} = -\frac{\boldsymbol{y}}{\boldsymbol{\rho}} = -\boldsymbol{\kappa y}$$

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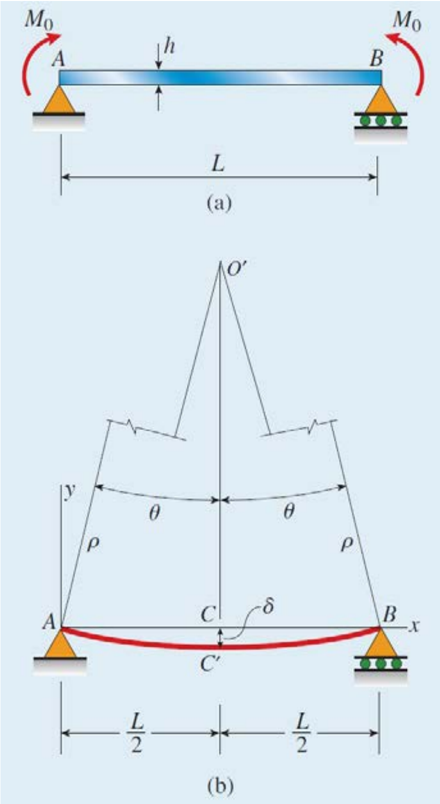
4. Longitudinal strain ε_x varies l_________

5. For positive curvature, longitudinal strain is negative above the neutral surface (shortening) and positive below (elongation)

6. The formula $\varepsilon_x = -\kappa y$ is derived solely from the geometry, so it works for any materials

⊙ **Example 5-1**: Simply supported beam with length $L = 8.0 \text{ ft}$ and height $h = 6.0 \text{ in.}$ under pure bending is deformed to a circular arc. The longitudinal strain at the bottom is 0.00125, and the distance from the neutral axis to the bottom surface is 3.0. Determine the radius of curvature, the curvature and the deflection of the beam.

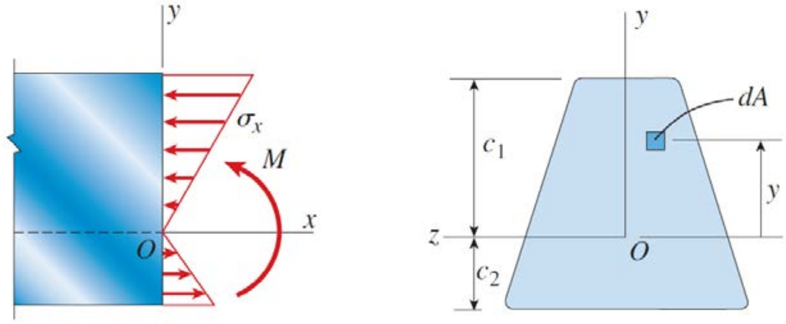


5.5 Normal Stresses in Beams (Linearly Elastic Materials)

⦿ Normal Stresses in Beams Consisting of Linearly Elastic Material

1. From Hooke's law, i.e. $\sigma = E\varepsilon$ and $\varepsilon_x = -\kappa y$

$$\boldsymbol{\sigma_x} = E\varepsilon_x = \boldsymbol{-E\kappa y} = -\frac{Ey}{\rho}$$



2. Observation: normal stress varies linearly as well (if the material is linearly elastic); For positive curvature and bending moment, compression above the neutral axis and tension below

3. However, the formula $\sigma_x = -E\kappa y$ is not useful in practice because we need to also know (1) the location of neutral axis to determine y and (2) the relationship between the curvature κ and bending moment M

4. Two conditions to find (1) and (2) above:

   - The force acting in the x direction is __________ ($\because$ pure bending)
   - The resultant of the longitudinal strain is the __________ ___________

⦿ Location of Neutral Axis

1. The normal force acting on the entire cross section is zero, i.e.

$$\int_A \sigma_x dA = -\int_A E\kappa y dA = 0$$

2. Therefore, we obtain

$$\int_A y dA = 0$$

3. Conclusion: **the neutral axis passes through the c**___________ of the cross-sectional area (conditions: pure bending, linearly elastic material)

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⦿ Moment-Curvature Relationship

1. Moment created by the normal stress acting on the infinitesimal area dA:

$$dM = -\sigma_x y dA$$

2. Moment acting on the entire cross section:

$$M = \int_A dM = -\int_A \sigma_x y dA$$

3. Substituting $\sigma_x = -E\kappa y$,

$$M = \int_A \kappa E y^2 dA = \kappa E \int_A y^2 dA$$

4. Introducing **moment of inertia** (unit: *length*^{4})

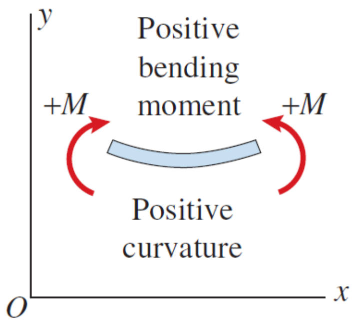
$$\boldsymbol{I = \int_A y^2 dA}$$

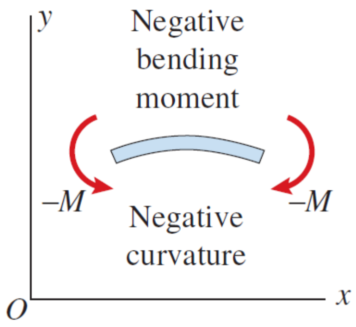
5. Now we obtain **the moment-curvature equation**

$$\boldsymbol{\kappa = \frac{1}{\rho} = \frac{M}{EI}}$$

6. EI: flexural rigidity (cf. axial rigidity, torsional rigidity)

7. Sign conventions: positive bending moment causes positive curvature



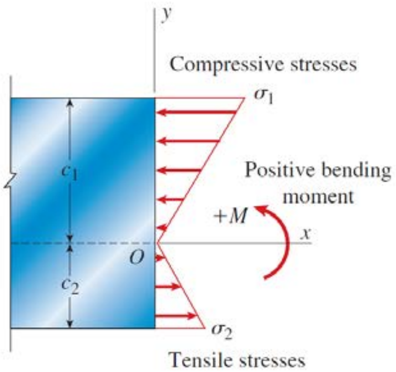


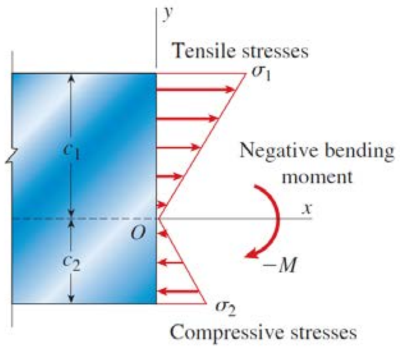
⦿ Flexure Formula

1. From $\sigma_x = -E\kappa y$ and $\kappa = M/EI$, the flexure formula is derived as

$$\boldsymbol{\sigma_x = -\frac{My}{I}}$$

2. "Bending stress" or "flexural stress" is proportional to the bending moment M and inversely proportional to the moment of inertia I; The stress varies linearly with the distance y from the neutral axis.





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⦿ Maximum Stresses at a Cross Section

1. Maximum normal stresses σ_1 and σ_2 located at the top (distance c_1) and bottom (distance c_2) surfaces of the beam (why?) are

$$\sigma_1 = -\frac{Mc_1}{I} = -\frac{M}{S_1}$$

$$\sigma_2 = -\frac{M(-c_2)}{I} = \frac{M}{S_2}$$

2. **Section moduli** S_1 and S_2 (unit: *length*^{3}): useful for design procedure

$$S_1 = \frac{I}{c_1}, \quad S_2 = \frac{I}{c_2}$$

⦿ Doubly Symmetric Shapes

1. If the cross section of a beam is symmetric with respect to the z axis as well as the y axis, then $c_1 = c_2 = c$

2. Maximum stresses

$$\sigma_1 = -\sigma_2 = -\frac{Mc}{I} = -\frac{M}{S}$$

where $S = I/c$

3. Example 1: rectangular cross section

$$I = \frac{bh^3}{12} \qquad S = \frac{bh^3}{6}$$

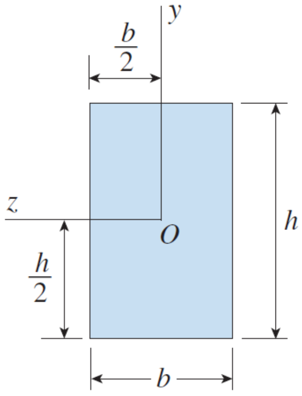
4. Example 2: circular cross section

$$I = \frac{\pi d^4}{64} \qquad S = \frac{\pi d^3}{32}$$

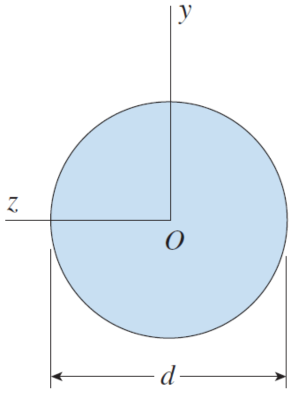
5. Appendix E: Moments of inertia

6. Appendixes F and G: dimensions and properties of standard beams

7. Chapter 12: techniques for location of the neutral axis, the moment of inertia, and section moduli



(a)



(b)

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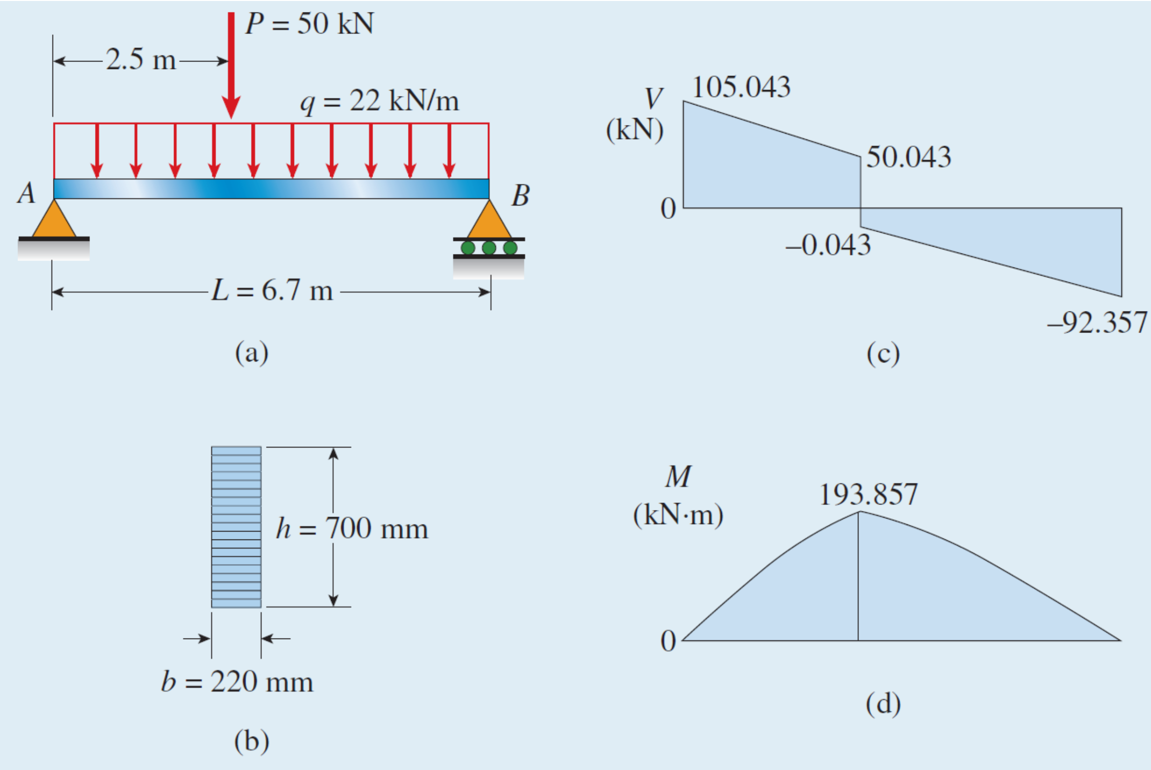
⊙ **Example 5-3**: Simple beam *AB* of span length $L = 6.7\ \mathrm{m}$ supports a uniform load of intensity $q = 22\ \mathrm{kN/m}$ and a concentrated load $P = 50\ \mathrm{kN}$. The concentrated load acts at a point $2.5\ \mathrm{m}$ from the left-hand end of the beam. The beam is constructed of glued laminated wood and has a cross section of width $b = 220\ \mathrm{mm}$ and a height $h = 700\ \mathrm{mm}$. (a) Determine the maximum tensile and compressive stresses in the beam due to the bending (b) If load q is unchanged, find the maximum permissible value of load P if the allowable normal stress in tension and compression is $\sigma_a = 13\ \mathrm{MPa}$.



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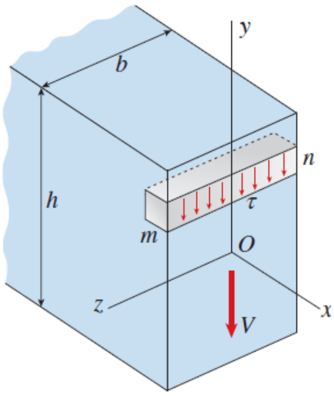
5.8 Shear Stresses in Beams of Rectangular Cross Section

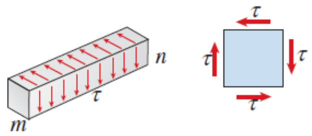
1. Pure bending → Bending moment only → _______ stresses only on cross sections (Sections 5.4 and 5.5)



2. Nonuniform bending → Normal stresses (flexure formula) <u>and</u> shear stresses (this section)

⊙ Vertical and Horizontal Shear Stresses

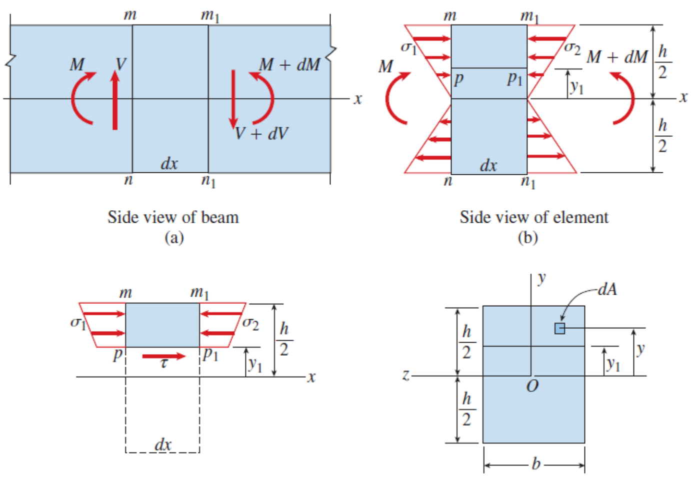
1. Assumptions: (1) shear stress p_______ to the shear force, and (2) the shear stresses are uniformly distributed across the width of the beam (may vary over the height)



2. As discussed in Section 1.7, the shear stresses around a rectangular element should have the same magnitude and have directions as shown in (→)

3. Shear stresses at the top and bottom of the beam, i.e. at $y = \pm h/2$, is _______.

⊙ Derivation of Shear Formula



Side view of beam
(a)

Side view of element
(b)

Side view of subelement
(c)

Cross section of beam at subelement
(d)

1. The normal stresses (caused by bending moment M) at cross sections mn and m_1n_1 are

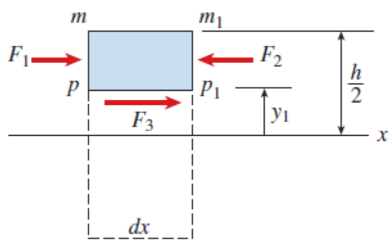
$$\sigma_1 = -\frac{My}{I} \quad \sigma_2 = -\frac{(M+dM)y}{I}$$

2. If the bending moments at cross sections mn and m_1n_1 are equal, i.e. pure bending, the normal stresses σ_1 and σ_2 acting over the sides will be equal, so τ on pp_1 will be zero → Shear stress occurs under non-uniform bending

3. The horizontal (normal) forces acting on the sides mp and m_1p_1 (absolute values):

$$F_1 = \int \sigma_1 dA = \int \frac{My}{I} dA$$

$$F_2 = \int \sigma_2 dA = \int \frac{(M+dM)y}{I} dA$$

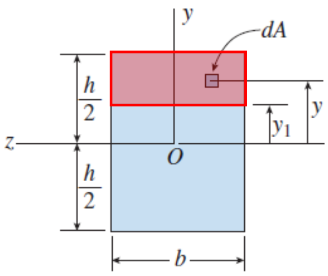


4. The horizontal (shear) force acting on the side pp_1:

$$F_3 = F_2 - F_1 = \int \frac{(M+dM)y}{I} dA - \int \frac{My}{I} dA = \int \frac{dMy}{I} dA = \frac{dM}{I} \int y dA = \tau b(dx)$$

5. Thus, the shear stress is

$$\tau = \frac{dM}{dx}\left(\frac{1}{Ib}\right)\int y dA = \frac{V}{Ib}\int y dA$$



6. Introducing "first moment" Q (for the area above the level of interest), $Q = \int y dA$ (= negative of the first moment of the area below the level. Why?)

7. **Shear formula**:

$$\boldsymbol{\tau = \frac{VQ}{Ib}}$$

⊙ Distribution of Shear Stresses in a Rectangular Beam

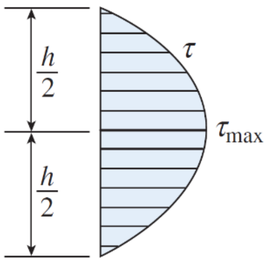
1. The first moment Q of a rectangular beam at $y = y_1$:

$$Q = \int_{\text{above } y=y_1} y dA = \int_{y_1}^{h/2} ybdy = b\left(\frac{h}{2} - y\right)\left(y_1 + \frac{h/2 - y_1}{2}\right) = \frac{b}{2}\left(\frac{h^2}{4} - y_1^2\right)$$

2. Substituting this into the shear formula

$$\tau = \frac{V}{2I}\left(\frac{h^2}{4} - y_1^2\right)$$

- varies parabolically over the height

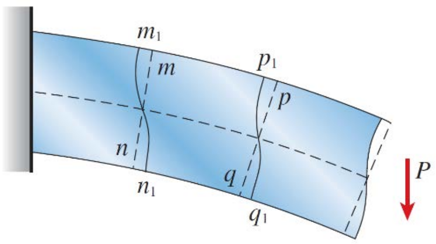


3. The maximum shear stress (at $y = 0$)

$$\tau_{\max} = \frac{Vh^2}{8I} = \frac{3V}{2A}$$

4. Accuracy of the shear formula depends on b/h: accurate for narrow beams; e.g. underestimated by 13% for the square beam.

5. Effects of shear strain: parabolic distribution of shear stress → parabolic distribution of shear strain over the cross section → warping



But advanced analysis shows the effects of warping on flexural stresses are not substantial. Therefore, one can use flexural formula for beams under non-uniform bending.

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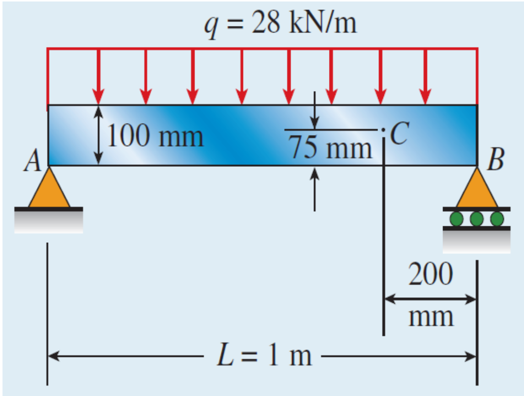
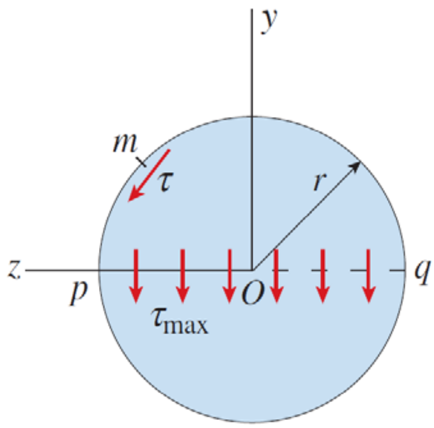
⊙ **Example 5-11**: A metal simply supported beam with span $L = 3\ \mathrm{ft}$ under uniform load $q = 160\ \mathrm{lb/in}$. The rectangular cross section has width $b = 1\ \mathrm{in.}$ and height $h = 4\ \mathrm{in.}$ Determine the normal stress σ_C and shear stress τ_C at point C located $1\ \mathrm{in.}$ below the top of the beam and $8\ \mathrm{in.}$ from the right-hand support. Show these stresses on a sketch of a stress element at point C.



5.9 Shear Stresses in Beams of Circular Cross Section

⊙ Shear stresses in beams of circular cross section

1. The assumption of “shear stress act parallel to the y axis” does not work
2. For example, at point m, there must be no shear stress in radial direction because there is zero shear stress on the outer surface.
3. The assumption of “parallel shear stress” work at least at the neutral axis



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4. Applying the same procedure used for deriving shear formula to the neutral axis location, we can use $\tau = VQ/Ib$ with

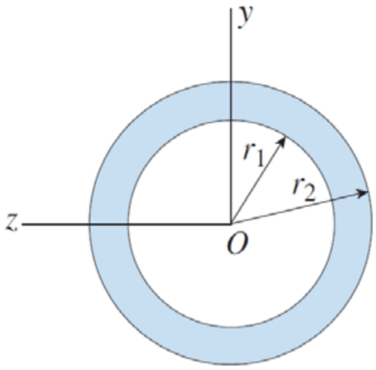
$$I = \frac{\pi r^4}{4} \quad Q = A\bar{y} = \left(\frac{\pi r^2}{2}\right)\left(\frac{4r}{3\pi}\right) = \frac{2r^3}{3} \quad b = 2r$$

5. As a result, the maximum shear stress at the neutral axis is

$$\tau_{\max} = \frac{4V}{3\pi r^2} = \frac{4V}{3A}$$

$4/3$ times the average shear stress V/A

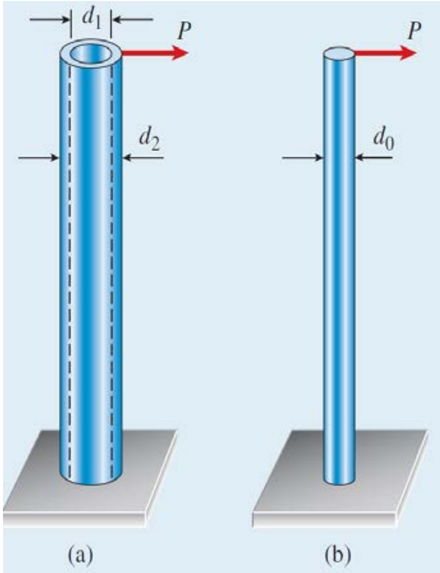
6. Shear stress for a hollow circular cross section: use the same procedure, but with $I = \frac{\pi}{4}(r_2^4 - r_1^4)$, $Q = \frac{2}{3}(r_2^3 - r_1^3)$, $b = 2(r_2 - r_1)$



$$\tau_{\max} = \frac{VQ}{Ib} = \frac{4V}{3A}\left(\frac{r_2^2 + r_2 r_1 + r_1^2}{r_2^2 + r_1^2}\right)$$

in which $A = \pi(r_2^2 - r_1^2)$

⊙ **Example 5-13:** A vertical pole consisting of a circular tube of outer diameter $d_2 = 4.0\text{ in.}$ and inner diameter $d_1 = 3.2\text{ in.}$ is loaded by a horizontal force $P = 1{,}500\text{ lb.}$ (a) Determine the maximum shear stress in the pole; and (b) For the same load P and the same maximum shear stress, what is the diameter d_0 of a solid circular pole?



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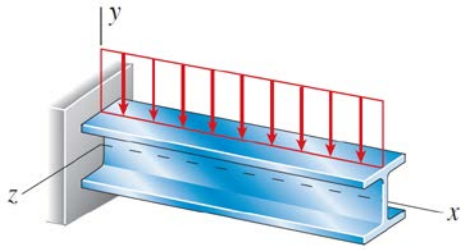
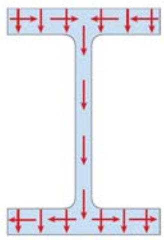
5.10 Shear Stresses in the Webs of Beams with Flanges



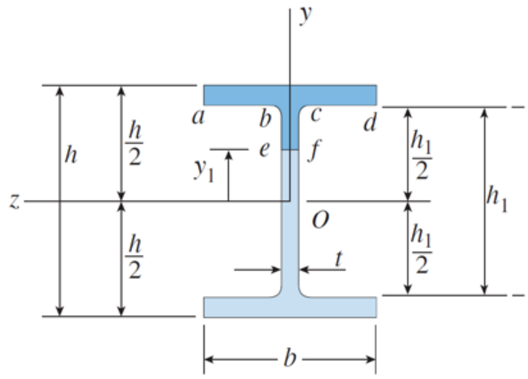
(a)


(b)

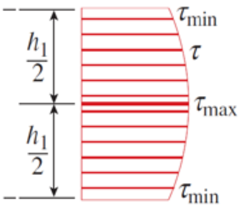
The shear stress distribution in the cross sections of a beam consisting of wide and flange is more complicated than in a rectangular beam. The (vertical) shear stresses in the web are much larger than flange (Shear stresses in the flanges: Section 6.7).

⦿ Shear Stresses in the Web

1. Based on the same assumptions used for deriving the shear formula, we can find approximate estimates on the shear stress in the web by $\tau = VQ/Ib$.



(a)



(b)

2. The first moment Q for the location ef is computed from the flange (A_1) and a part of the web (A_2) as

$$Q = A_1\left(\frac{h_1}{2} + \frac{h/2 - h_1/2}{2}\right) + A_2\left(y_1 + \frac{h_1/2 - y_1}{2}\right)$$

with $A_1 = b\left(\frac{h}{2} - \frac{h_1}{2}\right) \quad A_2 = t\left(\frac{h_1}{2} - y_1\right)$

3. Using the shear formula and the first moment above, the shear stress τ is

$$\tau = \frac{VQ}{It} = \frac{V}{8It}\left[b(h^2 - h_1^2) + t(h_1^2 - 4y_1^2)\right]$$

Note: the shear force varies quadratically throughout the web (as shown in (b))

$$I = \frac{bh^3}{12} - \frac{(b-t)h_1^3}{12} = \frac{1}{12}(bh^3 - bh_1^3 + th_1^3)$$

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⦿ Maximum and Minimum Shear Stresses

1. Maximum shear stress at $y_1 =$.

$$\tau_{\max} = \frac{V}{8It}(bh^2 - bh_1^2 + th_1^2)$$

2. Minimum shear stress at $y_1 = \pm h_1/2$

$$\tau_{\min} = \frac{Vb}{8It}(h^2 - h_1^2)$$

3. Total shear force in the web:

$$V_{\text{web}} = \frac{th_1}{3}(2\tau_{\max} + \tau_{\min})$$

⦿ **Example 5-14**: A beam of wide-flange shape in the figure is subjected to a vertical shear force $V = 45$ kN. Determine the maximum shear stress, minimum shear stress, and total shear force in the web.