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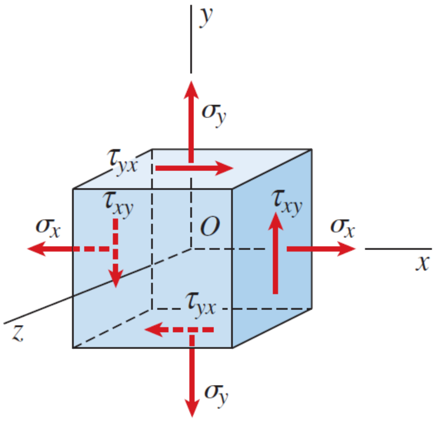
Chapter 7 Analysis of Stress and Strain

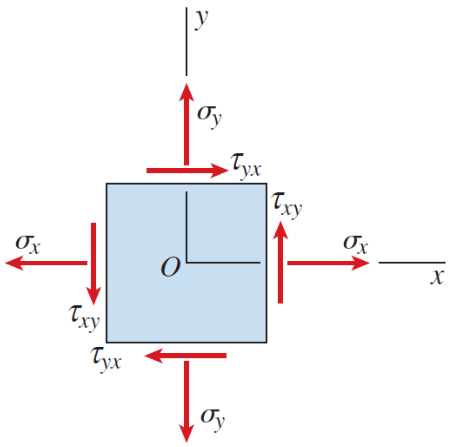
7.1 Introduction

1. Flexure and shear formula ($\sigma = -My/I$ and $\tau = VQ/Ib$), torsion formula ($\tau = T\rho/I_P$), etc. help determine the stresses on cross sections
2. However, larger stresses may occur on inclined sections
3. Example 1: Uniaxial stress (Section 2.6) – maximum shear at $45°$ and maximum normal at cross sections
4. Example 2: Pure shear (Section 3.5) – maximum tensile and compressive stresses occur on $45°$
5. Generalization of these examples → need theories for "**Plane Stress**"
6. Transformation equations help determine the stresses in any general direction from the given **state of stress**

7.2 Plane Stress



(a)



(b)

1. Stress element under "plane stress" condition, e.g. in the xy plane: only the x and y faces of the element are subjected to stresses, and all stresses act parallel to the x and y axis.
2. Normal stress (σ)
   - Subscript identifies the face on which the stress acts, e.g. σ_x and σ_y
   - For e___________, equal normal stresses act on the opposite faces
   - Sign convention: _________ is positive while ____________ is negative
3. Shear stress (τ)
   - Two subscripts: the first indicates the face, and the second direction
   - Sign convention: positive for plus(face)-plus(direction), and negative otherwise

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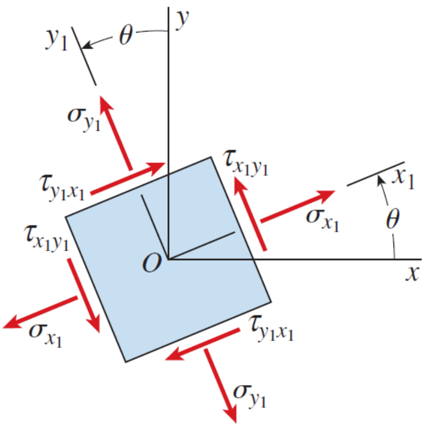
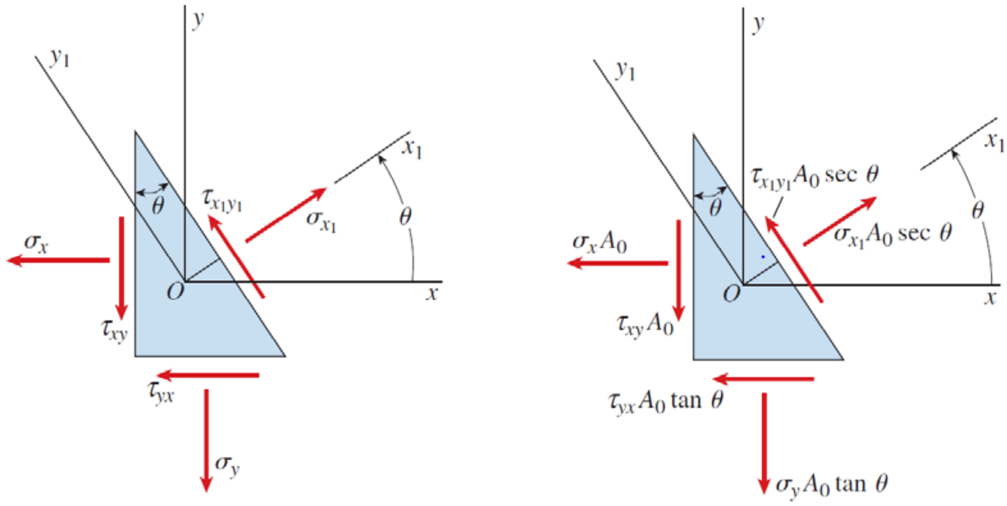
- The sign convention described above is consistent with the shear stress pattern discussed in Section 1.7 (derived from the equilibrium equation)
- Thus, $\tau_{xy} = \tau_{yx}$



⦿ Stresses on Inclined Sections

1. To express the stresses acting on the inclined x_1y_1 element in terms of those on the xy element, consider the e__________ of the forces on the wedge-shaped element



(a) Stresses (b) Forces

2. Equilibrium equation in x_1 direction:

$$\sigma_{x_1}A_0 \sec\theta - \sigma_x A_0 \cos\theta - \tau_{xy}A_0 \sin\theta - \sigma_y A_0 \tan\theta \sin\theta - \tau_{yx}A_0 \tan\theta \cos\theta = 0$$

3. Equilibrium equation in y_1 direction:

$$\tau_{x_1y_1}A_0 \sec\theta + \sigma_x A_0 \sin\theta - \tau_{xy}A_0 \cos\theta - \sigma_y A_0 \tan\theta \cos\theta + \tau_{yx}A_0 \tan\theta \sin\theta = 0$$

4. Using the relationship $\tau_{xy} = \tau_{yx}$, and also simplifying and rearranging, we obtain

$$\sigma_{x_1} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta$$

$$\tau_{x_1y_1} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

5. For $\theta = 0$, $\sigma_{x_1} =$ and $\tau_{x_1y_1} =$

6. For $\theta = 90°$, $\sigma_{x_1} =$ and $\tau_{x_1y_1} =$

⦿ Transformation Equations for Plane Stress

1. Using the following trigonometric identities:

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta),\ \ \sin^2\theta = \frac{1}{2}(1-\cos 2\theta),\ \ \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

2. The transformation equation is expressed in a more convenient form

$$\sigma_{x_1} = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x_1y_1} = -\frac{\sigma_x-\sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

→ **transformation equation for plane stress**

3. Normal stress on the y_1 face – can be obtained by substituting $\theta+90°$

$$\sigma_{y_1} = \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

4. It is noted that

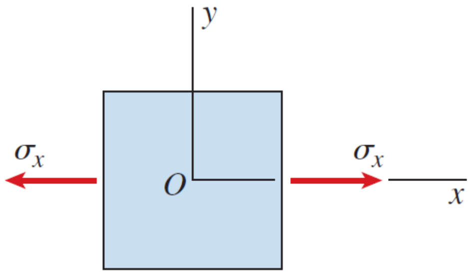
$$\sigma_{x_1}+\sigma_{y_1} = \sigma_x+\sigma_y$$

⦿ Special Cases of Plane Stress

1. Uniaxial stress, i.e. $\sigma_y = \tau_{xy} =$

$$\sigma_{x_1} = \frac{\sigma_x}{2}(1+\cos 2\theta)$$

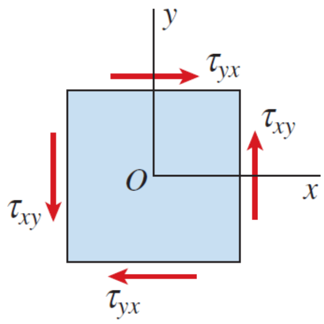
$$\tau_{x_1y_1} = -\frac{\sigma_x}{2}(\sin 2\theta)$$



2. Pure shear, i.e. $\sigma_x = \sigma_y =$

$$\sigma_{x_1} = \tau_{xy}\sin 2\theta$$

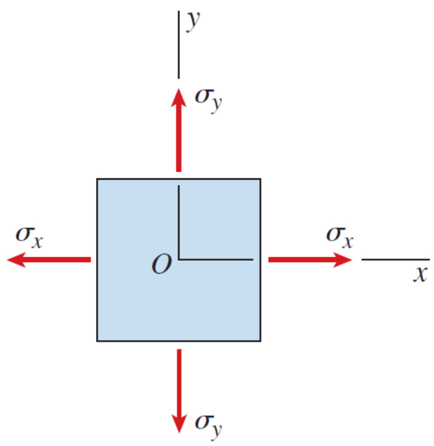
$$\tau_{x_1y_1} = \tau_{xy}\cos 2\theta$$



3. Biaxial stress, i.e. $\tau_{xy} =$

$$\sigma_{x_1} = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta$$

$$\tau_{x_1y_1} = -\frac{\sigma_x-\sigma_y}{2}\sin 2\theta$$



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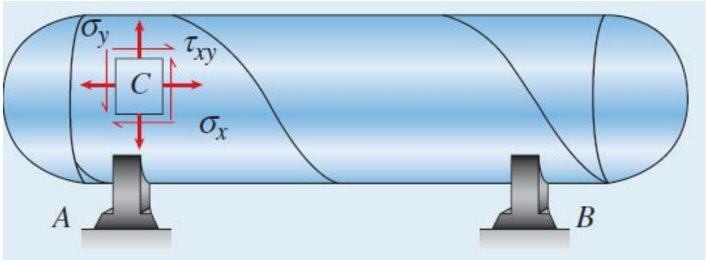
⦿ **Example 7-1**: Internal pressure results in longitudinal stress $\sigma_x = 6{,}000 \text{ psi}$ and circumferential stress $\sigma_y = 12{,}000 \text{ psi}$. Differential settlement after an earthquake → rotation at support B → shear stress $\tau_{xy} = 2{,}500 \text{ psi}$. Find the stresses acting on the element when rotated through angle $\theta = 45\,°$



7.3 Principal Stresses and Maximum Shear Stresses

⦿ Principal Stresses

1. **Principal stresses**: maximum and minimum stresses (→ occurs at every $90°$)

2. Setting the derivative to be zero, i.e.

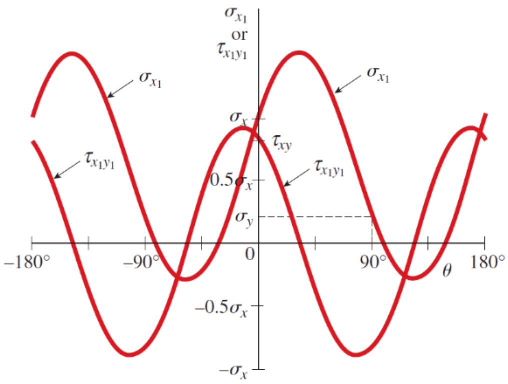
$$\frac{d\sigma_{x_1}}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

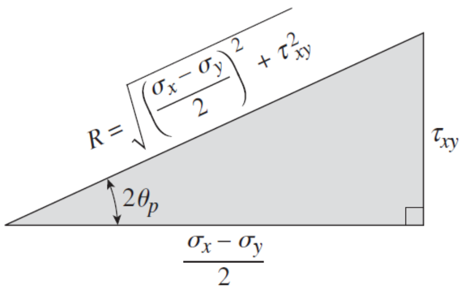
from which we get the **principal angle** θ_p

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

Note: the principal angles for minimum and maximum stresses are perpendicular to each other (why?)

3. Substituting θ_p into the transformation formula via (→)





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$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}, \qquad \cos 2\theta_p = \frac{\sigma_x - \sigma_y}{2R}, \qquad \sin 2\theta_p = \frac{\tau_{xy}}{R}$$

4. The larger of the two principal stresses, σ_1

$$\sigma_1 = \sigma_{x_1}(\theta_p) = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta_p + \tau_{xy}\sin 2\theta_p$$

$$= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\left(\frac{\sigma_x - \sigma_y}{2R}\right) + \tau_{xy}\left(\frac{\tau_{xy}}{R}\right)$$

$$= \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

5. The smaller of the principal stresses, σ_2 → From the property $\sigma_1 + \sigma_2 =$

$$\sigma_2 = \sigma_x + \sigma_y - \sigma_1$$

$$= \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

6. A single formula for the principal stresses

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

⊙ Principal Angles

1. Principal angles θ_{p1} and θ_{p2} (corresponding to σ_1 and σ_2, respectively) are roots of the equation $\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$ → Check the normal stresses to determine θ_{p1} and θ_{p2}

2. Alternatively, θ_{p1} is the root that satisfies both $\cos 2\theta_{p1} = \frac{\sigma_x - \sigma_y}{2R}$ and $\sin 2\theta_{p1} = \frac{\tau_{xy}}{R}$. Then θ_{p2} is $90°$ larger or smaller than θ_{p1}

⊙ Shear Stresses on the Principal Planes

1. If we set $\tau_{x_1y_1} = 0$ for the transformation equation $\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$, we get the equation $\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$, which is the same as the condition for having principal stresses

2. "The shear stresses are zero on the principal planes"

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⦿ Maximum Shear Stresses

1. Setting the derivative to be zero, i.e.

$$\frac{d\tau_{x_1y_1}}{d\theta} = -(\sigma_x - \sigma_y)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

2. Relationship between θ_s and θ_p:

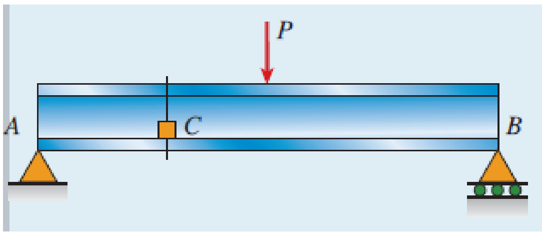
$$\tan 2\theta_s = -\frac{1}{\tan 2\theta_p}$$

3. Can show (See textbook for the derivation) $\theta_s = \theta_p \pm 45°$

4. Maximum shear stress

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{\sigma_1 - \sigma_2}{2}$$

⦿ **Example 7-3**: The state of stress in the beam web at element *C* is $\sigma_x = 86\ \mathrm{MPa}$, $\sigma_y = -28\ \mathrm{MPa}$, and $\tau_{xy} = -32\ \mathrm{MPa}$. (a) Determine the principal stresses and show them on a sketch of a properly oriented element; and (b) Determine the maximum shear stresses and show them on a sketch of a properly oriented element.



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7.4 Mohr's Circle for Plane Stress

⦿ Equations of Mohr's Circle

1. Recall the transformation equation

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\rightarrow \quad \sigma_{x_1} - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

2. It can be shown that

$$\left(\sigma_{x_1} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x_1y_1}^2 =$$

3. Note from the previous note that

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

4. Now the equation above becomes

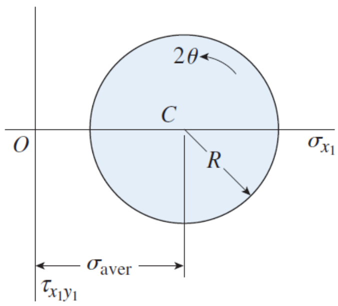
$$\left(\sigma_{x_1} - \sigma_{\text{aver}}\right)^2 + \tau_{x_1y_1}^2 = R^2$$

5. In words, $(\sigma_{x_1}, \tau_{x_1y_1})$ is located on a circle whose center is (,) and the radius is ____

⦿ Construction of Mohr's Circle

1. What's required: _____, _____ and _____

2. Sign conventions (consistent with the transformation formula, etc.)



- Positive shear stress: d________
- Positive normal stress: to the r______
- Positive rotation: c___________

3. Construction procedure

1) Draw a set of coordinate axes with σ_{x_1} as abscissa and $\tau_{x_1y_1}$ as ordinate

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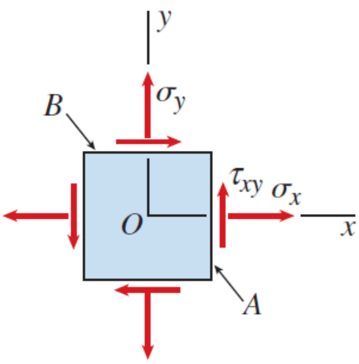
2) Locate the center *C*, (,   )

3) Locate point *A* representing the stress condition (,) shown in Figure (a), $\theta =$

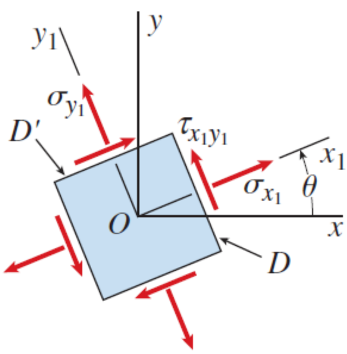
4) Locate point *B* representing the stress condition on the y face, i.e. (,   ), $\theta =$

5) Draw a line *AB*. This goes through *C* (why?), i.e. opposite ends of the diameter of the circle

6) Using Point *C* as the center, draw Mohr's circle through points *A* and *B*. The radius is the length of the line segments *AC* and *BC*, which is $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$



(a)



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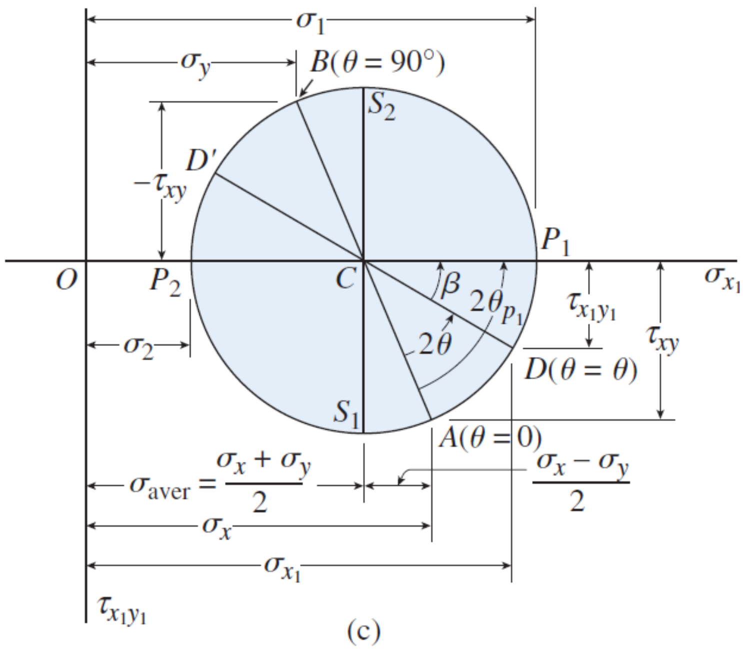

(c)

⊙ Stresses on an Inclined Element

1. Consider the new axes x_1 and y_1 after rotation θ

2. From the point *A* representing the original state (σ_x, σ_y), rotate by 2θ c___________wise to locate the point *D* representing the inclined element

3. The coordinates of *D* are the normal and shear stresses of the inclined element

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4. Proof: available in the textbook

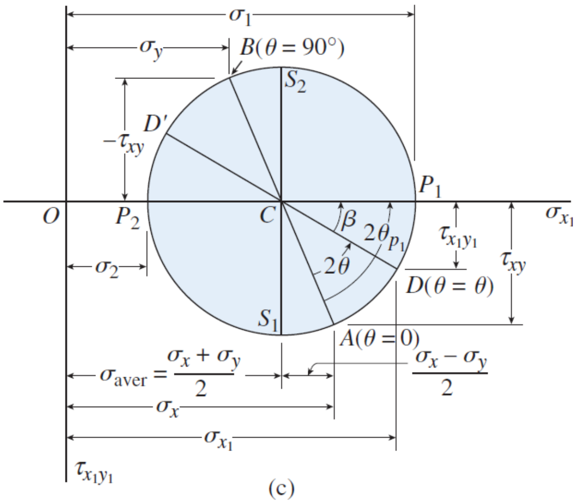
⦿ Principal Stresses

1. The points P_1 and P_2 on Mohr's circle represent m________ and m__________ normal stresses, respectively → **principal stresses**

2. Principal stresses:

$$\sigma_1 = OC + CP_1 = \frac{\qquad}{2} +$$

$$\sigma_2 = OC - CP_2 = \frac{\qquad}{2} -$$



(c)

3. The angle of rotation to achieve the principal planes = the angle between A and P_1 (or P_2) divided by _____

4. The angle θ_{p_1} can be obtained from

$$\cos 2\theta_{p_1} = \frac{\sigma_x - \sigma_y}{2R} \qquad \sin 2\theta_{p_1} = \frac{\tau_{xy}}{R}$$

5. From Mohr's circle, it is clear that

$$\theta_{p_2} = \theta_{p_1} + \qquad °$$

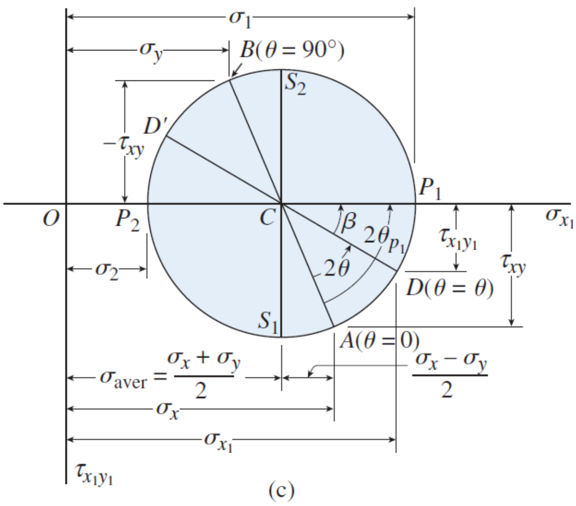
⦿ Maximum Shear Stresses

1. The points S_1 and S_2 on Mohr's circle represent maximum positive and negative shear stresses, respectively.

2. The angle between these points and P_1 and P_2 (on Mohr's circle) =

3. This confirms once again that the angle between principal stress and maximum shear stresses is

4. The normal stresses under maximum shear stresses =



(c)

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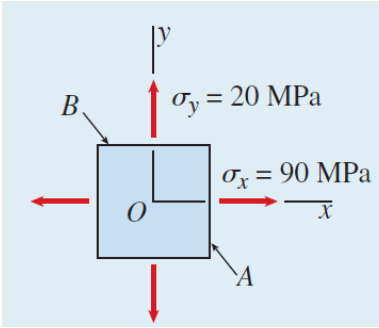
⦿ **Example 7-4**: At a point on the surface of a hydraulic ram on a piece of construction equipment, the material is subjected to biaxial stresses $\sigma_x = 90\ \mathrm{MPa}$ and $\sigma_y = 20\ \mathrm{MPa}$. Using Mohr's circle, determine the stresses acting on an element inclined at an angle $\theta = 30°$.



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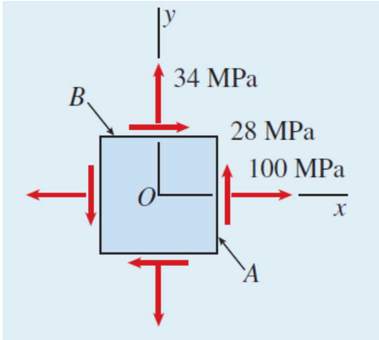
⊙ **Example 7-5**: An element in plane stress on the surface of an oil-drilling pump arm is subjected to stresses $\sigma_x = 100\ \mathrm{MPa}$, $\sigma_y = 34\ \mathrm{MPa}$, and $\tau_{xy} = 28\ \mathrm{MPa}$. Using Mohr's circle, determine (a) the stresses acting on an element inclined at an angle $\theta = 40°$, (b) the principal stresses, and (c) the maximum shear stresses.



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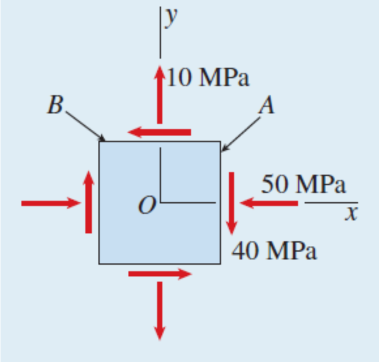
⦿ **Example 7-6**: At a point on the surface of a metal-working lathe the stresses are $\sigma_x = -50\ \mathrm{MPa}$, $\sigma_y = 10\ \mathrm{MPa}$, and $\tau_{xy} = -40\ \mathrm{MPa}$. Using Mohr's circle, determine (a) the stresses acting on an element inclined at an angle $\theta = 45°$, (b) the principal stresses, and (c) the maximum shear stresses.



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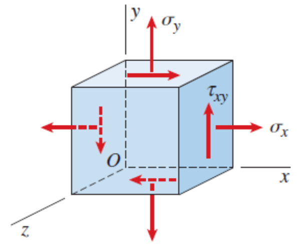
7.5 Hooke's Law for Plane Stress

⦿ Hooke's Law for Plane Stress



1. Conditions (in addition to being in "Plane Stress")

   1) Material properties uniform throughout the body and in all directions (h__________ and i_________)

   2) The material is l______ e______, i.e. follows Hooke's law

2. "Resultant" strains

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \quad \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \quad \varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

3. Shear strain: $\gamma_{xy} = \frac{\tau_{xy}}{G}$

4. Solving the equations of resultant strains simultaneously for σ_x and σ_y, we can obtain **"Hooke's law for plane stress"** as

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)$$

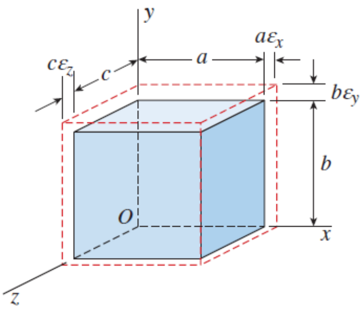
$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x)$$

$$\tau_{xy} = G\gamma_{xy}$$

5. Hooke's law for plane stress contain three material constants: $E, G,$ and ν, but only two are independent because of the relationship

$$G = \frac{E}{2(1+\nu)}$$

⦿ Volume Change (for Plane Stress)



1. Original volume: $V_0 = abc$

2. Final volume: $V_1 = (a + a\varepsilon_x)(b + b\varepsilon_y)(c + c\varepsilon_z)$

3. Volume change: $\Delta V = V_1 - V_0 \cong V_0(\varepsilon_x + \varepsilon_y + \varepsilon_z)$

4. Unit volume change ("dilatation"):

$$e = \Delta V / V_0 =$$

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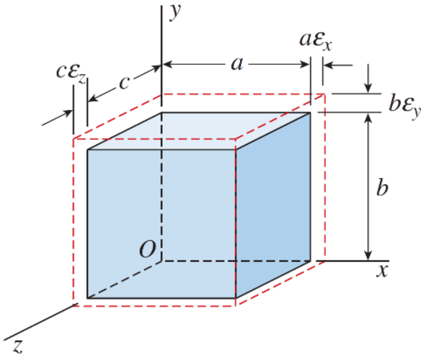
⦿ Strain-Energy Density (for Plane Stress)

1. Work done by the force on x-face and y-face:

$$\frac{1}{2}(\sigma_x bc)(a\varepsilon_x)+\frac{1}{2}(\sigma_y ac)(b\varepsilon_y)$$
$$=\frac{abc}{2}(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y)$$



2. Strain energy density by the normal stresses:

$$u_1=\frac{1}{2}(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y)$$

3. Strain energy density associated with the shear stresses:

$$u_2=\frac{\tau_{xy}\gamma_{xy}}{2}$$

4. Strain energy density in plane stress:

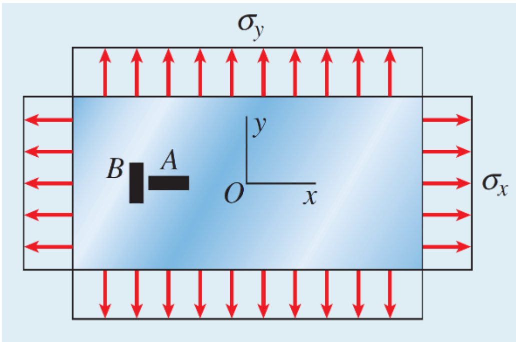
$$u=\frac{1}{2}(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y+\tau_{xy}\gamma_{xy})$$

5. Using Hooke's law, the strain energy density can be alternatively described as

$$u=\frac{1}{2E}(\sigma_x^2+\sigma_y^2-2\nu\sigma_x\sigma_y)+\frac{\tau_{xy}^2}{2G}$$

$$u=\frac{E}{2(1-\nu^2)}(\varepsilon_x^2+\varepsilon_y^2+2\nu\varepsilon_x\varepsilon_y)+\frac{G\gamma_{xy}^2}{2}$$

⦿ **Example 7-7**: Consider a rectangular plate with thickness $t=7\text{mm}$ under plane stress (biaxial) condition. The readings of the gages *A* and *B* give $\varepsilon_x=-0.00075$ and $\varepsilon_y=0.00125$. $E=73\text{ GPa}$ and $\nu=0.33$. Find the stresses σ_x and σ_y and the change Δt in the thickness. Find the volume change (or dilatation) e and the strain energy density u for the plate.



7.7 Plane Strain

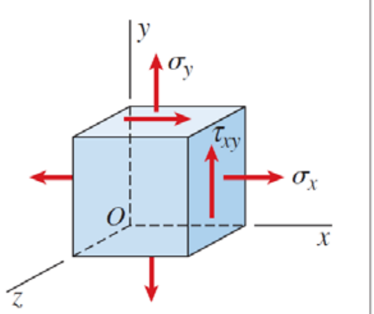
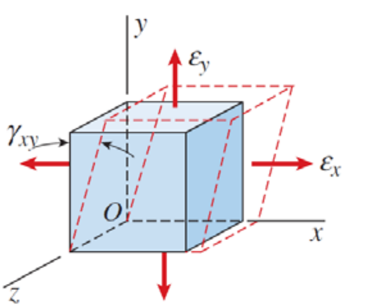
⦿ Plane Strain Versus Plane Stress

1. Recall "plane stress" condition:

$$\sigma_z = 0 \quad \tau_{xz} = 0 \quad \tau_{yz} = 0$$

2. "Plane strain" condition:

$$\varepsilon_z = 0 \quad \gamma_{xz} = 0 \quad \gamma_{yz} = 0$$

	Plane stress	**Plane strain**
		
Stresses	$\sigma_z = 0 \quad \tau_{xz} = 0 \quad \tau_{yz} = 0$ σ_x, σ_y, and τ_{xy} may have nonzero values	$\tau_{xz} = 0 \quad \tau_{yz} = 0$ σ_x, σ_y, σ_z, and τ_{xy} may have nonzero values
Strains	$\gamma_{xz} = 0 \quad \gamma_{yz} = 0$ ε_x, ε_y, ε_z, and γ_{xy} may have nonzero values	$\varepsilon_z = 0 \quad \gamma_{xz} = 0 \quad \gamma_{yz} = 0$ ε_x, ε_y, and γ_{xy} may have nonzero values

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3. Under ordinary conditions, plane stress and strain (do/do not) occur simultaneously.

4. The following exceptional cases of plane stress = plain strain (why?)

   1) Plane stress with $\sigma_x = -\sigma_y$

   2) Zero Poisson effect, i.e. $\nu = 0$

⊙ Transformation Equations for Plane Strain

1. Transformation equations (Proof in Textbook)

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\frac{\gamma_{x_1 y_1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

2. Sum of strains is conserved, i.e. $\varepsilon_x + \varepsilon_y = \text{const}$

3. Principal strains

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

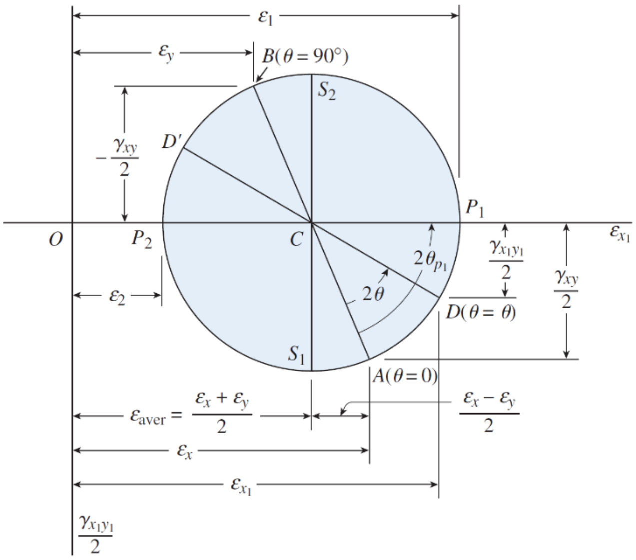
4. Maximum shear strains and corresponding normal strains

$$\frac{\gamma_{\max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\varepsilon_{\text{aver}} = \frac{\varepsilon_x + \varepsilon_y}{2}$$

5. Mohr's circle for strains (→)

6. Applications of transformation equations: one can use transformation equations for plane strain for plane stress conditions (and vice versa) because ε_z does not affect the strains



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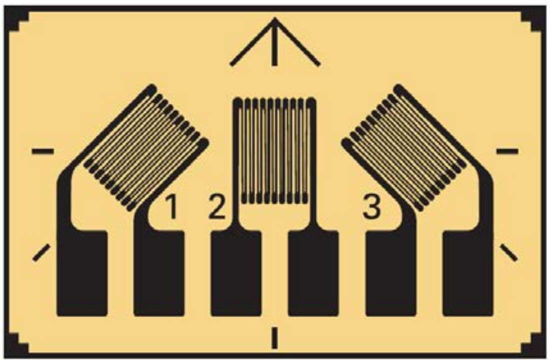
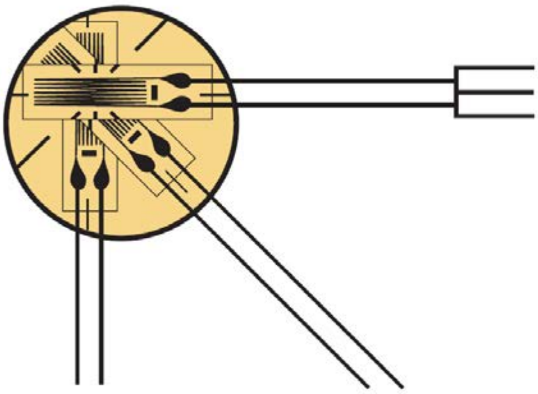
⦿ Strain Measurements

1. Electrical-resistance strain gage measures n_________ strain in t______ directions with $45°$ angle differences (why?)

2. How it works: electrical resistance of wire is altered when it stretches or shortens

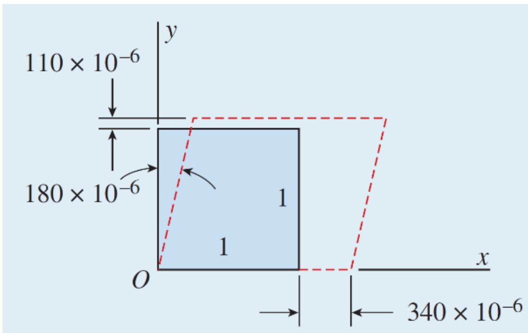


(a) 45° strain gages three-element rosette



(b) Three-element strain-gage rosettes prewired

⦿ **Example 7-8**: Consider a plane strain condition with $\varepsilon_x = 340 \times 10^{-6}$, $\varepsilon_y = 110 \times 10^{-6}$, $\gamma_{xy} = 180 \times 10^{-6}$. Determine (a) strains at $\theta = 30°$, (b) principal strains, and (c) maximum shear strains.



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⦿ **Example 7-9**: Explain how to obtain strains ε_{x_1}, ε_{y_1} and $\gamma_{x_1 y_1}$ associated with an angle θ from the gage readings ε_a, ε_b and ε_c.