Aircraft Structures

CHAPER 10.
Energy methods

Prof. SangJoon Shin

- 2 virtual work principles

  i) PVW : entirely equivalent to the equilibrium eqns. However, does not provide any information about the other

  2 sets of eqns. { Strain-displacement relationship
  Constitutive laws

  ii) PCVW : entirely equivalent to the strain-displacement relationships

  2 sets of eqns. { Equilibrium eqns
  Constitutive laws

◦ Type of forces

- In virtual work principles, various categories of forces are clearly defined and used.

  ① Internal, external forces

  ② Reaction forces : can be eliminated from the formulation since the work they perform vanishes when using kinematically admissible virtual displacements

  But, when arbitrary virtual displacements are used, the virtual work does not vanish

  ⇨ Become an integral part of the formulation

- Conservative forces
  - The work they perform always vanishes for a closed path displacement
  - Total mechanical energy of the system is preserved
  - If the externally applied forces are conservative, they can be derived from a potential ⇨ further simplify the calculation of VW
  - If the strain energy of an elastic component exists, the corresponding elastic forces can be derived from this strain energy ⇨ further simplify the calculation of VW

- combination of
  - PVW
  - Strain energy
  - Potential of external forces

  ⇨ Principle of minimum total potential energy

PVW is always valid

PMTPE is limited to systems involving conservative forces

10.1 Conservative forces

$\underline{r}$: position vector of a particle

$\underline{F}$: force acting on the particle, depends only upon the position of the particle,

$$\underline{F} = \underline{F}(\underline{r})$$

- Fig. 10.1 ··· two arbitrary paths ACB, ADB

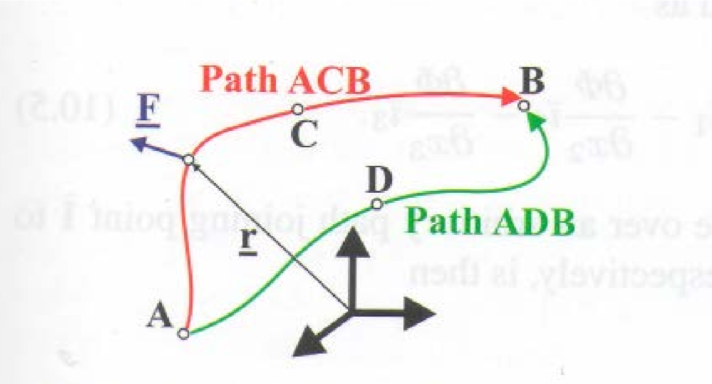


Fig. 10.1. Paths **ACB** and **ADB** join the same two points, **A** and **B**.

10.1 Conservative forces

◦ Definition

- $\underline{F}$ is conservative if the work it performs along any path joining the same initial and final points is identical

$$W = \int_{ACB} \underline{F} \cdot d\underline{r} = \int_{ADB} \underline{F} \cdot d\underline{r} \qquad (10.1)$$

- Work done along path ADB = (-). that along BDA

- Work over the closed path ACBDA = 0

$$W = \oint_{anypath} \underline{F} \cdot d\underline{r} = \oint_{C} \underline{F} \cdot d\underline{r} = 0 \qquad (10.2)$$

◦ Potential of a conservative force

- Stoke's theorem

$$W = \oint_{C} \underline{F} \cdot d\underline{r} = \int_{A} \bar{r} \cdot \nabla \times \underline{F} dA = 0 \qquad (10.3)$$

A : area enclosed by curve C

$\bar{r}$: outward normal to area A (Fig. 10.2)

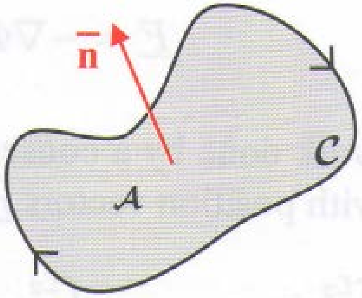


Fig. 10.2. Path enclosing a surface of area $\mathcal{A}$ with a normal $\bar{n}$.

⇨ $\nabla \times \underline{F} = 0$ ⇨ $\nabla \times \nabla \Phi = 0$ (Φ : arbitrary scalar function)

10.1 Conservative forces

- Solution of eqn. $\nabla \times \underline{F} = 0$

$$\underline{F} = -\nabla\Phi \tag{10.4}$$

justified later

“potential”

$$\underline{F} = -\nabla\Phi = -\frac{\partial\Phi}{\partial x_1}\overline{i}_1 - \frac{\partial\Phi}{\partial x_2}\overline{i}_2 - \frac{\partial\Phi}{\partial x_3}\overline{i}_3 \tag{10.5}$$

- Work done by a conservative force

$$\begin{aligned} W &= \int_{\underline{r}_1}^{\underline{r}_2} \underline{F}\cdot d\underline{r} = -\int_{\underline{r}_1}^{\underline{r}_2} \nabla\Phi\cdot d\underline{r} \\ &= \int_{\underline{r}_1}^{\underline{r}_2} \left(-\frac{\partial\Phi}{\partial x_1}dx_1 - \frac{\partial\Phi}{\partial x_2}dx_2 - \frac{\partial\Phi}{\partial x_3}dx_3\right) = -\int_{\underline{r}_1}^{\underline{r}_2} d\Phi = \Phi(\underline{r}_1) - \Phi(\underline{r}_2) \end{aligned}$$

… depends only on the position of initial/final points

can be evaluated as the difference between the values of the potential function

$$W = \Phi(\underline{r}_1) - \Phi(\underline{r}_2) = -\Delta\Phi \tag{10.6}$$

10.1 Conservative forces

- Examples of conservative forces

i) Gravity force ··· $\Phi = mg\underline{r}\,\overline{\underline{i}}_3 = mgx_3$

$$\underline{F}_g = -\nabla\Phi = -\partial\Phi / \partial x_3 \overline{\underline{i}}_3 = -mg\overline{\underline{i}}_3$$

$$W = \int_{x_{3a}}^{x_{3b}} \underline{F}_g \cdot d\underline{r} = -\int_{x_{3a}}^{x_{3b}} \frac{\partial\Phi}{\partial x_3} dx_3 = \Phi(x_{3a}) - \Phi(x_{3b})$$

ii) Restoring force of an elastic spring ···

restoring force $-ku$

Potential $A(u) = \frac{1}{2}ku^2$ ··· “strain energy”

elastic force $F_S = -\frac{\partial A}{\partial u} = -ku$

$$W = \int_{u_a}^{u_b} F_S \cdot du = -\int_{u_a}^{u_b} \frac{\partial A}{\partial u} du = A(u_a) - A(u_b)$$

10.1 Conservative forces

10.1.1 Potential for internal and external forces

- In PVW, a distinction is made between
  - Internal forces
  - Externally applied loads
- In elastic systems, internal forces
  - Stresses acting in a body
  - Elastic forces in structural components

⇨ Potential of internal forces = “strain energy”, “deformation energy”, “internal energy” ··· *A*

$$W_I = -\Delta A \qquad (10.7)$$

- Potential of external forces ··· Φ

$$W_E = -\Delta \Phi \qquad (10.8)$$

- Total potential energy

$$\Pi = A + \Phi \qquad (10.9)$$

10.1 Conservative forces

- Total work done by both internal and external forces

$$W = W_I + W_E = -\Delta A - \Delta\Phi = -\Delta\Pi \tag{10.10}$$

… "for conservative systems, the work done by the internal and external forces = negative change in total potential energy"

- Adding an arbitrary constant to the potential fn. will not alter the work done

10.1 Conservative forces

10.1.2 Calculation of the potential fns

- Potential of internal forces ⋯ "strain energy", $A = A(\underline{\in})$

  It is convenient to select $A(\underline{\in} = 0) = 0$, undeformed or unstrained state

$$W_I = \Delta A = -[A(\underline{\in}) - A(\underline{\in} = 0)] = -A(\underline{\in})$$

$$A(\underline{\in}) = -W_I \qquad (10.11)$$

- It is cumbersome to compute the work done within a solid as the negative product of the internal stress component acting through strains or deformations

  ⇨ alternative approach

  Eq. (9.19), $W_I = -W_E$ ⇨ $A(\underline{\in}) = W_E$ (10.12)

  ⋯ if the internal forces in a solid are conservative, the work done by the externally applied forces = strain energy stored in a body

10.1 Conservative forces

- assumption ··· the forces are applied slowly, in a quasi-steady manner associated

  kinetic energy is negligible

  ··· potential of the externally applied loads, Φ ··· negative of the work done by the external forces acting through the displacements.

  N_P forces, P_i, const. magnitude, line of action fixed in space ⇨ “dead loads”

$$\Phi = -W_E = -\sum_{i=1}^{N_P} P_i d_i - \sum_{i=1}^{N_Q} Q_j \phi_j \qquad (10.13)$$

- Non-conservative forces

i) Aerodynamic force ··· Lift $\propto$ AOA, non-conservative, cannot be derived from potential

ii) Follower force ··· Const. magnitude, but the orientation of their line of action changes with the rotation of structures

Ex) thrust of a rocket jet engine

10.2 Principle of minimum total potential energy

- System represented by N generalized coord. $\underline{q} = \{q_1, q_2, \ldots.., q_N\}^T$
- If the system is conservative, strain energy $A = A(\underline{q})$

  potential of the externally applied loads $\Phi = \Phi(\underline{q})$

  ⇨ Infinitesimal increment

$$dA = \frac{\partial A}{\partial q_1} dq_1 + \frac{\partial A}{\partial q_2} dq_2 + \ldots. + \frac{\partial A}{\partial q_N} dq_N = \sum_{i=1}^{N} \frac{\partial A}{\partial q_i} dq_i$$

$$d\Phi = \frac{\partial \Phi}{\partial q_1} dq_1 + \frac{\partial \Phi}{\partial q_2} dq_2 + \ldots. + \frac{\partial \Phi}{\partial q_N} dq_N = \sum_{i=1}^{N} \frac{\partial \Phi}{\partial q_i} dq_i \tag{10.14}$$

- VW done by the internal forces $\delta W_I = -\delta A(\underline{q})$

  external forces $\delta W_E = -\delta \Phi(\underline{q})$

$$\delta W_I = -\delta A(\underline{q}) = -\sum_{i=1}^{N} \frac{\partial A}{\partial q_i} dq_i$$

$$\delta W_E = -\delta \Phi(\underline{q}) = -\sum_{i=1}^{N} \frac{\partial \Phi}{\partial q_i} dq_i \tag{10.15}$$

10.2 Principle of minimum total potential energy

- Comparing Eq.(9.24) and (10.15)

$$Q_i^I = -\frac{\partial A}{\partial q_i} \quad , \quad Q_i^E = -\frac{\partial \Phi}{\partial q_i} \qquad (10.16)$$

- PVW : $Q_i^I + Q_i^E = 0$, by introducing Eq.(10.16)

$$-\frac{\partial A}{\partial q_i} - \frac{\partial \Phi}{\partial q_i} = \frac{\partial (A+\Phi)}{\partial q_i} = \frac{\partial \Pi}{\partial q_i} = 0 \qquad (10.17)$$

$$\delta W = -\delta \Pi$$

where, Π is total potential.

◦ Principle 4 : A system is in static equilibrium if the sum of the VW done by the internal and external forces vanishes for all arbitrary virtual displacements. $\rightarrow \delta W = -\delta \Pi = 0$

$$\rightarrow \delta \Pi = 0 \qquad (10.18)$$

$$\delta \Pi = \sum_{i=1}^{N} \left[\frac{\partial \Pi}{\partial q_i} \right] \delta q_i = 0 \quad , \quad \frac{\partial \Pi}{\partial q_i} = 0 \rightarrow Eq.(10.17) \qquad (10.19)$$

10.2 Principle of minimum total potential energy

◦ Principle 8 : A conservative system is in equilibrium if virtual changes in the total PE vanish for all virtual displacements.

“Principle of stationary TPE”

- Kinematically admissible virtual displacements are used

  → reaction forces are eliminated from the formulation.

  Arbitrary virtual displacements are used

  → reaction forces must be treated as externally applied loads.

- Graphical illustration of Principle 8 (Fig. 10.3)

  ... TPE is stationary at points A, B and C.

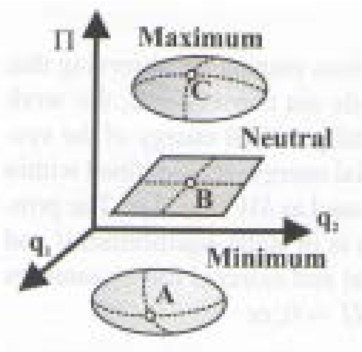


Fig. 10.3 Total potential energy.

10.2 Principle of minimum total potential energy

- Increments in TPE by Taylor series

$$d\Pi \simeq \sum_{i=1}^{N} \frac{\partial \Pi}{\partial q_i} dq_i + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial^2 \Pi}{\partial q_i \partial q_j} dq_i dq_j$$

in the neighborhood of static equilibrium

$$d\Pi \simeq \sum_{i=1}^{N}\sum_{i=1}^{N} \underbrace{\frac{\partial^2 \Pi}{\partial q_i \partial q_j} dq_i dq_j}$$

① $\underbrace{\qquad}$ >0 for all dq_i → TPE is minimum at equilibrium

“stable” (A) … TPE cannot increase without an external source of E

② $\underbrace{\qquad}$ =0 → “neutrally stable” (B)

③ $\underbrace{\qquad}$ <0 → “unstable” (C)… released PE is converted to KE, leading to spontaneous motion of the system

◦ Principle 9 : A conservative system is in a “stable” state of equilibrium if the TPE is a min. *w.r.t. changes in the generalized coord.*

10.2 Principle of minimum total potential energy

10.2.1 Non-conservative external forces

- If the externally applied loads are not conservative

$$\delta W = \delta W_I + \delta W_E = -\delta A + \delta W_E^{nc} = 0$$

◦ Principle 10 : A system is in equilibrium if virtual changes in the strain energy equal the

VW done by the externally applied loads for all arbitrary virtual displacements.

- If externally applied forces are a mixture of $\begin{cases} \text{conservative} \\ \text{non-conservative} \end{cases}$ forces

$$\delta W_E = -\delta W_E^c + \delta W_E^{nc}$$

$$\delta(A + \Phi) = \delta W_E^{nc}$$

VW done by the non-conservative forces

10.3 Strain energy in springs

- Strain energy … function of deformation of the structure

$$A = A(\varepsilon)$$

$$\text{deformation field} \rightarrow \text{function of} \begin{cases} \text{diplcement field} \\ \text{generalized coord.} \end{cases}$$

$$\text{spring} \begin{cases} \text{rectilinear spring} \\ \text{torsional rotational spring} \end{cases}$$

10.3.1 Rectilinear springs

- $\text{2 primary lumped properties} \begin{cases} \text{stiffness constant} \\ \text{unstretched length : } u_0 \end{cases}$

- force applied to the spring : F , force in the spring : F_S

constitutive behavior : $F = F(\Delta), \quad \Delta = u - u_0$: extension

$$F(\Delta = 0) = F(u = u_0) = 0$$

10.3 Strain energy in springs

◦ Linearly elastic spring

- Relationship between an applied load and the resulting extension is linear $(F = k\Delta) \rightarrow$ spring is linear

  k : stiffness constant, unit : force/length, N/m

- Strain energy in the spring

$$A = W_E = \int_{u_0}^{u} F du = \int_{u_0}^{u} k\Delta du = \int_{0}^{\Delta} k\Delta d\Delta = \frac{1}{2} k\Delta^2 = \frac{1}{2} F\Delta \qquad (10.21)$$

: positive-definite fn. of Δ, i.e. A>0 for any (+) or (-) Δ

vanishes only when $\Delta = 0$

- internal force in the spring $F_S = -\frac{\partial A}{\partial u} = -k\Delta$

  (-) : force in the spring opposes the externally applied force.

- constitutive law : straight line in the force vs. extension plot (Fig. 10.5)

  strain energy (A) : shaded area under the curve

10.3 Strain energy in springs

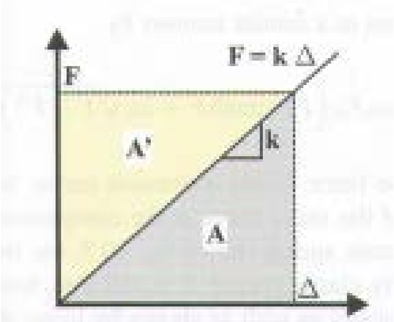


Fig. 10.5. Constitute law a linearly elastic spring

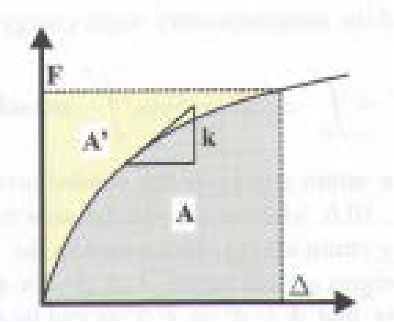


Fig. 10.6. Constitute law a nonlinearly elastic spring

- Complimentary strain energy (A’), stress energy : shaded area to the left of the straight line, “force energy”

$$A' = \int_0^F (u - u_0)dF = \int_0^F \Delta dF = \int_0^F \frac{F}{k} dF = \frac{1}{2}\frac{F^2}{k} = \frac{1}{2}F\Delta \qquad (10.22)$$

$$A' = \frac{1}{2}\frac{F^2}{k} = \frac{1}{2}F\Delta = \frac{1}{2}k\Delta^2 = A$$

$$A = A' = \frac{1}{2}F\Delta,\ A + A' = F\Delta \qquad (10.23)$$

10.3 Strain energy in springs

◦ Nonlinearly elastic spring

- metals(aluminum, copper) … slight amount of nonlinearly elastic behavior prior to yield point

  elastomers … quite obvious nonlinearly elastic behavior

- analytical models, the simplest form

$$F = F_0 \tanh\left(\frac{\Delta}{u_0}\right) \tag{10.24}$$

F_0 : ref. force, u_0 : ref. displacement

-Fig.(10.6) … aluminum, no sharp transition from linear to nonlinear behavior

$$\kappa = \frac{\partial F_0}{\partial \Delta} = \frac{F_0}{u_0}\operatorname{sech}^2\left(\frac{\Delta}{u_0}\right) = \kappa_0 \operatorname{sech}^2\left(\frac{\Delta}{u_0}\right)$$

$\kappa_0 : \dfrac{F_0}{u_0},$: stiffness of the spring at zero elongation

10.3 Strain energy in springs

- Strain energy

$$A = \int_0^\Delta F du = F_0 u_0 \int_0^\Delta \tanh \bar{\Delta} du = F_0 u_0 \ln(\cosh \bar{\Delta})$$

complementary strain energy

$$A' = \int_0^F \Delta dF = F_0 u_0 \int_0^{\bar{F}} \operatorname{arctanh}(\bar{F}) d\bar{F} = u_0 F_0 (\bar{F} \operatorname{arctanh} \bar{F} + \ln \sqrt{1 - \bar{F}^2})$$

- in contrast to the linearly elastic spring, $A \neq A'$, however, $A + A' = F\Delta$

- elastic force in the spring

$$F = \frac{\partial A}{\partial \Delta} = \frac{1}{u_0} \frac{\partial}{\partial \bar{\Delta}} \left[F_0 u_0 \ln(\cosh \bar{\Delta}) \right] = F_0 \tanh\left(\frac{\Delta}{u_0} \right) \qquad (10.25)$$

- Fig.(10.7),

upper … strain energy or potential

middle … force-extension relationship

$\rightarrow$: “softening spring”, decreasing stiffness

at higher extensions

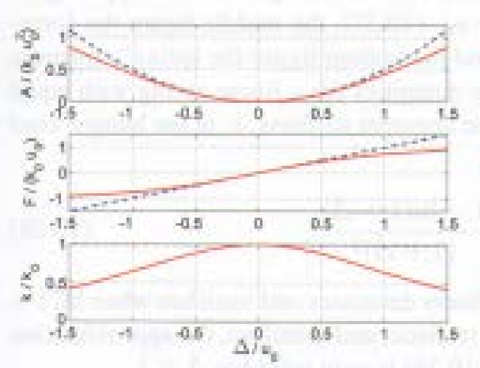


Fig. 10.7. Nonlinear spring with the constitutive law given by eq.(10.24). Top figure : strain energy; middle figure : force; bottom figure: stiffness. Solid line: nonlinear spring; dashed line: linear spring.

10.3 Strain energy in springs

10.3.2 Torsional springs

- Angular motion, θ , under the action of an externally applied torque, M (Fig. 10.9)
- linearly elastic torsional spring : $M = k\theta$
- k : unit $\cdots N \cdot m / rad, \quad N \cdot m / deg$

Fig. 10.9. Tosional spring subjected to a moment, *M*.

10.3.2 Bars

- strain energy

$$A = \frac{1}{2} k e^2 = \frac{1}{2} \frac{EA}{L} e^2 \tag{10.29}$$

e : bar elongation

10.4 Strain energy in beams

10.4.1 Beam under axial loads

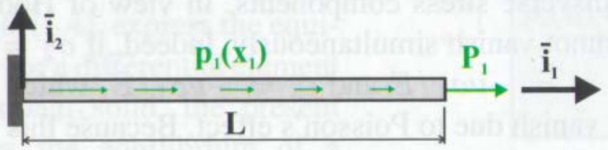


Fig. 5.6. Beam subjected to axial loads.

- Beam subjected only to axial loads (Fig. 5.6)
- infinitesimal slice, left face displacement $\bar{u}_1$
- infinitesimal slice, right face displacement $\bar{u}_1 + \left(\frac{d\bar{u}_1}{dx_1}\right)dx_1$
- Left face, axial force N, displacement from 0 to $\bar{u}_1$, work :
- $-\frac{1}{2}N_1\bar{u}_1$, (-) due to that displacement and force are counted positive in opposite directions
- right face, work : $\frac{1}{2}N_1\left[\bar{u}_1 + \left(\frac{d\bar{u}_1}{dx_1}\right)dx_1\right]$
- total work : $\frac{1}{2}N_1\left(\frac{d\bar{u}_1}{dx_1}\right)dx_1 = \frac{1}{2}N_1\bar{\varepsilon}_1 dx_1$
- external work : $dW_E = \frac{1}{2}N_1\bar{\varepsilon}_1 dx_1 = \frac{1}{2}S\bar{\varepsilon}_1^2 dx_1$ (10.33)

10.4 Strain energy in beams

$$a\left(\overline{\varepsilon}_1\right)=\frac{1}{2}S\overline{\varepsilon}_1^{2} \qquad (10.34)$$

: “strain energy density function”

... potential of the axial force, $N_1=-\dfrac{\partial a(\overline{\varepsilon}_1)}{\partial\overline{\varepsilon}_1}=-S\overline{\varepsilon}_1$

Internal force in the beam

- total strain energy by the axial force distribution

$$A(\overline{\varepsilon}_1)=\int_0^L a(\overline{\varepsilon}_1)dx_1=\frac{1}{2}\int_0^L S\overline{\varepsilon}_1^{2}dx_1 \qquad (10.35)$$

- in terms of the axial force

$$A(\overline{\varepsilon}_1)=\int_0^L \frac{N_1^2}{2S}dx_1=A'(N_1) \qquad (10.36)$$

"total stress E"

"complementary E"

$$a'(N_1)=\frac{N_1^2}{2S}$$

: “stress energy density function”

“complementary strain energy density”

10.4 Strain energy in beams

10.4.2 Beam under transverse loads

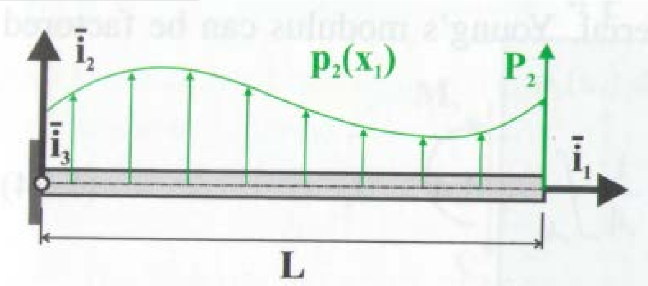


Fig. 5.14. Beam subjected to transverse loads.

- Beams subjected only to transverse loads (Fig. 5.14)
- left face rotation : $\frac{d\bar{u}_2}{dx_1}$
- right face rotation : $\frac{d\bar{u}_2}{dx_1} + \left(\frac{d^2\bar{u}_2}{dx_1^2}\right) dx_1$
- work by bending moment M_3 at left face : $-\frac{1}{2} M_3 \frac{d\bar{u}_2}{dx_1}$

  (-) due to that rotation and moment are counted positive in opposite directions
- work by bending moment M_3 at right face : $\frac{1}{2} M_3 \left[\frac{d\bar{u}_2}{dx_1} + \left(\frac{d^2\bar{u}_2}{dx_1^2}\right) dx_1\right]$
- total work : $\frac{1}{2} M_3 \left(\frac{d^2\bar{u}_2}{dx_1^2}\right) dx_1 = \frac{1}{2} M_3 \kappa_3 dx_1$

  sectional curvature
- external work : $dW_E = \frac{1}{2} M_3 \kappa_3 dx_1 = \frac{1}{2} H_{33}^c \kappa_3^2 dx_1$ (10.37)

10.4 Strain energy in beams

$$a(\kappa_3) = \frac{1}{2} H_{33}^c \kappa_3^2$$: "strain energy density fn" (10.38)

... potential of the bending moment : $M_3 = -\frac{\partial a(\kappa_3)}{\partial \kappa_3} = -H_{33}^c \kappa_3$

Internal moment in the beam

- Total strain E by the bending moment distribution

$$A(\kappa_3) = \int_0^L a(\kappa_3) dx_1 = \frac{1}{2}\int_0^L H_{33}^c \kappa_3^2 dx_1 \qquad (10.39)$$

or

$$A(u_2(x_1)) = \frac{1}{2}\int_0^L H_{33}^c \left(\frac{d^2\bar{u}_2}{dx_1^2}\right)^2 dx_1 \qquad (10.40)$$

or

$$A(M_3) = \int_0^L \frac{M_3^2}{2H_{33}^c} dx_1 = A'(M_3) \qquad (10.41)$$

$$a'(M_3) = \frac{1}{2}\frac{M_3^2}{H_{33}^c}$$: "stress *E* density fn"

10.4 Strain energy in beams

10.4.3 Beam under torsional loads

- circular cylindrical beam subjected to torsion

- rotation of the left face : ϕ_1

- rotation of the left face : $\phi_1 + \left(\frac{d\phi_1}{dx_1}\right) dx_1$

- work by the torque M_1 at the left face : $-\frac{1}{2} M_1 \phi_1$

  (-) due to that rotation and torque are counted positive in opposite directions

- work by the torque M_1 at the right face : $\frac{1}{2} M_1 \left[\phi_1 + \left(\frac{d\phi_1}{dx_1}\right) dx_1 \right]$

- total work : $\frac{1}{2} M_1 \left(\frac{d\phi_1}{dx_1}\right) dx_1 = \frac{1}{2} M_1 \kappa_1 dx_1$

  sectional twist rate

- external work : $dW_E = \frac{1}{2} M_1 \kappa_1 dx_1 = \frac{1}{2} H_{11} \kappa_1^2 dx_1$ (10.42)

10.4 Strain energy in beams

$$a(\kappa_1) = \frac{1}{2} H_{11} \kappa_1^2$$: "strain energy density fn" (10.43)

... potential of the torque: $$M_1 = -\frac{\partial a(\kappa_1)}{\partial \kappa_1} = -H_{11}\kappa_1$$ (10.44)

- Total strain energy by the torque distribution

or $$A(\kappa_1) = \int_0^L a(\kappa_1) dx_1 = \frac{1}{2}\int_0^L H_{11}\kappa_1^2 dx_1$$ (10.39)

or $$A(M_1) = \int_0^L \frac{M_1^2}{2H_{11}} dx_1 = A'(M_1)$$ "total complementary strain E stored" (10.40)

$$a'(M_1) = \frac{1}{2}\frac{M_1^2}{H_{11}}$$: "stress *E* density fn" (10.41)

10.4 Strain energy in beams

10.4.4 Relationship with VW

- internal VW by a bending moment M_3 : $dW_I = -M_3\kappa_3 dx_1$,Eq.(9.69)

$$dW_E = -dW_I = M_3\kappa_3 dx_1$$

However, in Sec.10.4, strain energy stored in beam is

$$dW_E = \frac{1}{2} M_3\kappa_3 dx_1$$

$\frac{1}{2}$ factor difference

- internal VW : bending moment is assumed to remain constant while undergoing a curvature

$$dW_E = \left[\int_0^{\kappa_3} M_3\kappa_3\right] dx_1 = \left[M_3\int_0^{\kappa_3} d\kappa_3\right] dx_1 = M_3\kappa_3 dx_1$$

10.4 Strain energy in beams

- Strain energy stored in beam : bending moment is assumed grow in proportion to the curvature

$$dW_E = \left[\int_0^{\kappa_3} M_3 \kappa_3 \right] dx_1 = \left[\int_0^{\kappa_3} k\kappa_3 d\kappa_3 \right] dx_1 = \frac{1}{2} k\kappa_3 dx_1$$

$$= \frac{1}{2} M_3 \kappa_3 dx_1$$

- Same reasoning for torsion

Internal, external VW : $dW_E = -dW_I = M_1 \kappa_1 dx_1$

Strain energy : $dW_E = \frac{1}{2} H_{11} \kappa_1^2 dx_1$

$\swarrow \frac{1}{2}$ factor difference

- When computing VW and CVW : virtual displacements do not affect the forces or stresses in the system

  Strain energy stored in the structure : internal forces and moments increase in proportion to the deformation

10.5 Strain energy in solids

10.5.1 3-D solid

- Sec. 9.7.3, work done by the constant, external stress

$$W_E = \int_V \sigma^T \underline{\varepsilon}\, dV \qquad \text{Eq.(9.76)}$$

- Then, if the stresses increase in proportion to the deformations

$$W_E = \frac{1}{2}\int_V \sigma^T \underline{\varepsilon}\, dV \qquad (10.46)$$

- Hook's law, $\underline{\sigma} = \underline{\underline{C}}\,\underline{\varepsilon}$

$$\underline{\underline{C}} = \frac{E}{(1+\nu)(1-2\nu)} \left[\begin{array}{ccc|ccc} 1-\nu & \nu & \nu & & & \\ \nu & 1-\nu & \nu & & 0 & \\ \nu & \nu & 1-\nu & & & \\ \hline & & & {}^{1-2\nu}\!/_{2} & & \\ & 0 & & & {}^{1-2\nu}\!/_{2} & \\ & & & & & {}^{1-2\nu}\!/_{2} \end{array} \right] \qquad (2.14)$$

10.5 Strain energy in solids

$$W_E = \frac{1}{2}\int_V \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2)+2\nu(\varepsilon_1\varepsilon_2+\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_3)$$

$$+\frac{1-2\nu}{2}(\gamma_{23}^2+\gamma_{31}^2+\gamma_{12}^2)]dV = \int_V a(\varepsilon)dV = A(\underline{\varepsilon})$$

$a(\varepsilon)$: “strain E density *fn* for a 3-D solid”

- more compact form

$$a(\varepsilon) = \frac{1}{2}\frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)I_1^2 - 2(1-2\nu)I_2] \qquad (10.48)$$

I_1 , I_2 : first 2 invariants of the strain tensor, Eqs.(1.86)

$$a(\varepsilon) = \frac{1}{2}\underline{\varepsilon}^T\underline{\underline{C}}\,\underline{\varepsilon} \qquad (10.49)$$

- Hook’s law is a linear relationship $\Rightarrow a(\varepsilon) = a'(\sigma)$

- complementary strain E density

$$a'(\sigma) = \frac{1}{2E}[\sigma_1^2+\sigma_2^2+\sigma_3^2-2\nu(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3)$$

$$+2(1+\nu)(\tau_{12}^2+\tau_{23}^2+\tau_{31}^2)] \qquad (10.50)$$

10.5 Strain energy in solids

$$\underline{\varepsilon} = \underline{\underline{S}}\,\underline{\sigma} \tag{2.10}$$

$$\underline{\underline{S}} = \frac{1}{E}\left[\begin{array}{ccc:ccc} 1 & -\nu & -\nu & & & \\ -\nu & 1 & -\nu & & 0 & \\ -\nu & -\nu & 1 & & & \\ \hdashline & & & 2(1+\nu) & & \\ & 0 & & & 2(1+\nu) & \\ & & & & & 2(1+\nu) \end{array}\right] \tag{2.12}$$

$$a'(\sigma) = \frac{1}{2}\underline{\sigma}^T\underline{\underline{S}}\,\underline{\sigma} \tag{10.52}$$

10.5 Strain energy in solids

10.5.2 3-D beams

- Eq.(9.78) : internal W done by const. stress results in 3-D beams
- W done by the same stress resultants when they increase in proportion to the deformation

$$W_E = \frac{1}{2}\int_0^L (N_1\overline{\varepsilon}_1 + M_2\kappa_2 + M_3\kappa_3)dx_1 \qquad (10.53)$$

- Hooke's law → sectional constitutive laws, Eq.(6.12)

$$A = \frac{1}{2}\int_0^L (S\overline{\varepsilon}_1^{\,2} + H_{22}^c\kappa_2^2 - 2H_{23}^c\kappa_2\kappa_3 + H_{33}^c\kappa_3^c)dx_1 \qquad (10.54)$$

- complementary strain E … using the compliance form, Eq.(6.13)

$$A' = \frac{1}{2}\int_0^L \left(\frac{N_1^2}{S} + \frac{H_{33}^c}{\Delta H} M_2^2 + 2\frac{H_{33}^c}{\Delta H} M_2 M_3 + \frac{H_{22}^c}{\Delta H} M_3^2 \right) dx_1$$

where, $\Delta H = H_{22}^c H_{33}^c - H_{23}^{c^2}$

assuming that the origin must be located at the section's centroid

10.6 Applications to trusses and beams

10.6.1 Application to trusses

◦ 3-bar, hyperstatic truss (Fig. 10.16)

- bar length : $L_1 = L_3 = L/\cos\theta,\ L_2 = L$
- bar elongations : Eq.(9.27), $e_1 = u_1\cos\theta + u_2\sin\theta,\ e_2 = u_2,$

$$e_3 = -u_1\cos\theta + u_2\sin\theta$$

- bar strain E : $A = \frac{1}{2}ke^2$, Eq.(10.29), $k = \frac{EA}{L}$ (bar stiffness)

$$A = \frac{1}{2}\left(\frac{EA\cos\theta}{L}e_1^2 + \frac{EA}{L}e_2^2 + \frac{EA\cos\theta}{L}e_3^2\right)$$

$$= \frac{1}{2}\frac{EA}{L}[(u_1\cos\theta + u_2\sin\theta)^2\cos\theta + u_2^2$$

$$+ (-u_1\cos\theta + u_2\sin\theta)^2\cos\theta]$$

$$= \frac{1}{2}\frac{EA}{L}\left[2u_1^2\cos^3\theta + \left(1 + 2\sin^2\theta\cos\theta\right)u_2^2\right]$$

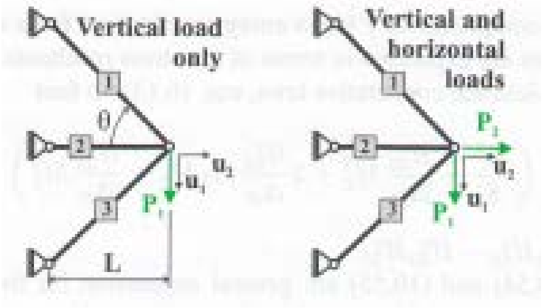


Fig. 10.16. Simple 3-bar truss

10.6 Applications to trusses and beams

- potential of externally applied load, $P_1 \rightarrow \Phi = -P_1 u_1$

  total potential $\Pi = A + \Phi = A - P_1 u_1$

- 2 D.O.F.'s, PMTPE, Eq.(10.17) $\rightarrow$

$$\frac{\partial \Pi}{\partial u_1} = \frac{EA}{L} 2u_1 \cos^3 \theta - P_1 = 0$$

$$\frac{\partial \Pi}{\partial u_2} = \frac{EA}{L} (1 + 2\sin^2 \theta \cos \theta) u_2 = 0$$

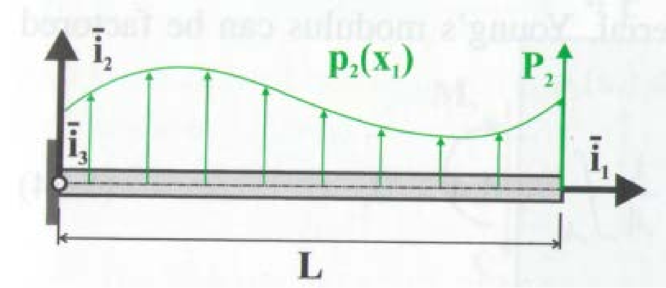
- Matrix form … two linear eqn.s for the 2 generalized coord.

$$\begin{bmatrix} z\cos^3 \theta & 0 \\ 0 & 1 + 2\sin^2 \theta \cos \theta \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{L}{EA} \begin{Bmatrix} P_1 \\ 0 \end{Bmatrix}$$

$$\rightarrow u_1 = \frac{P_1 L}{2EA\cos^3 \theta} \text{ , } u_2 = 0$$

10.6 Applications to Truss and Beams

10.6.3 Applications to beams



◦beam under a distributed transverse load, $p_2(x_1)$, Fig. 5.14

- Potential of the externally applied loads

$$\Phi = -\int_0^L p_2(x_1)\bar{u}_2(x_1)dx_1 \qquad (10.58)$$

- Total Potential Π of the beam ……………………..from Eq.(10.9)

$$\Pi = A + \Phi = \underbrace{\frac{1}{2}\int_0^L H_{33}^c \left(\frac{d^2\bar{u}_2}{d^2x_1^2}\right)^2 dx_1}_{\text{Eq.(10.40)}} - \int_0^L p_2\bar{u}_2 dx_1$$

…. now $\Pi = \Pi(\bar{u}_2(x_1))$, a function of another function ⟶ "functional"

⇨ Beam problems are *infinite dimensional or continuous problems* since determination of the transverse displacement field, $\bar{u}_2(x_1)$

⟷ planar truss w/ 2N unknowns, " finite dimensional, discrete"

10.6 Applications to Truss and Beams

◦ Minimization of the TPE of finite dimension → standard calculus

functional → calculus of variations

◦ Reduction of infinite # of D.O.F → finite # ……………..by choosing specific functions for $u_2(x_1)$ → Chap.11

3-D beam under complex loading condition

distributed loads $p_1(x_1), p_2(x_1), p_3(x_1)$

concentrated loads $P_1,\ P_2,\ P_3$

distributed moment $q_1(x_1), q_2(x_1), q_3(x_1)$

concentrated moment Q_1, Q_2, Q_3

→
$$\begin{aligned}\Phi = &-\int_0^L p_1\bar{u}_1 dx_1 - P_1\bar{u}_1(\alpha L) - \int_0^L q_1\Phi_1 dx_1 - Q_1\Phi_1(\alpha L)\\ &-\int_0^L p_2\bar{u}_2 dx_1 - P_2\bar{u}_2(\alpha L) + \int_0^L q_2\frac{d\bar{u}_3}{dx_1}dx_1 + Q_2\frac{d\bar{u}_3}{dx_1}(\alpha L) \qquad (10.59)\\ &-\int_0^L p_3\bar{u}_3 dx_1 - P_3\bar{u}_3(\alpha L) - \int_0^L q_3\frac{d\bar{u}_2}{dx_1}dx_1 - Q_2\frac{d\bar{u}_2}{dx_1}(\alpha L)\end{aligned}$$

10.6 Applications to Truss and Beams

Euler-Bernoulli assumption $\Phi_3 = \frac{d\bar{u}_2}{dx_1}$, $-Q_3\Phi_3(\alpha L) \rightarrow -Q_3\frac{d\bar{u}_2}{dx_1}(\alpha L)$

$$\Phi_2 = -\frac{d\bar{u}_3}{dx_1},\ -Q_2\Phi_2(\alpha L) \rightarrow Q_2\frac{d\bar{u}_3}{dx_1}(\alpha L)$$

10.8 Principle of minimum complementary energy

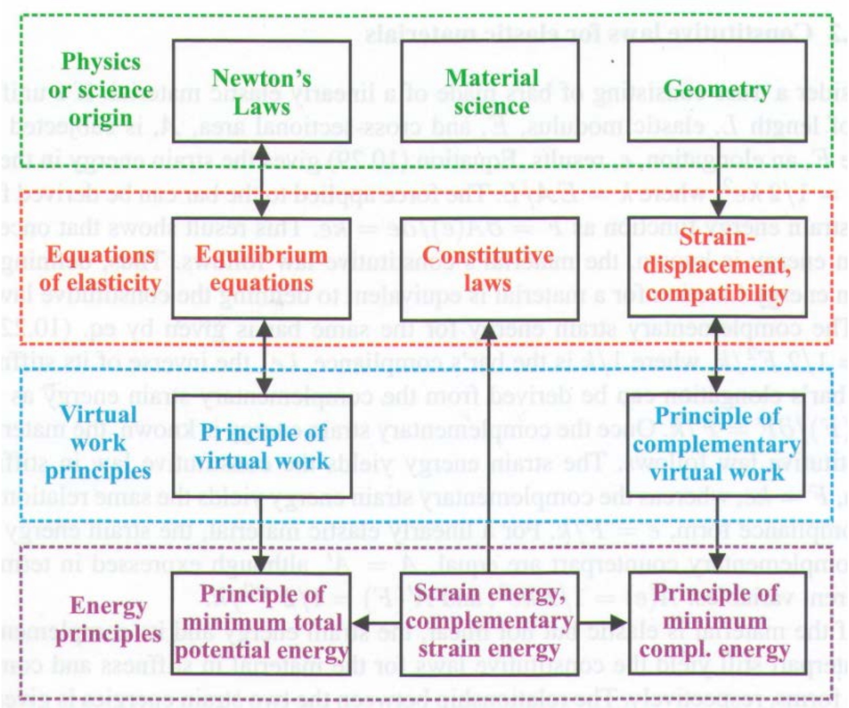
- Sec 10.2 ---- Principle of Virtual Work → Principle of Minimum Total Potential Energy

two assumptions ---- ① internal forces are conservative ← strain Energy

② external forces are also conservative ← potential of the externally applied loads

- Figure 10.27 -- constitutive relationship → strain energy

2nd assumption not shown



10.8 Principle of minimum complementary energy

Principle of minimum complementary energy → Principle of complementary virtual work

two assumptions ---- ① complementary strain energy function

② prescribed displacements can be derived from a potential

→ Sec. 10.8.1

10.8.1 The potential of the prescribed displacements

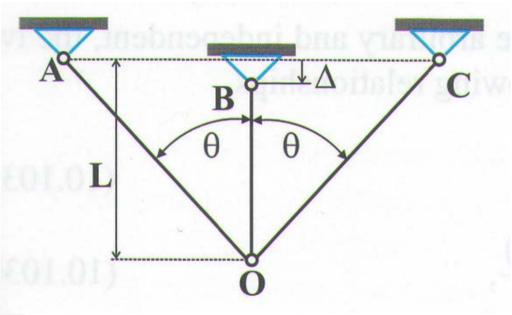


Fig. 10.28 Three-bar truss with prescribed displacement

- 3- bar truss, prescribed displacement Δ at B driving force D, unknown quantity

- Principle of complementary virtual work, Eq.(9.57)

$$\delta W_E^{'} = \Delta \delta D$$

now it is assumed that the prescribed displacement can be derived from a potential, $\Phi^{'}$

"potential of the prescribed displacement" or "dislocation potential"

$$\Delta = -\frac{\partial \Phi'(D)}{\partial D} \qquad (10.101)$$

$$\delta W_E^{'} = \Delta \delta D = -\frac{\partial \Phi'}{\partial D}\delta D = -\delta \Phi'(D) \qquad (10.102)$$

10.8 Principle of minimum complementary energy

10.8.2 Constitutive laws for elastic materials

◦ strain energy for a bar …… $A = \frac{1}{2}ke^2, \quad k = \frac{EA}{L}$

bar forces $F = \frac{\partial A(e)}{\partial e} = ke$

complementary strain energy ……… $A' = \frac{1}{2}\frac{1}{k}F^2, \quad \frac{1}{k}$: compliance

elongation $e = \frac{\partial A(F)}{\partial F} = \frac{1}{k}F$

linearly elastic material, $A = A', \quad A(e) = \frac{1}{2}ke^2, \quad A'(F) = \frac{1}{2}\frac{1}{k}F^2$

10.8 Principle of minimum complementary energy

- elastic, but not linear

Eq.(10.23) → $A(e) + A'(F) = eF$

differentiate, $\left(\frac{\partial A}{\partial e}\right) de + \left(\frac{\partial A'}{\partial F}\right) dF = F de + e dF$

Regrouping $\left(F - \frac{\partial A}{\partial e}\right) de + \left(e - \frac{\partial A'}{\partial F}\right) dF = 0$

- 2 bracketed terms must vanish

$$F = \frac{\partial A(e)}{\partial e} \quad , \quad e = \frac{\partial A'(F)}{\partial F} \qquad (10.103)$$

……….. Some constitutive laws {in stiffness} form
{in compliance}

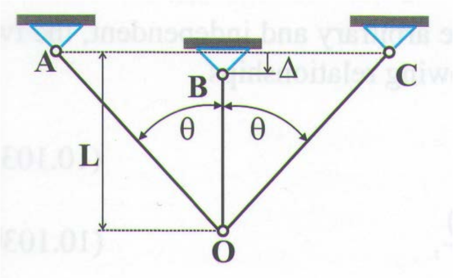
- existence of the {strain energy function} ⟺ assumption of constitutive law
{complementary counterpart}

10.8 Principle of minimum complementary energy

10.8.3 Principle of minimum complementary energy

◦ Principle of Complementary Virtual Work ……. $\delta W^{'} = \delta W_E^{'} + \delta W_I^{'} = 0$

◦ 3-bar truss, Fig 10.28



$$\delta W_I^{'} = -e_A \delta F_A - e_B \delta F_B - e_C \delta F_C$$

- Assuming elastic material, existence of complementary strain energy function

Eq. (10.103b) ⟶

$$\delta W_I^{'} = -\frac{\partial A_A^{'}(F_A)}{\partial F_A}\delta F_A - \frac{\partial A_B^{'}(F_B)}{\partial F_B}\delta F_B - \frac{\partial A_C^{'}(F_C)}{\partial F_C}\delta F_C$$

$$= -\delta A_A^{'} - \delta A_B^{'} - \delta A_C^{'} = -\delta A^{'}$$

$A^{'} = A_A^{'} + A_B^{'} + A_C^{'}$ total complementary strain energy

10.8 Principle of minimum complementary energy

- Prescribed displacement at B……..can be derived from a potential

$$\delta W_E^{'} = -\delta \Phi^{'}(D)$$

- Principle of Complementary Virtual Work →

$$\delta W^{'} = \delta W_E^{'} + \delta W_I^{'} = -\delta A^{'} - \delta \Phi^{'} = -\delta (A^{'} + \Phi^{'}) = 0$$

- total complementary energy, $\Pi^{'}$ ……… $\Pi^{'} = A^{'} + \Phi^{'}$ (10.104)
- Statement ……… $\delta \Pi^{'} = 0$ (10.105)

◦ Principle 11 (Principle of stationary complementary energy) *A conservative system undergoes compatible deformations if and only if the total complementary energy vanishes for all statically admissible virtual forces*

- Stationary = minimum value for stable equilibrium

  → Principle of minimum complementary energy

10.8 Principle of minimum complementary energy

- Principle 12 (Principle of Minimum complementary energy) *A conservative system undergoes compatible deformations if and only if the total complementary energy is a minimum with respect to arbitrary changes in statically admissible forces.*

10.8 Principle of minimum complementary energy

Example 10.8 Three-bar truss with prescribed displacement

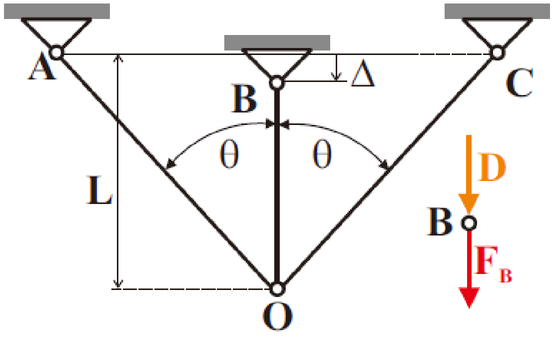


Fig. 10.29.

- only relevant equilibrium eqn: at joint **O**

$$F_A = F_C, F_A \cos\theta + F_B + F_C \cos\theta = 0$$

- complementary strain energy, first in terms of F_A, F_B, and F_C

$$A' = \frac{1}{2}\left(\frac{F_A^2}{k_A \cos\theta} + \frac{F_B^2}{k_B} + \frac{F_C^2}{k_C \cos\theta} \right)$$

- three bar forces are expressed in terms of one, say F_C

$$A' = \frac{1}{2}\left[\frac{F_C^2}{k_A \cos\theta} + \frac{(2F_C \cos\theta)^2}{k_B} + \frac{F_C^2}{k_C \cos\theta} \right] = \frac{\bar{k}F_C^2}{2\bar{k}_A k_B \bar{k}_C \cos\theta}$$

- Potential of the prescribed displacement

$$\Phi' = -D\Delta, D + F_B = 0, F_B = -2F_C \cos\theta, \Phi' = -2\Delta F_C \cos\theta$$

- Total complementary potential E

$$\Pi' = A' + \Phi' = \frac{\bar{k}F_C^2}{2\bar{k}_A k_B \bar{k}_C \cos\theta} - 2\Delta F_C \cos\theta,$$

10.8 Principle of minimum complementary energy

- PMCE

$$\frac{\partial \Pi'}{\partial F_C} = \frac{\bar{k} F_C}{\bar{k}_A k_B \bar{k}_C \cos\theta} - 2\Delta \cos\theta = 0,$$

- This yields F_A, F_B, and F_C

$$F_A = F_C = \frac{2\bar{k}_A \bar{k}_C \cos^2\theta}{\bar{k}} k_B \Delta, F_B = D = \left(1 - \frac{\bar{k}_A + \bar{k}_C}{\bar{k}}\right) k_B \Delta$$

- displacement at **O**: extension of the bar **B**

$$u_1^{(B)} = e_B + \Delta = \frac{\bar{k}_A + \bar{k}_C}{\bar{k}} \Delta$$

10.8 Principle of minimum complementary energy

10.8.4 The principle of least work

◦ total complementary energy=system's complementary energy + potential of the prescribed displacement

if prescribed displacement = 0, total complementary energy = complementary strain energy

→ Principle of least work

◦ Principle 13 (Principle of least work) In the absence of prescribed displacement , a conservative system undergoes compatible displacements if and only if the complementary strain energy is min. with respect to arbitrary changes in statically admissible forces.

◦ Principle 14 (Principle of least work) In the absence of prescribed displacement, a linearly elastic system undergoes compatible deformations if and only if the strain energy is a minimum with respect to arbitrary changes in statically admissible forces.

10.8 Principle of minimum complementary energy

Example 10.9. Three-bar truss with tip load

- only relevant equilibrium eqn: at joint **O**

$$F_A = F_C, F_A \cos\theta + F_B + F_C \cos\theta = P$$

- strain energy, first in terms of F_A, F_B, and F_C

$$A = \frac{1}{2}\left(\frac{F_A^2}{k_A \cos\theta} + \frac{F_B^2}{k_B} + \frac{F_C^2}{k_C \cos\theta} \right)$$

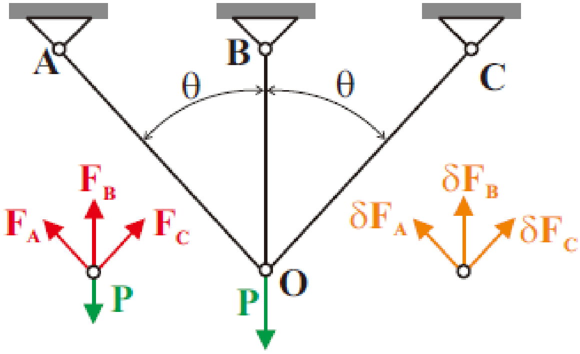


Fig. 10.30.

- three bar forces are expressed in terms of one, say F_C

$$A = \frac{1}{2}\left[\frac{F_C^2}{k_A \cos\theta} + \frac{\left(P - 2F_C \cos\theta\right)^2}{k_B} + \frac{F_C^2}{k_C \cos\theta} \right]$$

- Principle of least work, principle 14

$$\frac{\partial A}{\partial F_C} = \left[\frac{F_C}{k_A \cos\theta} - \frac{\left(P - 2F_C \cos\theta\right) 2\cos\theta}{k_B} + \frac{F_C}{k_C \cos\theta} \right] = 0$$

- can be solved for the bar force, F_C , and the equilibrium eqn then yield the other bar forces

$$\frac{F_A}{P} = \frac{F_C}{P} = \frac{2\bar{k}_A \bar{k}_C \cos^2\theta}{\bar{k}}, \frac{F_B}{P} = \frac{\bar{k}_A + \bar{k}_C}{\bar{k}}$$

- PMCE: derive the same condition in a more abstract but also systematic manner

10.9 Energy theorems

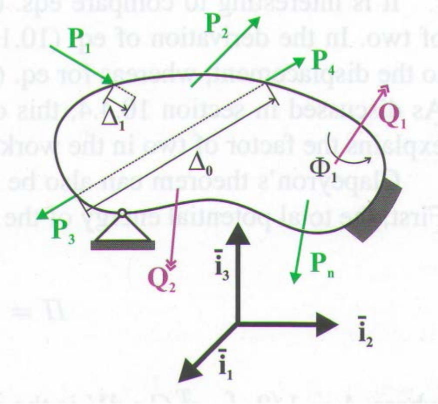


Fig. 10.40. Elastic body subjected to various loads

Properly constrained elastic body subjected to various concentrated loads and couples

P_i , $i = 1, 2, \ldots\ldots\ldots N$ ➞ displacement Δ_i

Q_j, $j = 1, 2, \ldots\ldots\ldots M$ ➞ rotation Φ_j

10.9.1 Clapeyron's theorem

◦ Eq.(10.12) ---- strain energy stored in the body=work done by the external forces as they are increased quasi-statically from zero to the final values.

$$A = W_E = \sum_{i=1}^{N} \int_0^{\Delta_i} P_i d\, u_i + \sum_{j=1}^{M} \int_0^{\Phi_i} Q_j d\theta_j$$

- lineary elastic -----applied loads are proportional to the displacements $P_i \propto u_i$, $Q_j \propto \theta_j$

$$A = W_E = \sum_{i=1}^{N} \frac{1}{2} P_i \Delta_i + \sum_{j=1}^{M} \frac{1}{2} Q_j \Phi_j \qquad (10.107)$$

10.9 Energy theorems

---- Clapeyron's theorem → useful for evaluating the strain energy as well as computing the deflection, Δ, at the point of application of a load, P

↔ Eq.(10.13) ----difference by a factor of ½.

load P is assumed to remain constant,

difference in the nature of the applied loading.

Example 10.13

10.9.2 Castigliano's first theorem

◦ Eq.(10.10) ---- $\Pi = A + \Phi = A - \sum_{i=1}^{N} P_i \Delta_i$

Dead loads

Principle of minimum total potential energy → stationarity of the total energy, Eq.(10.17)

$$\frac{\partial \Pi}{\partial \Delta_j} = \frac{\partial A}{\partial \Delta_j} - \frac{\partial}{\partial \Delta_j} \sum_{i=1}^{N} P_i \Delta_i = \frac{\partial A}{\partial \Delta_j} - P_j = 0$$

$$\rightarrow P_i = \frac{\partial A}{\partial \Delta_i}$$

Castigliano's first theorem (10.108)

10.9 Energy theorems

* All theorems are valid only for elastic structures

Clapeyron's theorem

Castigliano's 2nd theorem

← further limited to linearly elastic structures

10.9.3 Crotti-Engesser theorem

◦ Clapeyron's and Castigliano's first theorems ← Principle of minimum total potential energy

→ Parallel developments based on principle of mimimum complementary energy

- Eq (10.104): $\Pi^{'} = A^{'} + \Phi^{'}$

$$\Phi^{'} = -\sum_{i=1}^{N} P_i \Delta_i$$

P_i : driving forces required to obtain the prescribed displacements

$$\rightarrow \Pi^{'} = A^{'} + \Phi^{'} = A^{'} - \sum_{i=1}^{N} P_i \Delta_i$$

10.9 Energy theorems

- Statically admissible stress field → $A' = A'(P_i)$

Principle of minimum complementary energy → $$\frac{\partial \Pi'}{\partial P_j} = \frac{\partial A'}{\partial P_j} - \frac{\partial}{\partial P_j}\sum_{i=1}^{N} P_i \Delta_i = \frac{\partial A'}{\partial P_j} - \Delta_j = 0$$

→ $\Delta_i = \dfrac{\partial A'}{\partial P_i}$: Crotti-Engesser theorem (10.109)

... can be applied to multiple applied loads

10.9.4 Castigliano's 2nd theorem

- In the derivation of the Crotti-Engesser theorem, existence of complementary energy is assumed for elastic material

If linearly elastic, $A = A'$

→ $\underline{\Delta_i} = \dfrac{\partial A}{\partial P_i}$: Castigliano's 2nd theorem (10.110)

prescribed deflection

10.9 Energy theorems

10.9.5 Applications of energy theorems

◦ Castigliano's 2^{nd} theorem…. also useful for hyper static problems

- cantilevered beam with a tip support

… a prescribed tip displacement, which is required to vanish

P_i : driving force, → Reaction force at the support

◦ Castiglian's 2^{nd} theorem → $\Delta_i = 0, \ \dfrac{\partial A}{\partial P_i} = 0$

Compatibility equation at the tip support → Principle of least work (Principle 13)

Example 10.14

10.9 Energy theorems

10.9.6 The dummy load method

◦ Is it possible to use Castigliano's 2nd theorem to compute the deflection at a point where no load is applied?

◦ 1st step …… a fictitious or "dummy load," $\mathcal{P}$, is applied to the structure at the point where the displacement is to be computed.

◦ 2nd step ……. $\hat{\Delta} = \dfrac{\partial A}{\partial \mathcal{P}}$ By Castigliano's 2nd theorem

◦ last step ….. $\Delta = \lim\limits_{\mathcal{P}\rightarrow 0} \hat{\Delta}$

$$\Delta = \lim_{\mathcal{P}\rightarrow 0} \frac{\partial A}{\partial \mathcal{P}} \qquad (10.111)$$

◦ if elastic, but nonlinear, A' must be used instead of A.

10.9 Energy theorems

Example 10.19 Tip deflection of a cantilevered beam

$$\hat{\Delta} = \frac{\partial A}{\partial \mathcal{P}} = \frac{1}{2H_{33}^c}\left(\frac{p_0 L^4}{4} + \frac{2\mathcal{P}L^3}{3}\right)$$

$$\Delta = \lim_{\mathcal{P}\to 0}\hat{\Delta} = \frac{p_0 L^4}{8H_{33}^c}$$

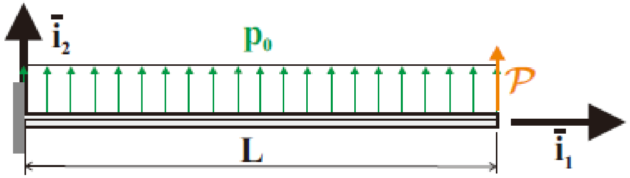


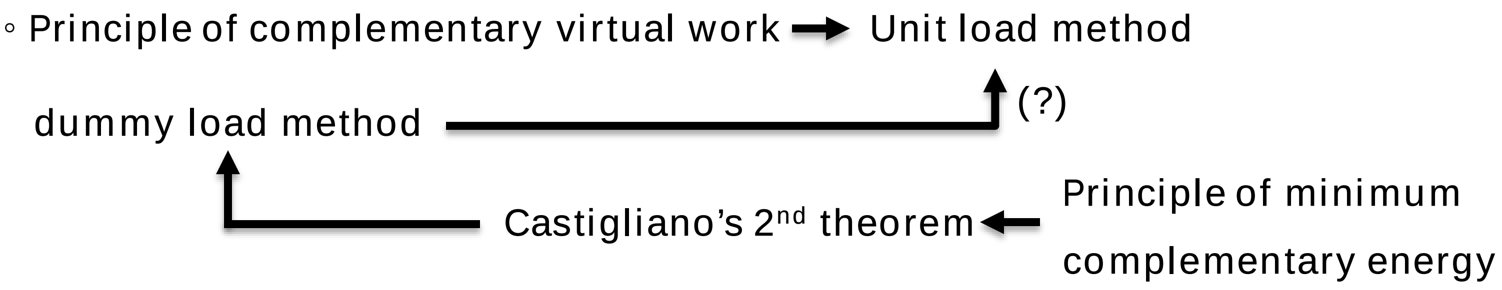
Fig. 10.45. Cantilevered beam under uniform loading.

or, can be obtained by taking the limit before carrying out the integrations

$$\Delta = \left[\frac{\partial}{\partial\mathcal{P}}\int_0^L \frac{M_3^2}{2H_{33}^c}dx_1\right]_{\mathcal{P}=0} = \int_0^L \frac{M_3}{H_{33}^c}\left[\frac{\partial M_3}{\partial \mathcal{P}}\right]_{\mathcal{P}=0} dx_1 \qquad (10.112)$$

10.9 Energy theorems

10.9.7 Unit load method revisited



◦ Principle of complementary virtual work → Unit load method

(?)

dummy load method

Castigliano's 2nd theorem

Principle of minimum complementary energy

- dummy load method ……strain energy in an isostatic beam

$$A=\int_0^L \frac{\mathcal{M}_3^2}{2H_{33}^c} d x_1$$

$\mathcal{M}_3(x_1)$ …… bending moment distribution generated by the externally applied loads and dummy load

10.9 Energy theorems

- Castiglano's 2nd theorem

$$\Delta = \lim_{\mathcal{P}\to 0} \frac{\partial A}{\partial \mathcal{P}} = \lim_{\mathcal{P}\to 0} \int_0^L \frac{\mathcal{M}_3}{H_{33}^c} \frac{\partial \mathcal{M}_3}{\partial \mathcal{P}} d x_1 \qquad (10.113)$$

$\lim_{\mathcal{P}\to 0} \mathcal{M}_3 = M_3$ = bending moment due to externally applied loads only

$\lim_{\mathcal{P}\to 0} \frac{\partial \mathcal{M}_3}{\partial P} = \hat{M}_3$ = bending moment due to a unit load only

Eq. (10.113) ➡ unit load method, Eq.(9.83)

$$\Delta = \int_0^L \frac{\hat{M}_3 M_3}{H_{33}^c} d x_1 \qquad (10.114)$$

10.9 Energy theorems

M_3 is identical for
- unit
- Dummy load method

However, $\hat{M}_3$ has a difference

- dummy load method…..bending moment acting in the structure subjected to a unit dummy load
- unit load method …. "any statically admissible" bending moment distribution in equilibrium with unit load

→ not necessarily the actual bending moment distribution acting in the structure subjected to the unit load

→ more versatile, can result in a significant simplification of the procedure