Chapter 11. Kinematics of Particles

Contents

Curvilinear Motion: Position, Velocity & Acceleration

Derivatives of Vector Functions

Rectangular Components of Velocity and Acceleration

Sample Problem 11.10

Motion Relative to a Frame in Translation

Sample Problem 11.14

Tangential and Normal Components

Sample Problem 11.16

Radial and Transverse Components

Sample Problem 11.18

Introduction

Kinematic relationships are used to help us determine the trajectory of a snowboarder completing a jump, the orbital speed of a satellite, and accelerations during acrobatic flying.

©Purestock/SuperStock RF

©iLexx/Getty Images RF

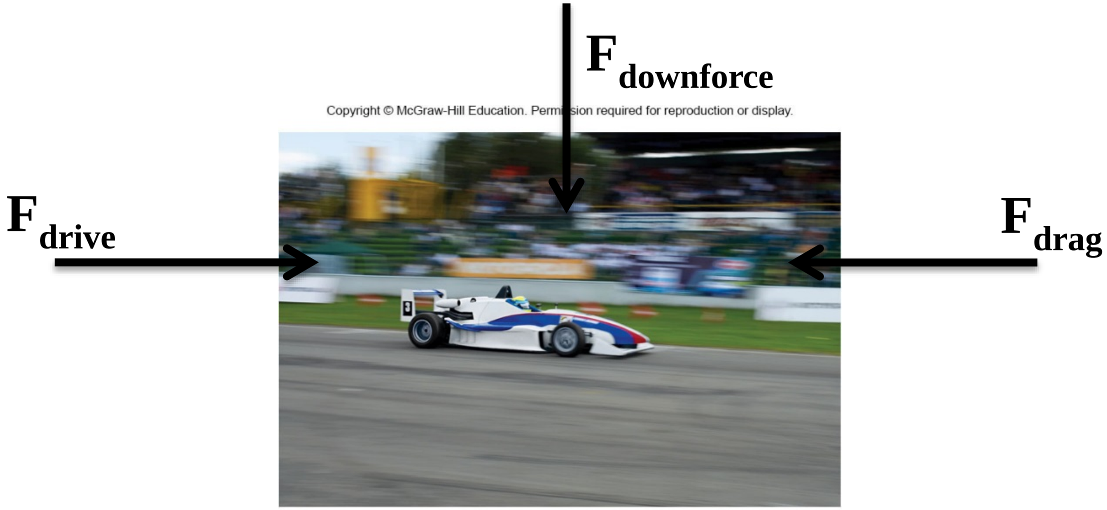
©Purestock/SuperStock RF

NASA/JSC

Dynamics includes:

Kinematics: study of the geometry of motion.

Relates displacement, velocity, acceleration, and time *without reference* to the cause of motion.



Kinetics: study of the relations existing between the forces acting on a body, the mass of the body, and the motion of the body. Kinetics is used to predict the motion caused by given forces or to determine the forces required to produce a given motion.

Particle kinetics includes :

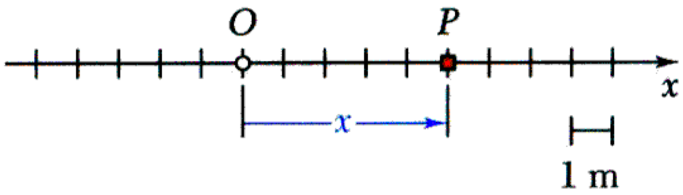
- *Rectilinear motion*: position, velocity, and acceleration of a particle as it moves along a straight line.

Stefano Paltera/NREL

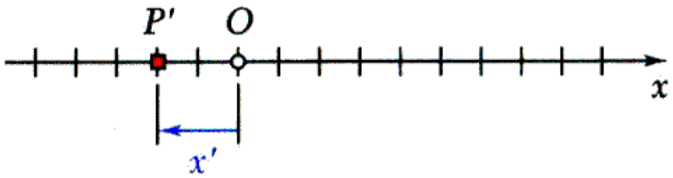
©Tony Hertz/Alamy

- *Curvilinear motion*: position, velocity, and acceleration of a particle as it moves along a curved line in two or three dimensions.

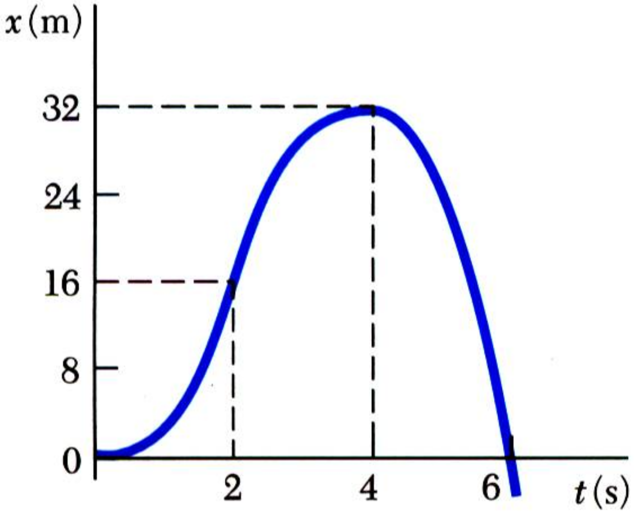
Rectilinear Motion: Position, Velocity & Acceleration



Rectilinear motion: particle moving along a straight line

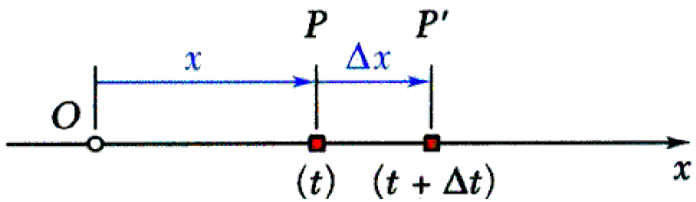


Position coordinate: defined by positive or negative distance from a fixed origin on the line.

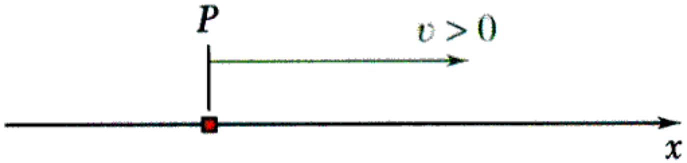


- The *motion* of a particle is known if the position coordinate for particle is known for every value of time *t*.

- May be expressed in the form of a function, e.g., $x = 6t^2 - t^3$ or in the form of a graph *x* vs. *t*.

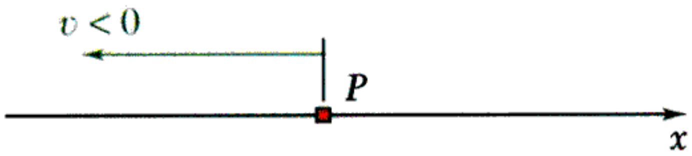


- Consider particle which occupies position *P* at time *t* and *P′* at *t*+*Dt*,

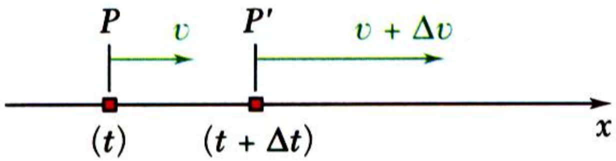


$$\textit{Average velocity} = \frac{\Delta x}{\Delta t}$$

$$\textit{Instantaneous velocity} = v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$



- Instantaneous velocity may be positive or negative. Magnitude of velocity is referred to as *particle speed*.

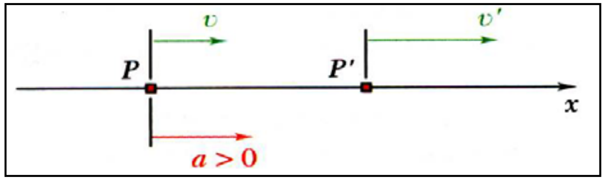


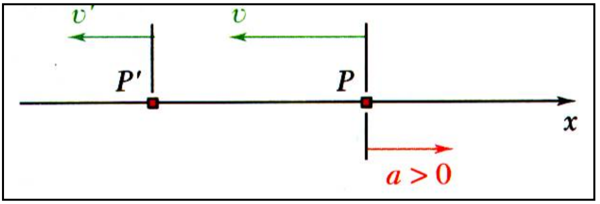
- Consider particle with velocity *v* at time *t* and *v′* at *t*+⊿*t*,

$$\textit{Instantaneous acceleration} = a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

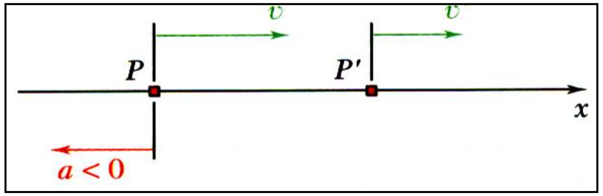
•Instantaneous acceleration may be:

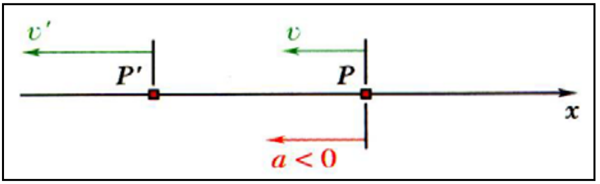
– positive: increasing positive velocity
or decreasing negative velocity





– negative: decreasing positive velocity
or increasing negative velocity.



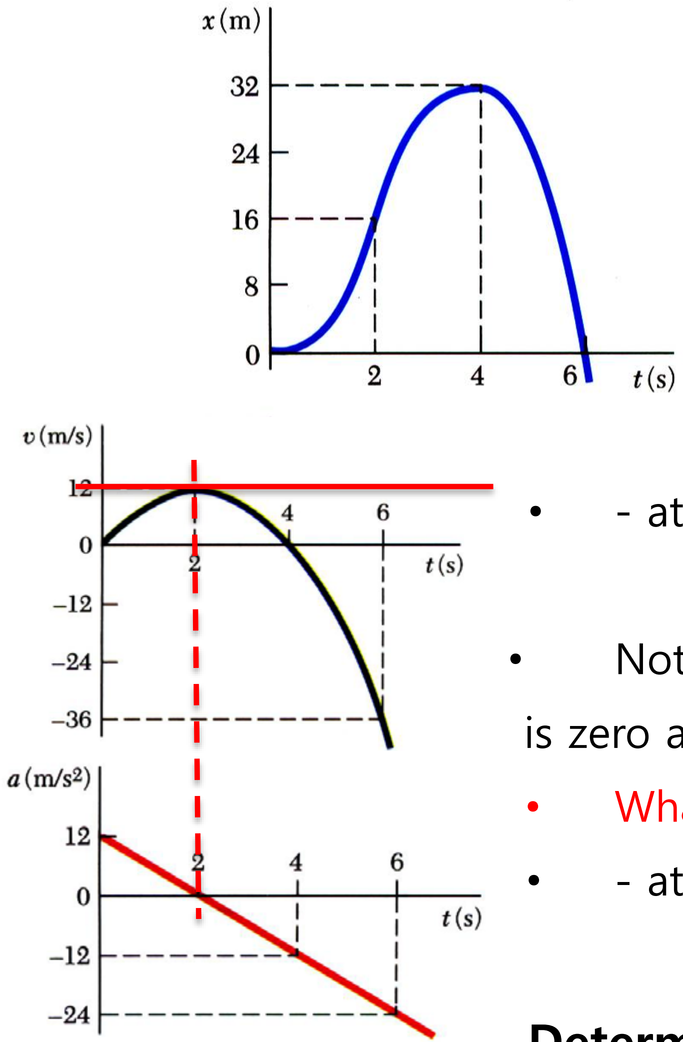


Concept Quiz

What is true about the kinematics of a particle?

a) The velocity of a particle is always positive
b) The velocity of a particle is equal to the slope of the position-time graph
c) If the position of a particle is zero, then the velocity must zero
d) If the velocity of a particle is zero, then its acceleration must be zero

answer : (b)



- From our example,

$$x = 6t^2 - t^3$$

$$v = \frac{dx}{dt} = 12t - 3t^2$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = 12 - 6t$$

- What are *x*, *v*, and *a* at t = 2 s ?
- \- at t = 2 s, x = 16 m, $v = v_{max}$ = 12 m/s, a = 0

- Note that v_{max} occurs when a=0, and that the slope of the velocity curve is zero at this point.
- What are *x*, *v*, and *a* at t = 4 s ?
- \- at t = 4 s, $x = x_{max}$ = 32 m, v = 0, a = -12 m/s^2

Determining the Motion of a Particle

- We often determine accelerations from the forces applied (kinetics will be covered later)

- Generally have three classes of motion
  - acceleration given as a function of *time*, $a = f(t)$
  - acceleration given as a function of *position*, $a = \mathrm{f}(x)$
  - acceleration given as a function of *velocity*, $a = \mathrm{f}(v)$

Can you think of a physical example of when force is a function of position? When force is a function of velocity?

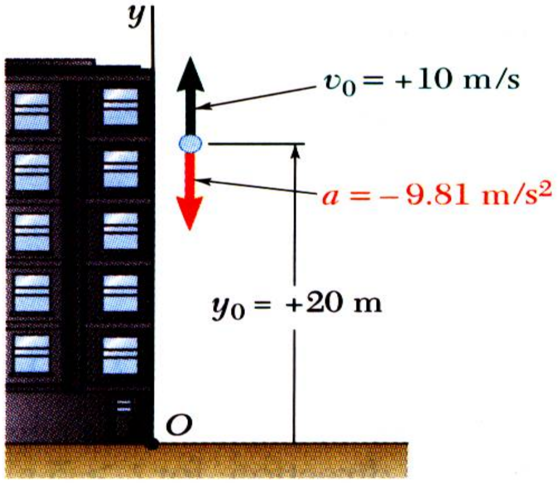
A Spring

Drag

Acceleration as a function of time, position, or velocity

If….	Kinematic relationship	Integrate
$a = a(t)$	$\frac{dv}{dt} = a(t)$	$\int_{v_0}^{v} dv = \int_{0}^{t} a(t)\,dt$
$a = a(x)$	$dt = \frac{dx}{v}$ and $a = \frac{dv}{dt}$ $v\,dv = a(x)\,dx$	$\int_{v_0}^{v} v\,dv = \int_{x_0}^{x} a(x)\,dx$
$a = a(v)$	$\frac{dv}{dt} = a(v)$	$\int_{v_0}^{v} \frac{dv}{a(v)} = \int_{0}^{t} dt$
	$v\frac{dv}{dx} = a(v)$	$\int_{x_0}^{x} dx = \int_{v_0}^{v} \frac{v\,dv}{a(v)}$

Sample **Problem 11.2**



STRATEGY:

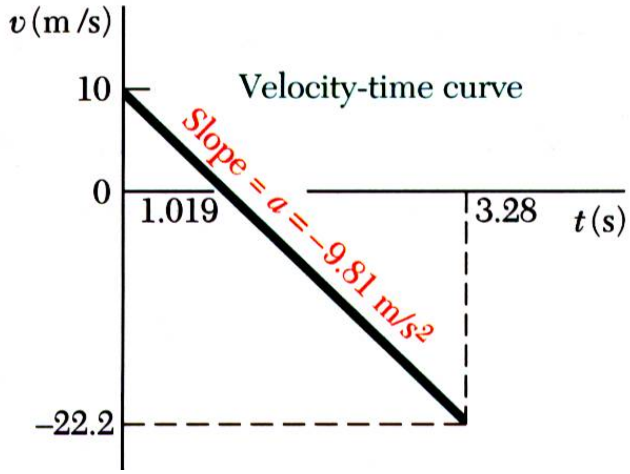
•Integrate twice to find *v*(*t*) and *y*(*t*).

•Solve for *t* when velocity equals zero (time for maximum elevation) and evaluate corresponding altitude.

•Solve for *t* when altitude equals zero (time for ground impact) and evaluate corresponding velocity.

Ball tossed with 10 m/s vertical velocity from window 20 m above ground. Determine:

- velocity and elevation above ground at time *t*,
- highest elevation reached by ball and corresponding time, and
- time when ball will hit the ground and corresponding velocity.



MODELING and ANALYSIS:

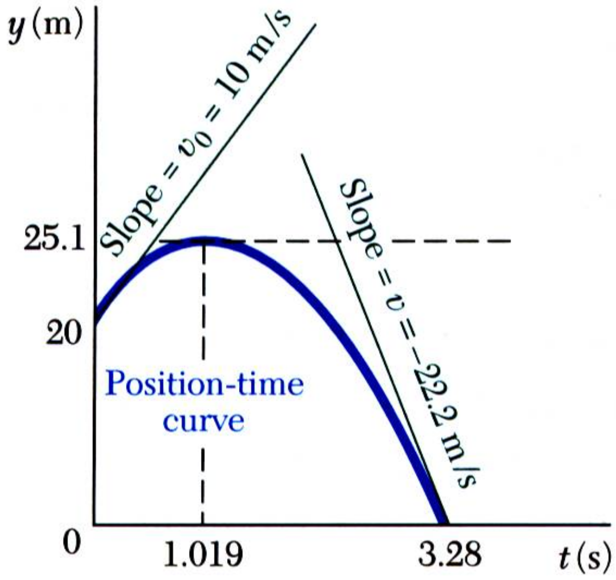
- Velocity and Elevation

Integrate twice to find $v(t)$ and $y(t)$.

$$\frac{dv}{dt} = a = -9.81 \,\mathrm{m}/\mathrm{s}^2$$

$$\int_{v_0}^{v(t)} dv = -\int_0^t 9.81\, dt \qquad v(t) - v_0 = -9.81t$$

$$\boxed{v(t) = 10\frac{\mathrm{m}}{\mathrm{s}} - \left(9.81\frac{\mathrm{m}}{\mathrm{s}^2}\right)t}$$



$$\frac{dy}{dt} = v = 10 - 9.81t$$

$$\int_{y_0}^{y(t)} dy = \int_0^t (10 - 9.81t)\,dt \qquad y(t) - y_0 = 10t - \tfrac{1}{2}9.81t^2$$

$$\boxed{y(t) = 20\,\mathrm{m} + \left(10\frac{\mathrm{m}}{\mathrm{s}}\right)t - \left(4.905\frac{\mathrm{m}}{\mathrm{s}^2}\right)t^2}$$

- Highest Elevation

Solve for t when velocity equals zero and evaluate corresponding altitude.

$$v(t) = 10\frac{\mathrm{m}}{\mathrm{s}} - \left(9.81\frac{\mathrm{m}}{\mathrm{s}^2}\right)t = 0$$

$$\boxed{t = 1.019\,\mathrm{s}}$$

Substitute t:

$$y(t) = 20\,\mathrm{m} + \left(10\,\frac{\mathrm{m}}{\mathrm{s}}\right)t - \left(4.905\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)t^2$$

$$y = 20\,\mathrm{m} + \left(10\,\frac{\mathrm{m}}{\mathrm{s}}\right)(1.019\,\mathrm{s}) - \left(4.905\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)(1.019\,\mathrm{s})^2$$

$$\boxed{y = 25.1\mathrm{m}}$$

- Ball Hits the Ground

Solve for t when altitude equals zero and evaluate corresponding velocity.

$$y(t) = 20\,\mathrm{m} + \left(10\,\frac{\mathrm{m}}{\mathrm{s}}\right)t - \left(4.905\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)t^2 = 0$$

$$\boxed{\begin{array}{l} t = -1.243\mathrm{s}\ \ (\mathrm{meaningless}) \\ t = 3.28\mathrm{s} \end{array}}$$

$$v(t) = 10\frac{\text{m}}{\text{s}} - \left(9.81\frac{\text{m}}{\text{s}^2}\right)t$$

$$v(3.28\,\text{s}) = 10\frac{\text{m}}{\text{s}} - \left(9.81\frac{\text{m}}{\text{s}^2}\right)(3.28\,\text{s})$$

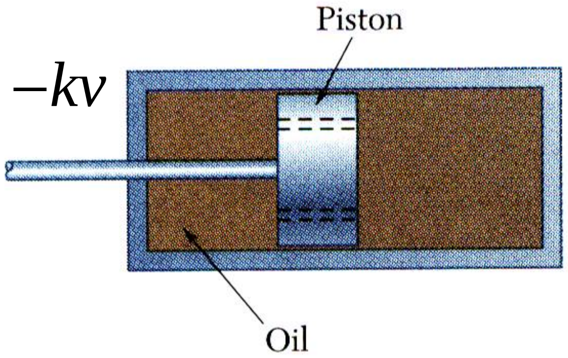
$$\boxed{v = -22.2\frac{\text{m}}{\text{s}}}$$

REFLECT and THINK:

When the acceleration is constant, the velocity changes linearly, and the position is a quadratic function of time.

Sample Problem 11.3

$$a = -kv$$



A mountain bike shock mechanism used to provide shock absorption consists of a piston that travels in an oil-filled cylinder. As the cylinder is given an initial velocity v_0, the piston moves and oil is forced through orifices in piston, causing piston and cylinder to decelerate at rate proportional to their velocity.

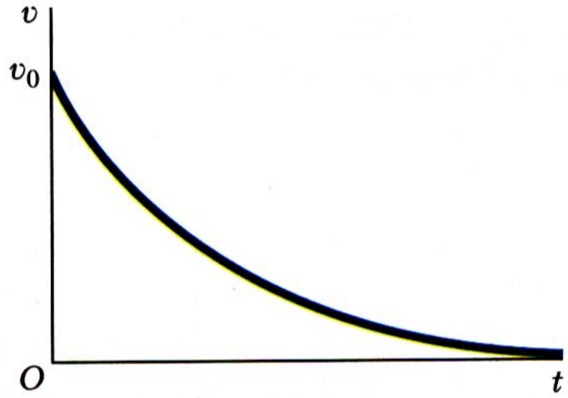
Determine $v(t)$, $x(t)$, and $v(x)$.

STRATEGY:

Integrate $a = dv/dt = -kv$ to find $v(t)$.

Integrate $v(t) = dx/dt$ to find $x(t)$.

Integrate $a = v\,dv/dx = -kv$ to find $v(x)$.

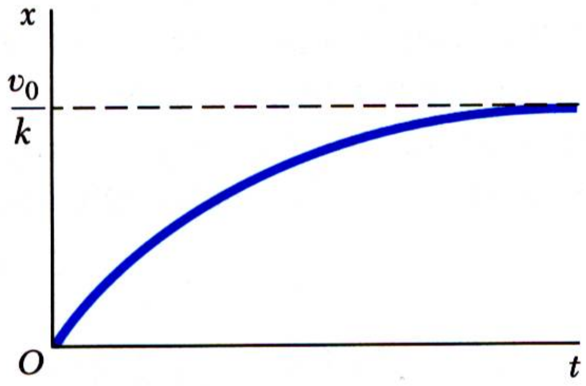


MODELING and ANALYSIS:

a. Integrate $a = dv/dt = -kv$ to find $v(t)$.

$$a = \frac{dv}{dt} = -kv \qquad \int_{v_0}^{v} \frac{dv}{v} = -k\int_{0}^{t} dt \qquad \ln\frac{v(t)}{v_0} = -kt$$

$$\boxed{v(t) = v_0 e^{-kt}}$$

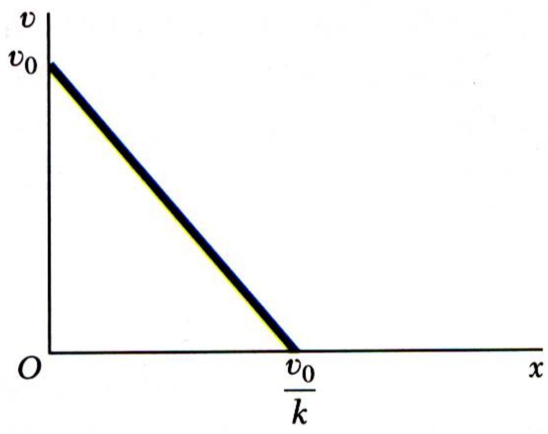


b. Integrate $v(t) = dx/dt$ to find $x(t)$.

$$v(t) = \frac{dx}{dt} = v_0 e^{-kt}$$

$$\int_0^x dx = v_0 \int_0^t e^{-kt} dt \qquad x(t) = v_0 \left[-\frac{1}{k} e^{-kt} \right]_0^t$$

$$\boxed{x(t) = \frac{v_0}{k}\left(1 - e^{-kt}\right)}$$



c. Integrate $a = v\,dv/dx = -kv$ to find $v(x)$.

$$a = v\frac{dv}{dx} = -kv \qquad dv = -k\,dx \qquad \int_{v_0}^{v} dv = -k\int_{0}^{x} dx$$

$$v - v_0 = -kx$$

$$\boxed{v = v_0 - kx}$$

- Alternatively,

with
$$x(t) = \frac{v_0}{k}\left(1 - e^{-kt}\right)$$

and
$$v(t) = v_0 e^{-kt} \text{ or } e^{-kt} = \frac{v(t)}{v_0}$$

then
$$x(t) = \frac{v_0}{k}\left(1 - \frac{v(t)}{v_0}\right)$$

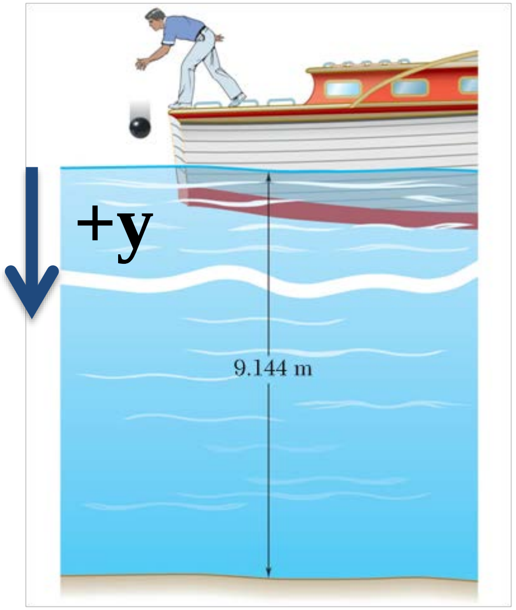
$$\boxed{v = v_0 - kx}$$

REFLECT and THINK:

You could have solved part *c* by eliminating *t* from the answers obtained for parts *a* and *b*. You could use this alternative method as a check. From part *a*, you obtain $e^{-kt} = v/v_0$; substituting into the answer of part *b*, you have:

$$x = \frac{v_0}{k}\left(1 - e^{-kt}\right) = \frac{v_0}{k}\left(1 - \frac{v}{v_0}\right) \qquad v = v_0 - kx \qquad \text{(checks)}$$

Group Problem Solving



A bowling ball is dropped from a boat so that it strikes the surface of a lake with a speed of 8 m/s. Assuming the ball experiences a downward acceleration of $a = 3 - 0.1v^2$ when in the water, determine the velocity of the ball when it strikes the bottom of the lake. (a and v expressed in m/s^2 and m/s respectively)

Which integral should you choose?

(a) $$\int_{v_0}^{v} dv = \int_{0}^{t} a(t)\,dt$$

(c) $$\int_{v_0}^{v} v\,dv = \int_{x_0}^{x} a(x)\,dx$$

(b) $$\int_{x_0}^{x} dx = \int_{v_0}^{v} \frac{v\,dv}{a(v)}$$

(d) $$\int_{v_0}^{v} \frac{dv}{a(v)} = \int_{0}^{t} dt$$

answer (b)

. RELATIVE MOTION

Uniform Rectilinear Motion

U.S. Air Force photo/Samuel King Jr.

Once a safe speed of descent for a vertical landing is reached, a Harrier jet pilot will adjust the vertical thrusters to equal the weight of the aircraft. The plane then travels at a constant velocity downward. If motion is in a straight line, this is uniform rectilinear motion.

For a particle in uniform rectilinear motion, the acceleration is zero and the velocity is constant.

Careful – these only apply to uniform rectilinear motion!

Uniformly Accelerated Rectilinear Motion

For a particle in uniformly accelerated rectilinear motion, the acceleration of the particle is constant. You may recognize these constant acceleration equations from your physics courses.

$$\frac{dv}{dt} = a = \text{constant} \qquad \int_{v_0}^{v} dv = a\int_{0}^{t} dt \qquad v = v_0 + at$$

$$\frac{dx}{dt} = v_0 + at \qquad \int_{x_0}^{x} dx = \int_{0}^{t} \left(v_0 + at\right) dt \qquad x = x_0 + v_0 t + \tfrac{1}{2} a t^2$$

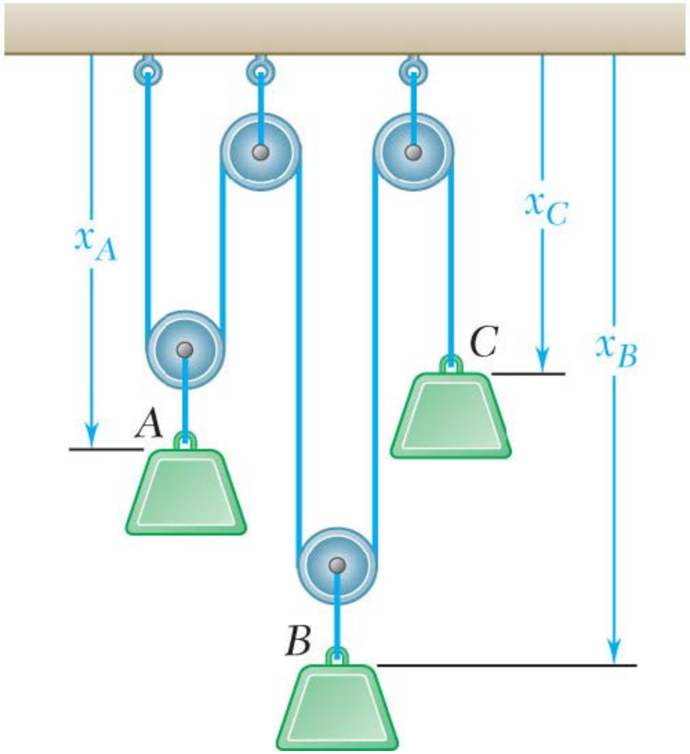
$$v\frac{dv}{dx} = a = \text{constant} \qquad \int_{v_0}^{v} v\, dv = a\int_{x_0}^{x} dx \qquad v^2 = v_0^2 + 2a\left(x - x_0\right)$$

Careful – these only apply to uniformly accelerated rectilinear motion!

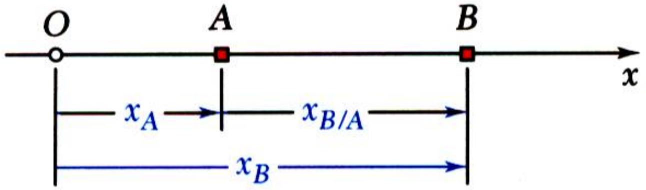
Motion of Several Particles

We may be interested in the motion of several different particles, whose motion may be independent or linked together.

©Chris Ryan/agefotostock RF



Motion of Several Particles: Relative Motion



For particles moving along the same line, time should be recorded from the same starting instant and displacements should be measured from the same origin in the same direction.

$x_{B/A} = x_B - x_A =$ relative position of *B* with respect to *A*

$x_B = x_A + x_{B/A}$

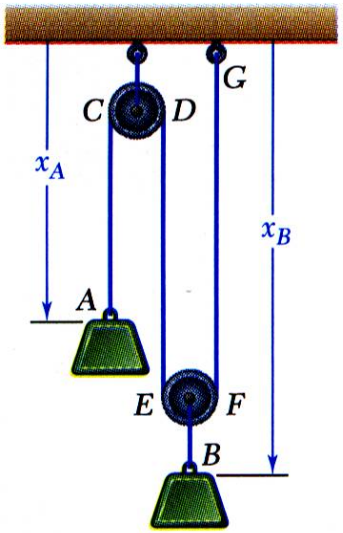
$v_{B/A} = v_B - v_A =$ relative velocity of *B* with respect to *A*

$v_B = v_A + v_{B/A}$

$a_{B/A} = a_B - a_A =$ relative acceleration of *B* with respect to *A*

$a_B = a_A + a_{B/A}$

Dependent Motion

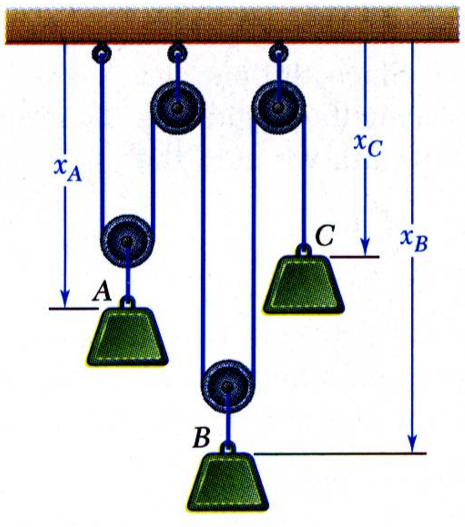


- Position of a particle may *depend* on position of one or more other particles.
- Position of block *B* depends on position of block *A*. Since rope is of constant length, it follows that sum of lengths of segments must be constant.

$x_A + 2x_B =$ constant (one degree of freedom)

- Positions of three blocks are dependent.

$2x_A + 2x_B + x_C =$ constant (two degrees of freedom)

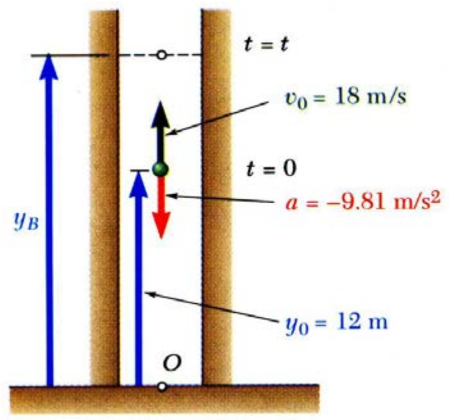


- For linearly related positions, similar relations hold between velocities and accelerations.

$$2\frac{dx_A}{dt} + 2\frac{dx_B}{dt} + \frac{dx_C}{dt} = 0 \quad \text{or} \quad 2v_A + 2v_B + v_C = 0$$

$$2\frac{dv_A}{dt} + 2\frac{dv_B}{dt} + \frac{dv_C}{dt} = 0 \quad \text{or} \quad 2a_A + 2a_B + a_C = 0$$

Sample Problem 11.5

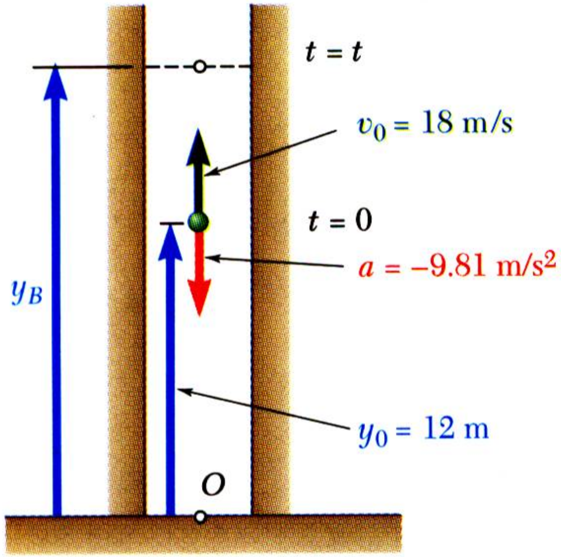


Ball thrown vertically from 12 m level in elevator shaft with initial velocity of 18 m/s. At same instant, open-platform elevator passes 5 m level moving upward at 2 m/s.

Determine *(a)* when and where ball hits elevator and *(b)* relative velocity of ball and elevator at contact.

STRATEGY:

- Substitute initial position and velocity and constant acceleration of ball into general equations for uniformly accelerated rectilinear motion.
- Substitute initial position and constant velocity of elevator into equation for uniform rectilinear motion.
- Write equation for relative position of ball with respect to elevator and solve for zero relative position, i.e., impact.
- Substitute impact time into equation for position of elevator and relative velocity of ball with respect to elevator.

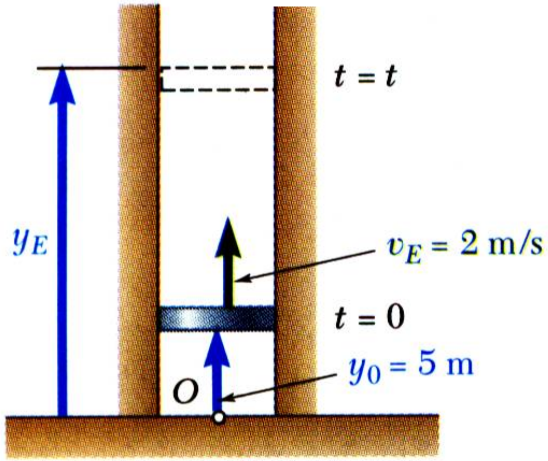


MODELING and ANALYSIS:

- Substitute initial position and velocity and constant acceleration of ball into general equations for uniformly accelerated rectilinear motion.

$$v_B = v_0 + at = 18\frac{\text{m}}{\text{s}} - \left(9.81\frac{\text{m}}{\text{s}^2}\right)t$$

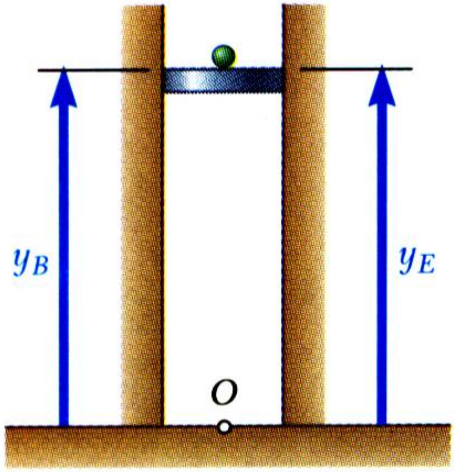
$$y_B = y_0 + v_0 t + \tfrac{1}{2}at^2 = 12\,\text{m} + \left(18\frac{\text{m}}{\text{s}}\right)t - \left(4.905\frac{\text{m}}{\text{s}^2}\right)t^2$$



- Substitute initial position and constant velocity of elevator into equation for uniform rectilinear motion.

$$v_E = 2\frac{\text{m}}{\text{s}}$$

$$y_E = y_0 + v_E t = 5\,\text{m} + \left(2\frac{\text{m}}{\text{s}}\right)t$$



• Write equation for relative position of ball with respect to elevator and solve for zero relative position, i.e., impact.

$$y_{B/E} = \left(12 + 18t - 4.905t^2\right) - \left(5 + 2t\right) = 0$$

$$\boxed{\begin{aligned} &t = -0.39\,\mathrm{s} \quad (\text{meaningless}) \\ &t = 3.65\,\mathrm{s} \end{aligned}}$$

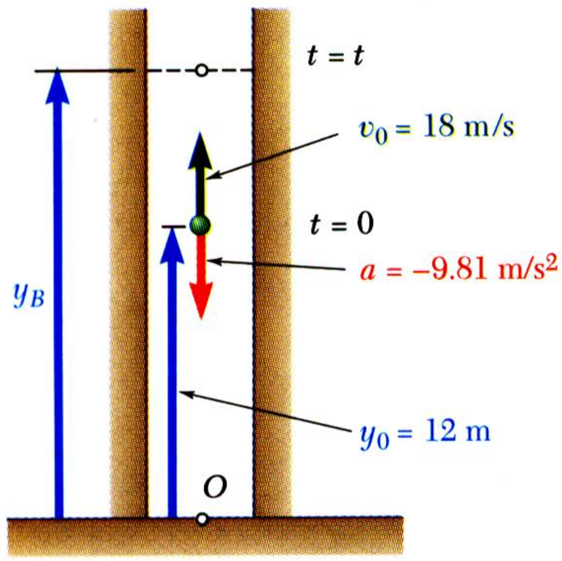
• Substitute impact time into equations for position of elevator and relative velocity of ball with respect to elevator.

$$y_E = 5 + 2(3.65)$$

$$\boxed{y_E = 12.3\,\mathrm{m}}$$

$$v_{B/E} = (18 - 9.81t) - 2$$
$$= 16 - 9.81(3.65)$$

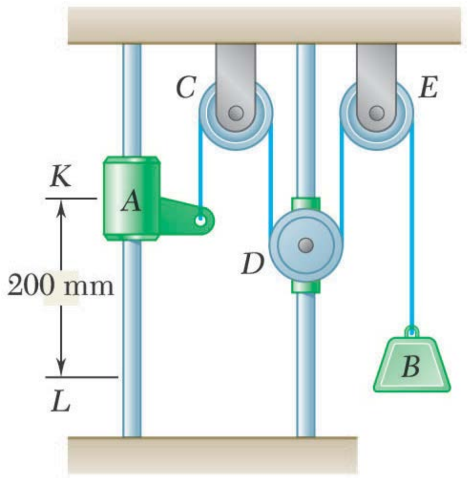
$$\boxed{v_{B/E} = -19.81\,\frac{\mathrm{m}}{\mathrm{s}}}$$



REFLECT and THINK:

The key insight is that, when two particles collide, their position coordinates must be equal. Also, although you can use the basic kinematic relationships in this problem, you may find it easier to use the equations relating *a*, *v*, *x*, and *t* when the acceleration is constant or zero.

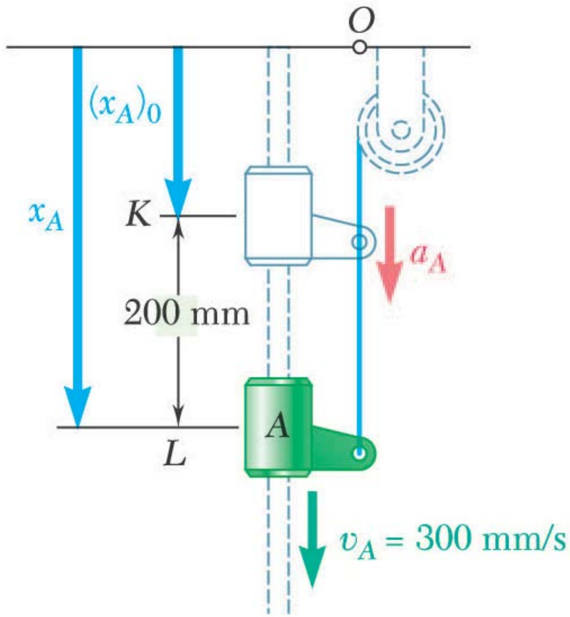
Sample Problem 11.7



Pulley *D* is attached to a collar which is pulled down at 75 mm/s. At $t = 0$, collar *A* starts moving down from *K* with constant acceleration and zero initial velocity. Knowing that velocity of collar *A* is 300 mm/s as it passes *L*, determine the change in elevation, velocity, and acceleration of block *B* when block *A* is at *L*.

STRATEGY

- Define origin at upper horizontal surface with positive displacement downward.
- Collar *A* has uniformly accelerated rectilinear motion. Solve for acceleration and time *t* to reach *L*.
- Pulley *D* has uniform rectilinear motion. Calculate change of position at time *t*.
- Block *B* motion is dependent on motions of collar *A* and pulley *D*. Write motion relationship and solve for change of block *B* position at time *t*.
- Differentiate motion relation twice to develop equations for velocity and acceleration of block *B*.



MODELING and ANALYSIS:

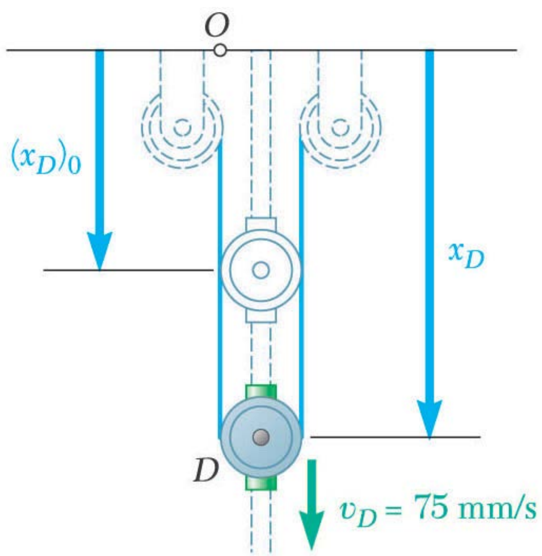
- Define origin at upper horizontal surface with positive displacement downward.
- Collar *A* has uniformly accelerated rectilinear motion. Solve for acceleration and time *t* to reach *L*.

$$v_A^2 = (v_A)_0^2 + 2a_A\left[x_A - (x_A)_0\right]$$

$$\left(300\frac{\text{mm}}{\text{s}}\right)^2 = 2a_A(200\,\text{mm}) \qquad a_A = 225\frac{\text{mm}}{\text{s}^2}$$

$$v_A = (v_A)_0 + a_A t$$

$$300\frac{\text{mm}}{\text{s}} = 225\frac{\text{mm}}{\text{s}^2}t \qquad t = 1.333\ \text{s}$$

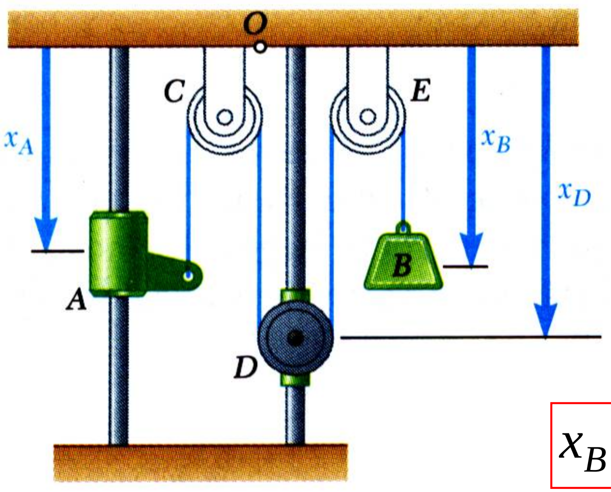


- Pulley *D* has uniform rectilinear motion. Calculate change of position at time *t*.

$$x_D = (x_D)_0 + v_D t$$

$$x_D - (x_D)_0 = \left(75\frac{\text{mm}}{\text{s}}\right)(1.333\,\text{s}) = 100\ \text{mm}$$

- Block *B* motion is dependent on motions of collar *A* and pulley *D*. Write motion relationship and solve for change of block *B* position at time *t*.



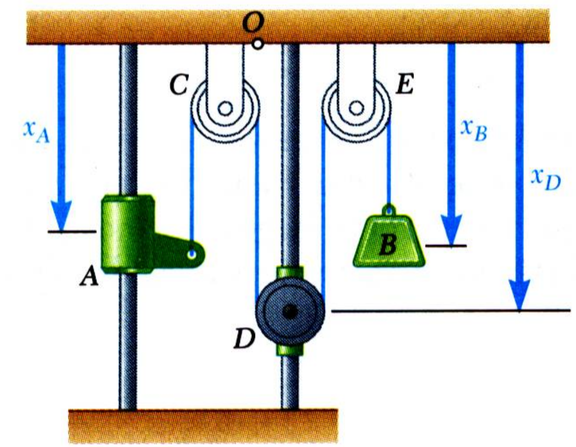
- Total length of cable remains constant,

$$x_A + 2x_D + x_B = (x_A)_0 + 2(x_D)_0 + (x_B)_0$$

$$\left[x_A - (x_A)_0\right] + 2\left[x_D - (x_D)_0\right] + \left[x_B - (x_B)_0\right] = 0$$

$$(200\,\text{mm}) + 2(100\,\text{mm}) + \left[x_B - (x_B)_0\right] = 0$$

$$\boxed{x_B - (x_B)_0 = -400\,\text{mm.}}$$



• Differentiate motion relation twice to develop equations for velocity and acceleration of block *B*.

$$x_A + 2x_D + x_B = \text{constant}$$

$$v_A + 2v_D + v_B = 0$$

$$\left(300\frac{\text{mm}}{\text{s}}\right) + 2\left(75\frac{\text{mm}}{\text{s}}\right) + v_B = 0$$

$$\boxed{v_B = 450\frac{\text{mm}}{\text{s}}}$$

$$a_A + 2a_D + a_B = 0$$

$$\left(225\frac{\text{mm}}{\text{s}^2}\right) + a_B = 0$$

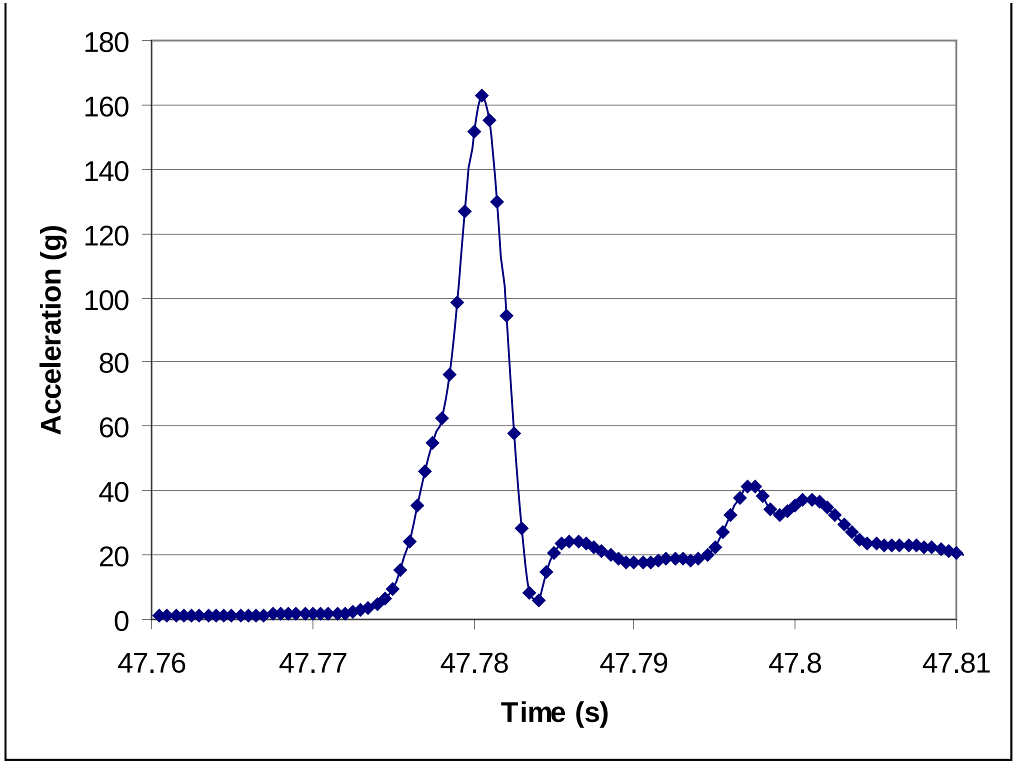
$$\boxed{a_B = -225\frac{\text{mm}}{\text{s}^2}}$$

REFLECT and THINK:

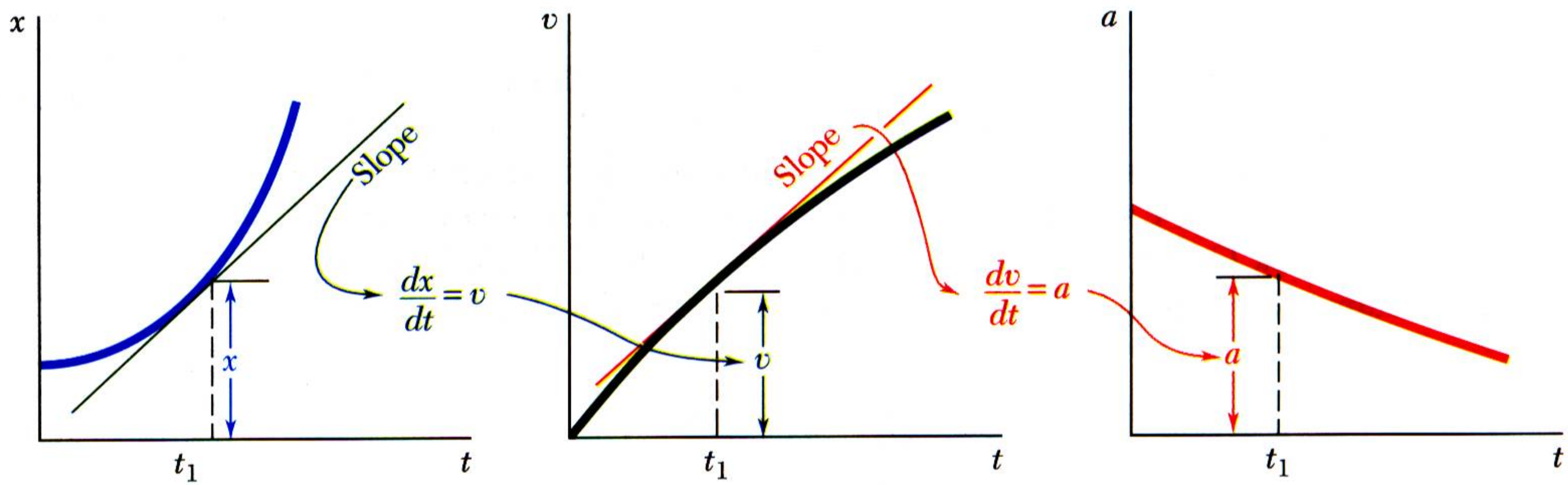
In this case, the relationship we needed was not between position coordinates, but between changes in position coordinates at two different times. The key step is to clearly define your position vectors. This is a two degree-of-freedom system, because two coordinates are required to completely describe it.

Graphical Solutions

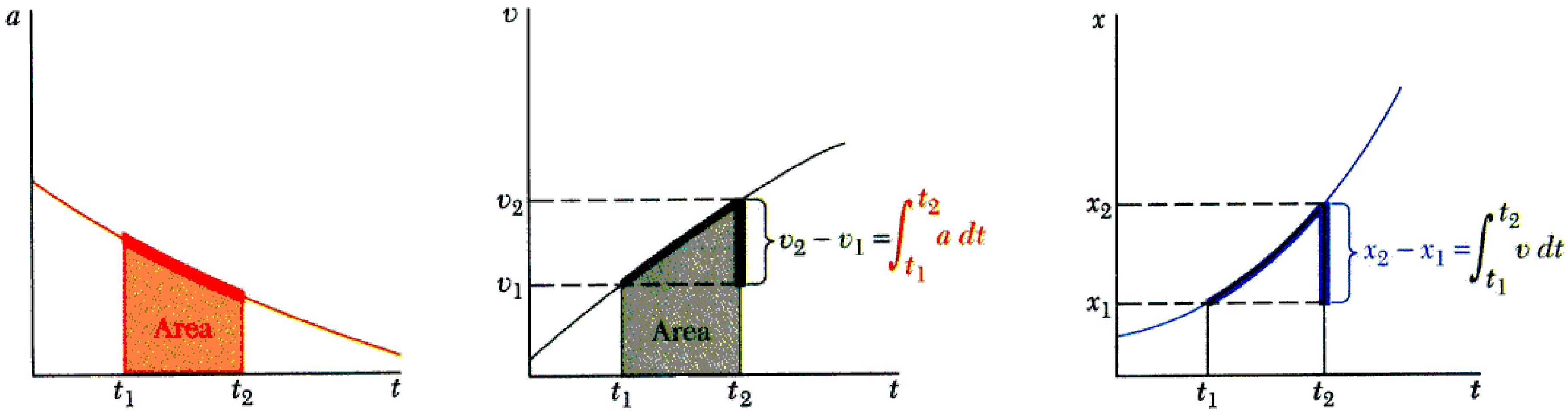
Engineers often collect position, velocity, and acceleration data.   Graphical solutions are often useful in analyzing these data.



Acceleration data from a head impact during a round of boxing.



- **Given the *x-t* curve, the *v-t* curve is equal to the *x-t* curve slope.**
- **Given the *v-t* curve, the *a-t* curve is equal to the *v-t* curve slope.**



- **Given the *a-t* curve, the change in velocity between t_1 and t_2 is equal to the area under the *a-t* curve between t_1 and t_2.**

- **Given the *v-t* curve, the change in position between t_1 and t_2 is equal to the area under the *v-t* curve between t_1 and t_2.**

Curvilinear Motion:

The snowboarder and the train both undergo curvilinear motion.

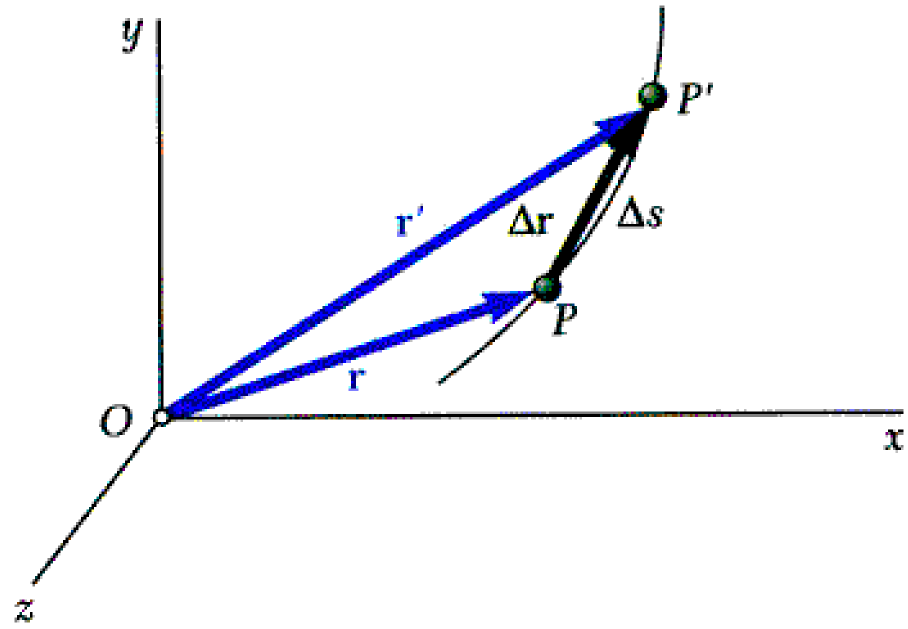
©Purestock/SuperStock RF

Copyright © McGraw-Hill Education. Permission required for reproduction or display.

©Tony Hertz/Alamy

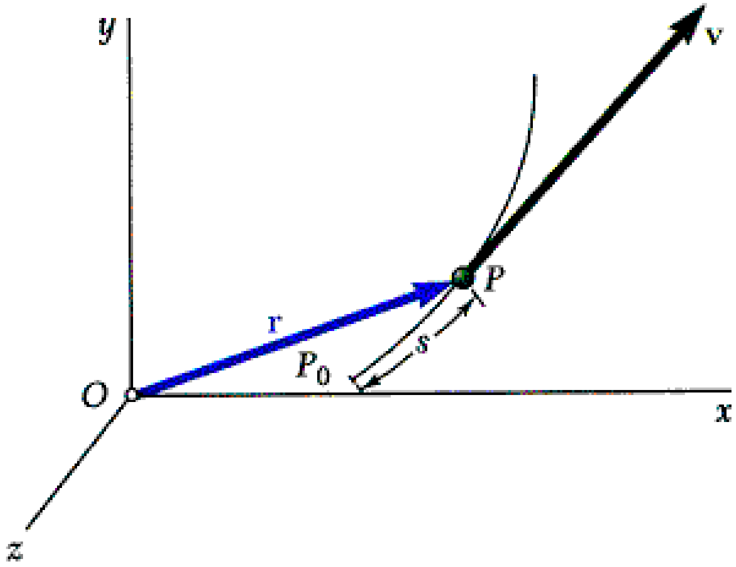
- A particle moving along a curve other than a straight line is in *curvilinear motion*.

- The *position vector* of a particle at time *t* is defined by a vector between origin *O* of a fixed reference frame and the position occupied by particle.
- Consider a particle which occupies position *P* defined by $\vec{r}$ at time *t* and *P′* defined by $\vec{r}\,'$ at $t + Dt$,



Instantaneous velocity
(vector)

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

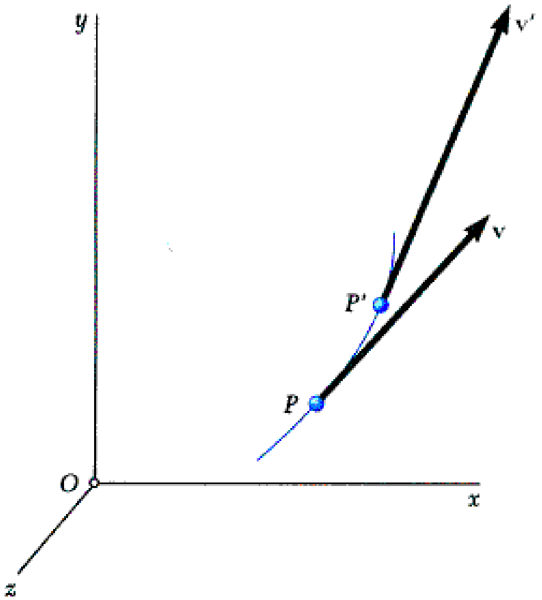


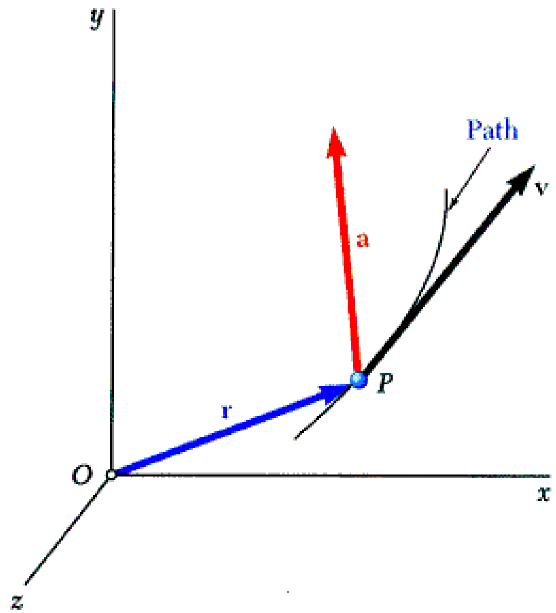
Instantaneous speed
(scalar)

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

- Consider velocity $\vec{v}$ of a particle at time *t* and velocity $\vec{v}'$ at *t + Dt*,

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} =$$ instantaneous acceleration (vector)

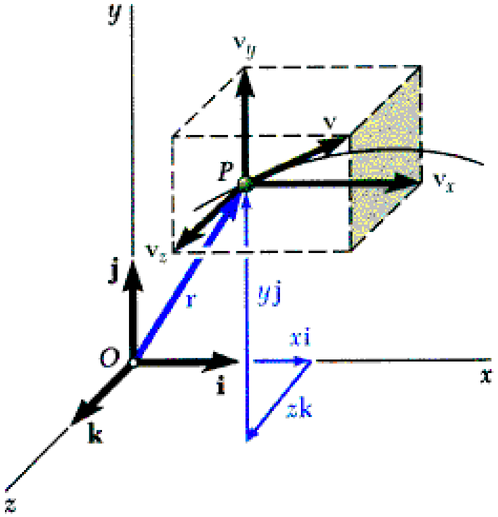




- In general, the acceleration vector is not tangent to the particle path and velocity vector.

11.4**B**

11.4C Rectangular Components of Velocity & Acceleration

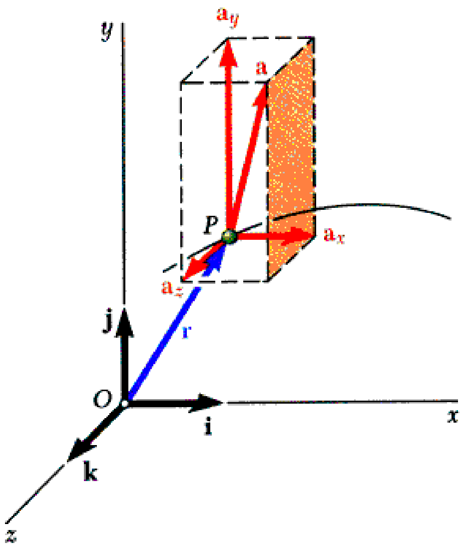


- When position vector of particle P is given by its rectangular components,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

- Velocity vector,

$$\vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}$$

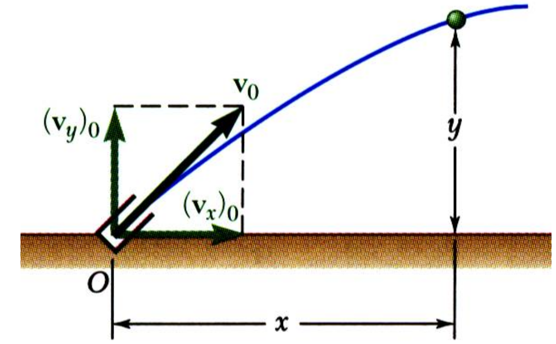


- $$= v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

- Acceleration vector,

$$\vec{a} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}$$

$$= a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$$

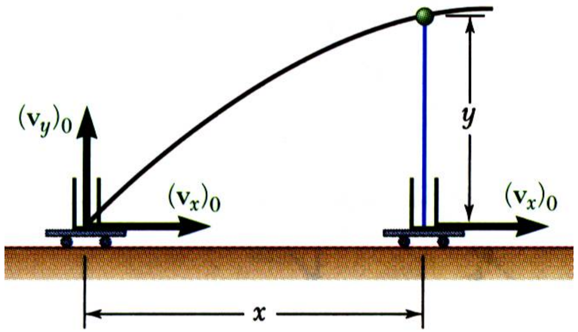


- Rectangular components particularly effective when component accelerations can be integrated independently, e.g., motion of a projectile,

$$a_x = \ddot{x} = 0 \qquad a_y = \ddot{y} = -g \qquad a_z = \ddot{z} = 0$$

with initial conditions,

$$x_0 = y_0 = z_0 = 0 \qquad (v_x)_0, (v_y)_0, (v_z)_0 = 0$$



Integrating twice yields

$$v_x = (v_x)_0 \qquad v_y = (v_y)_0 - gt \qquad v_z = 0$$

$$x = (v_x)_0 t \qquad y = (v_y)_0 y - \tfrac{1}{2} g t^2 \qquad z = 0$$

•Motion in horizontal direction is uniform.

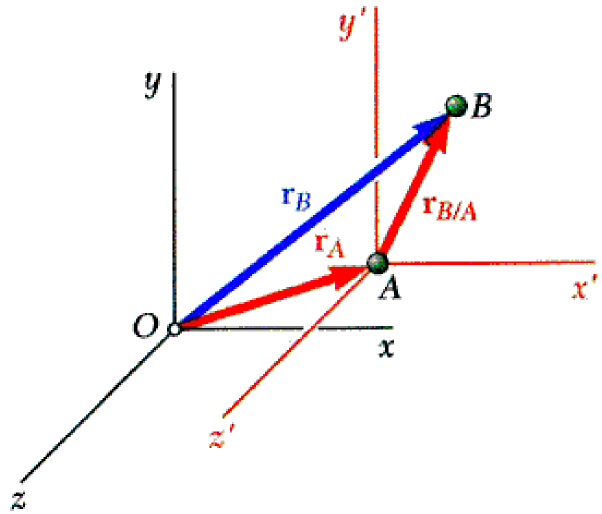
•Motion in vertical direction is uniformly accelerated.

- Motion of projectile could be replaced by two independent rectilinear motions.

11.4D Motion Relative to a Frame in Translation

It is critical for a pilot to know the relative motion of his helicopter with respect to the aircraft carrier to make a safe landing.

©Digital Vision/Getty Images RF



- Designate one frame as the *fixed frame of reference*. All other frames not rigidly attached to the fixed reference frame are *moving frames of reference*.
- Position vectors for particles *A* and *B* with respect to the fixed frame of reference *Oxyz* are
- Vector $\vec{r}_{B/A}$ joining *A* and *B* defines the position of *B* with respect to the moving frame *Ax′y′z′* and

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

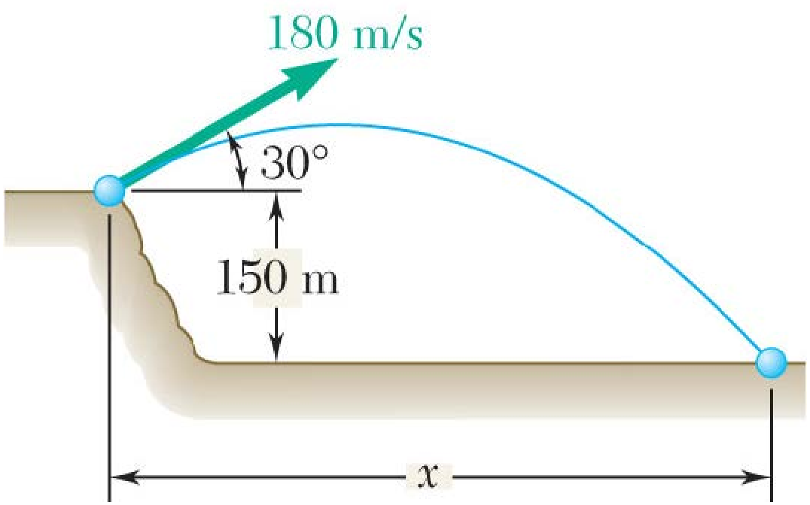
- Differentiating twice,

$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$ $\quad \vec{v}_{B/A} =$ velocity of *B* relative to *A*.

$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$ $\quad \vec{a}_{B/A} =$ acceleration of *B* relative to *A*.

- Absolute motion of *B* can be obtained by combining motion of *A* with relative motion of *B* with respect to moving reference frame attached to *A*.

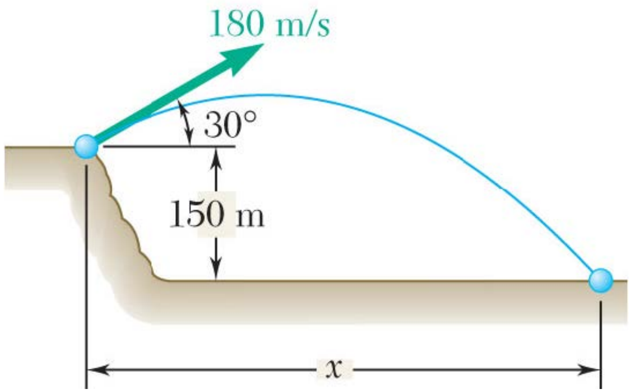
Sample Problem 11.10



A projectile is fired from the edge of a 150-m cliff with an initial velocity of 180 m/s at an angle of 30°with the horizontal. Neglecting air resistance, find (*a*) the horizontal distance from the gun to the point where the projectile strikes the ground, (*b*) the greatest elevation above the ground reached by the projectile.

STRATEGY:

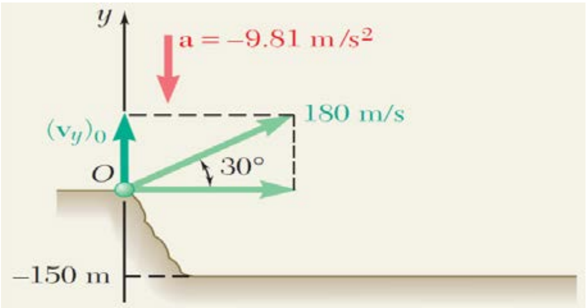
- Consider the vertical and horizontal motion separately (they are independent)
- Apply equations of motion in y-direction
- Apply equations of motion in x-direction
- Determine time *t* for projectile to hit the ground, use this to find the horizontal distance
- Maximum elevation occurs when $v_y=0$



MODELING and ANALYSIS:

Given: $(v)_o$ =180 m/s $(y)_o$ =150 m

$(a)_y$ = - 9.81 m/s^2 $(a)_x$ = 0 m/s^2



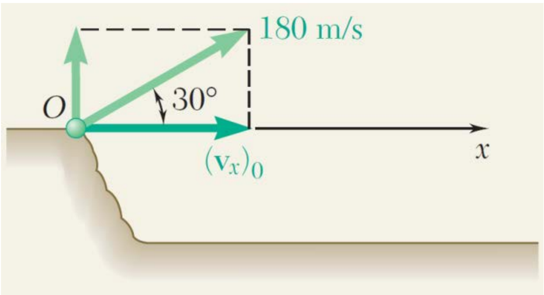
Vertical motion – uniformly accelerated: g

$$(v_y)_0 = (180 \text{ m/s}) \sin 30° = +90 \text{ m/s}$$

$$v_y = (v_y)_0 + at \qquad v_y = 90 - 9.81t \qquad (1)$$

$$y = (v_y)_0 t + \tfrac{1}{2}at^2 \qquad y = 90t - 4.90t^2 \qquad (2)$$

$$v_y^2 = (v_y)_0^2 + 2ay \qquad v_y^2 = 8100 - 19.62y \qquad (3)$$



Horizontal motion – zero acceleration:

Choose positive x to the right as shown

$$(v_x)_0 = (180 \text{ m/s}) \cos 30° = +155.9 \text{ m/s}$$

$$x = (v_x)_0 t \qquad x = 155.9t \qquad (4)$$

MODELING and ANALYSIS:

a. Horizontal distance

Projectile strikes the ground at: $y = -150 \text{ m}$

Substitute into equation (1) above

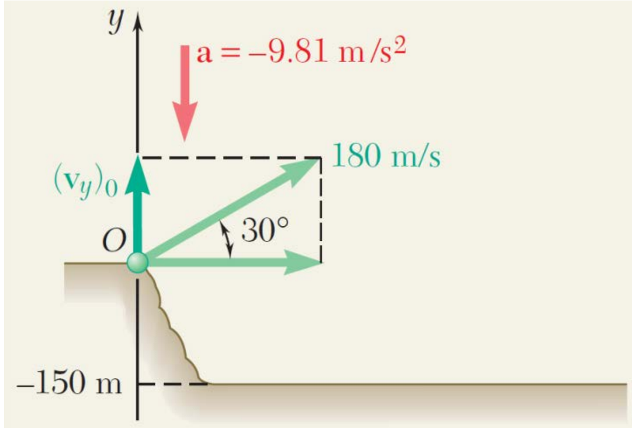
$$-150 = 90t - 4.90t^2$$

Solving for *t*, we take the positive root

$$t^2 - 18.37t - 30.6 = 0 \qquad t = 19.91 \text{ s}$$

Substitute t into equation (4)

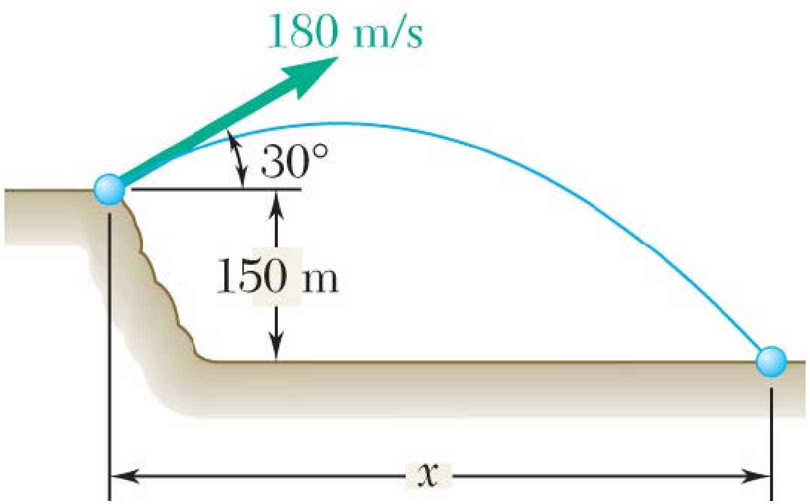
$$x = 155.9(19.91) \qquad x = 3100 \text{ m}$$



b. Maximum elevation occurs when $v_y = 0$

$$0 = 8100 - 19.62y \qquad y = 413 \text{ m}$$

Maximum elevation above the ground = $150 \text{ m} + 413 \text{ m} = 563 \text{ m}$

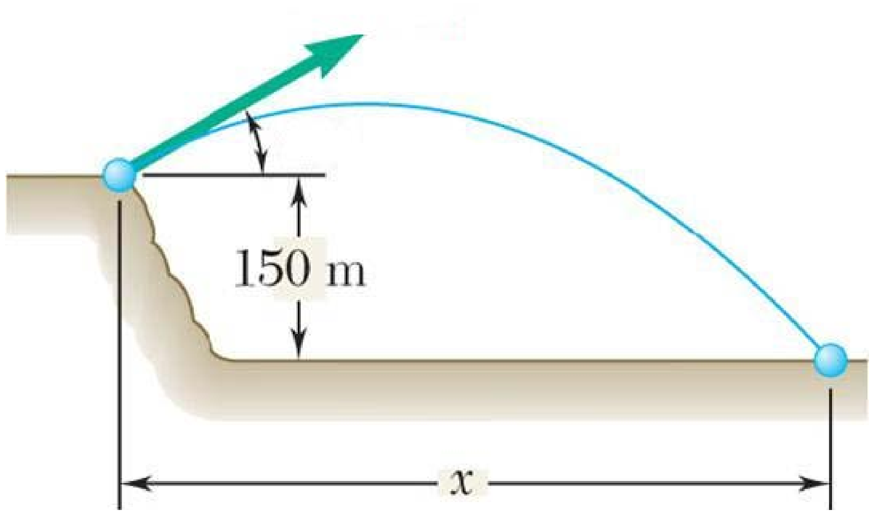


REFLECT and THINK:

Because there is no air resistance, you can treat the vertical and horizontal motions separately and can immediately write down the algebraic equations of motion. If you did want to include air resistance, you must know the acceleration as a function of speed (you will see how to derive this in Chapter 12), and then you need to use the basic kinematic relationships, separate variables, and integrate.

Concept Quiz

If you fire a projectile from 150 meters above the ground (see Ex Problem 11.7), what launch angle will give you the greatest horizontal distance *x*?



a) A launch angle of 45°

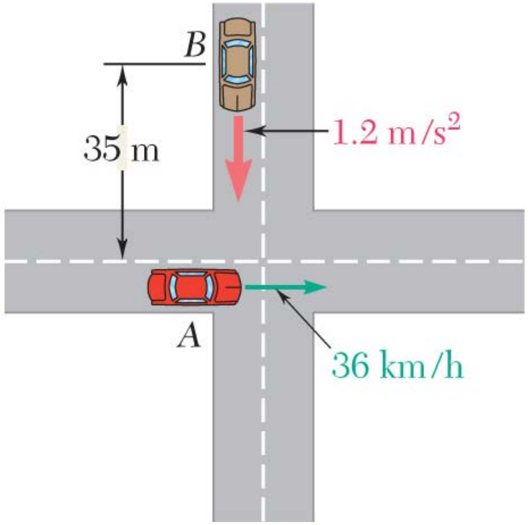
b) A launch angle less than 45°

c) A launch angle greater than 45°

d) It depends on the launch velocity

answer: (b)

Sample Problem 11.14

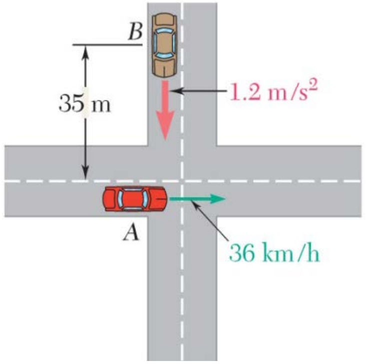


Automobile *A* is traveling east at the constant speed of 36 km/h. As automobile *A* crosses the intersection shown, automobile *B* starts from rest 35 m north of the intersection and moves south with a constant acceleration of 1.2 m/s^2. Determine the position, velocity, and acceleration of *B* relative to *A* 5 *s* after *A* crosses the intersection.

STRATEGY:

- Define inertial axes for the system
- Determine the position, speed, and acceleration of car A at t = 5 s
- Determine the position, speed, and acceleration of car B at t = 5 s
- Using vectors (Eqs 11.30, 11.32, and 11.33) or a graphical approach, determine

the relative position, velocity, and acceleration



MODELING and ANALYSIS:

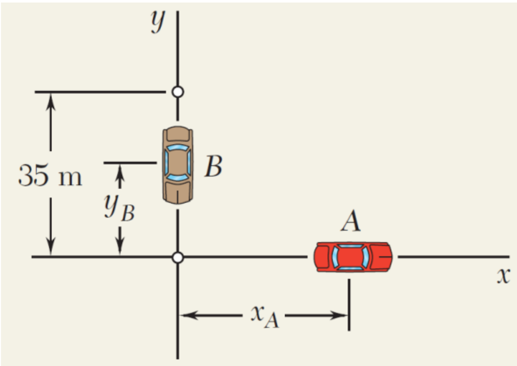
- Define axes along the road

Given: v_A=36 km/h, a_A= 0, $(x_A)_0$ = 0

$(v_B)_0$= 0, a_B= - 1.2 m/s^2, $(y_A)_0$ = 35 m

Determine motion of Automobile A:

$$v_A = \left(36\frac{\text{km}}{\text{h}}\right)\left(\frac{1000\text{ m}}{1\text{ km}}\right)\left(\frac{1\text{ h}}{3600\text{ s}}\right) = 10\text{ m/s}$$



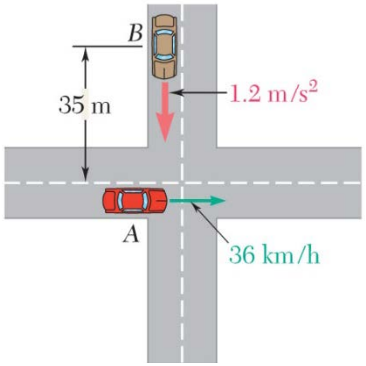
We have uniform motion for A so:

$$\begin{aligned} a_A &= 0 \\ v_A &= +10\text{ m/s} \\ x_A &= (x_A)_0 + v_A t = 0 + 10t \end{aligned}$$

At t = 5 s

$$\begin{aligned} a_A &= 0 \\ v_A &= +10\text{ m/s} \\ x_A &= +(10\text{ m/s})(5\text{ s}) = +50\text{ m} \end{aligned}$$

$$\begin{aligned} \mathbf{a}_A &= 0 \\ \mathbf{v}_A &= 10\text{ m/s} \rightarrow \\ \mathbf{r}_A &= 50\text{ m} \rightarrow \end{aligned}$$

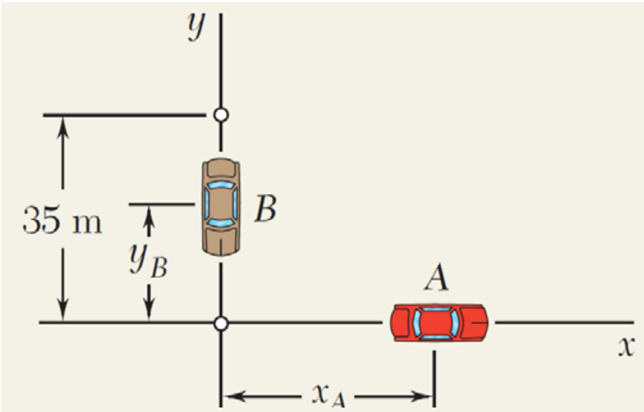


MODELING and ANALYSIS:

Determine motion of Automobile B:

We have uniform acceleration for B so:

$$a_B = -1.2 \text{ m/s}^2$$
$$v_B = (v_B)_0 + at = 0 - 1.2\,t$$
$$y_B = (y_B)_0 + (v_B)_0 t + \tfrac{1}{2}a_B t^2 = 35 + 0 - \tfrac{1}{2}(1.2)t^2$$



$$\mathbf{a}_B = 1.2 \text{ m/s}^2 \downarrow$$
$$\mathbf{v}_B = 6 \text{ m/s} \downarrow$$
$$\mathbf{r}_B = 20 \text{ m} \uparrow$$

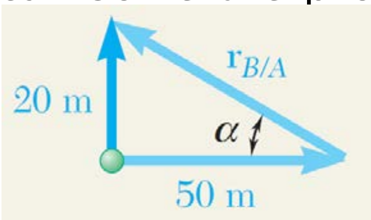
At t = 5 s

$$a_B = -1.2 \text{ m/s}^2$$
$$v_B = -(1.2 \text{ m/s}^2)(5 \text{ s}) = -6 \text{ m/s}$$
$$y_B = 35 - \tfrac{1}{2}(1.2 \text{ m/s}^2)(5 \text{ s})^2 = +20 \text{ m}$$

$$\mathbf{a}_A = 0 \qquad \mathbf{v}_A = 10\text{ m/s} \rightarrow \qquad \mathbf{r}_A = 50\text{ m} \rightarrow$$

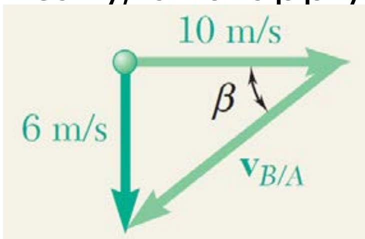
$$\mathbf{a}_B = 1.2\text{ m/s}^2 \downarrow \qquad \mathbf{v}_B = 6\text{ m/s} \downarrow \qquad \mathbf{r}_B = 20\text{ m} \uparrow$$

We can solve the problems geometrically, and apply the arctangent relationship:



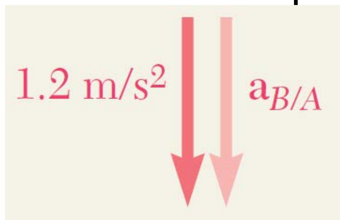
$\alpha = 21.8°$ $\quad r_{B/A} = 53.9\text{ m}$

$\mathbf{r}_{B/A} = 53.9\text{ m} \measuredangle 21.8°$



$\beta = 31.0°$ $\quad v_{B/A} = 11.66\text{ m/s}$

$\mathbf{v}_{B/A} = 11.66\text{ m/s} \measuredangle 31.0°$



$\mathbf{a}_{B/A} = 1.2\text{ m/s}^2 \downarrow$

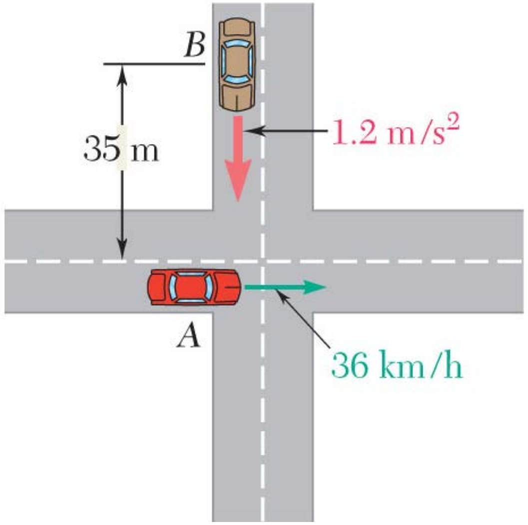
Or we can solve the problems using vectors to obtain equivalent results:

$$\mathbf{r_B} = \mathbf{r_A} + \mathbf{r_{B/A}}$$
$$20\mathbf{j} = 50\mathbf{i} + \mathbf{r_{B/A}}$$
$$\mathbf{r_{B/A}} = 20\mathbf{j} - 50\mathbf{i} \ \text{(m)}$$

$$\mathbf{v_B} = \mathbf{v_A} + \mathbf{v_{B/A}}$$
$$-6\mathbf{j} = 10\mathbf{i} + \mathbf{v_{B/A}}$$
$$\mathbf{v_{B/A}} = -6\mathbf{j} - 10\mathbf{i} \ \text{(m/s)}$$

$$\mathbf{a_B} = \mathbf{a_A} + \mathbf{a_{B/A}}$$
$$-1.2\mathbf{j} = 0\mathbf{i} + \mathbf{a_{B/A}}$$
$$\mathbf{a_{B/A}} = -1.2\mathbf{j} \ \text{(m/s}^2\text{)}$$

Physically, a rider in car A would "see" car B travelling south and west.



REFLECT and THINK:

Note that the relative position and velocity of B relative to A change with time; the values given here are only for the moment t =5 s.

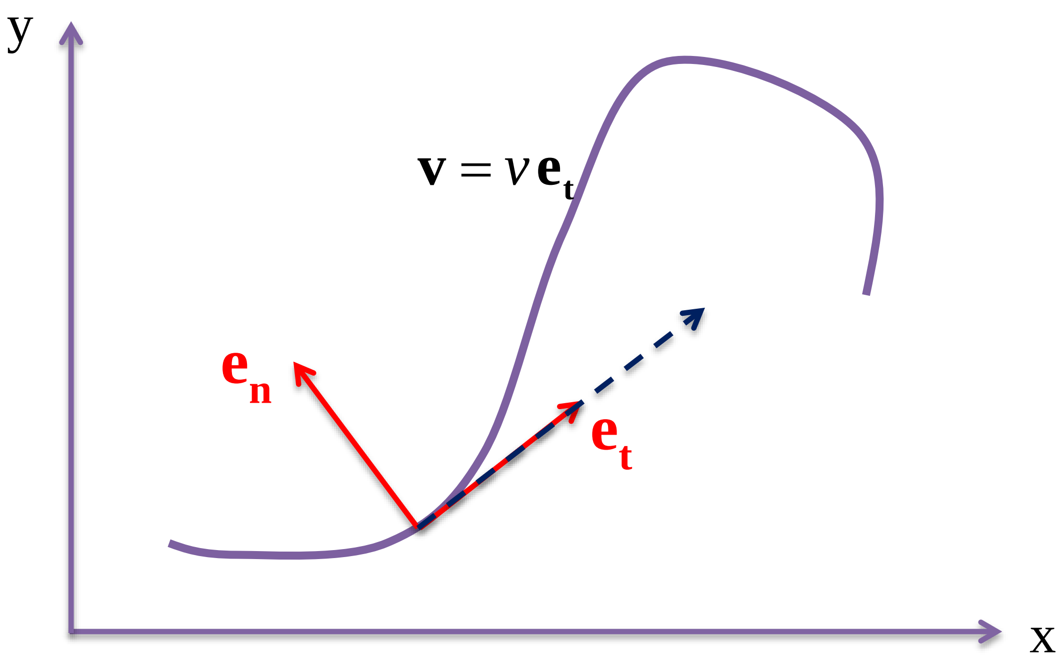
11.5 NON-RECTANGULAR COMPONENT

11.5A Tangential and Normal Components

If we have an idea of the path of a vehicle or object, it is often convenient to analyze the motion using tangential and normal components (sometimes called *path* coordinates).

©Russell Illig/Getty Images RF

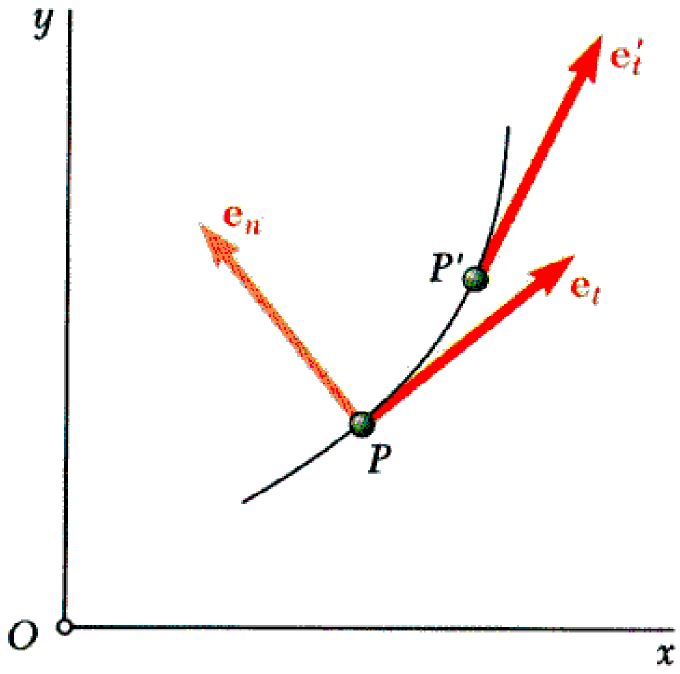
©Belinda Images/SuperStock



ρ= the instantaneous radius of curvature

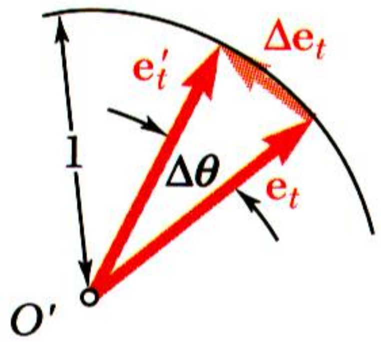
$$\mathbf{a} = \frac{dv}{dt}\mathbf{e}_t + \frac{v^2}{\rho}\mathbf{e}_n$$

- The tangential direction ($\mathbf{e}_t$) is tangent to the path of the particle. This velocity vector of a particle is in this direction
- The normal direction ($\mathbf{e}_n$) is perpendicular to $\mathbf{e}_t$ and points towards the inside of the curve.
- The acceleration can have components in both the $\mathbf{e}_n$ and $\mathbf{e}_t$ directions



- To derive the acceleration vector in tangential and normal components, define the motion of a particle as shown in the figure.

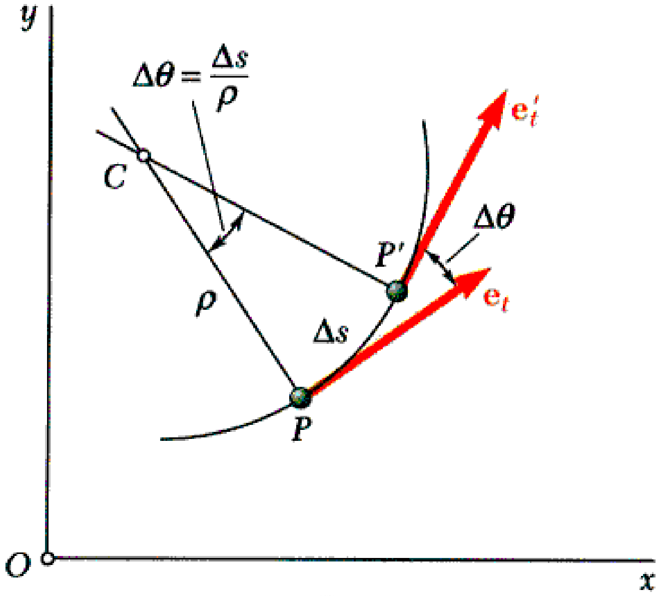
- $\vec{e}_t$ and $\vec{e}'_t$ are tangential unit vectors for the particle path at P and P'. When drawn with respect to the same origin, $\Delta\vec{e}_t = \vec{e}'_t - \vec{e}_t$ and $\Delta\theta$ is the angle between them.



$$\Delta e_t = 2\sin(\Delta\theta/2)$$

$$\lim_{\Delta\theta\to 0}\frac{\Delta\vec{e}_t}{\Delta\theta} = \lim_{\Delta\theta\to 0}\frac{\sin(\Delta\theta/2)}{\Delta\theta/2}\vec{e}_n = \vec{e}_n$$

$$\vec{e}_n = \frac{d\vec{e}_t}{d\theta}$$



- With the velocity vector expressed as $\vec{v} = v\vec{e}_t$

the particle acceleration may be written as

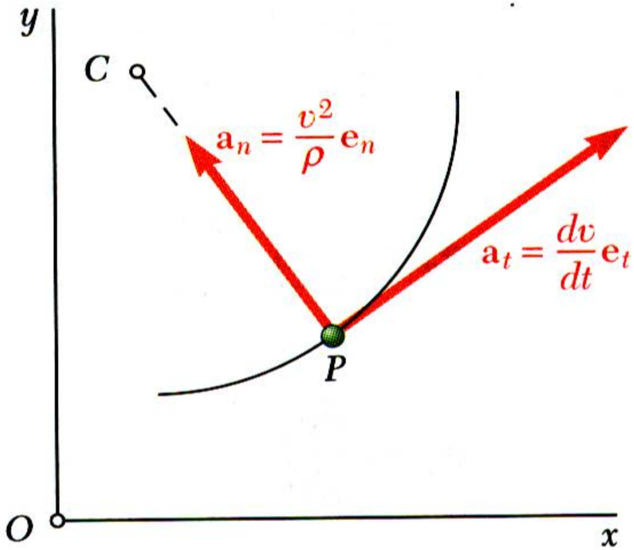
$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}\vec{e}_t + v\frac{d\vec{e}}{dt} = \frac{dv}{dt}\vec{e}_t + v\frac{d\vec{e}}{d\theta}\frac{d\theta}{ds}\frac{ds}{dt}$$

but

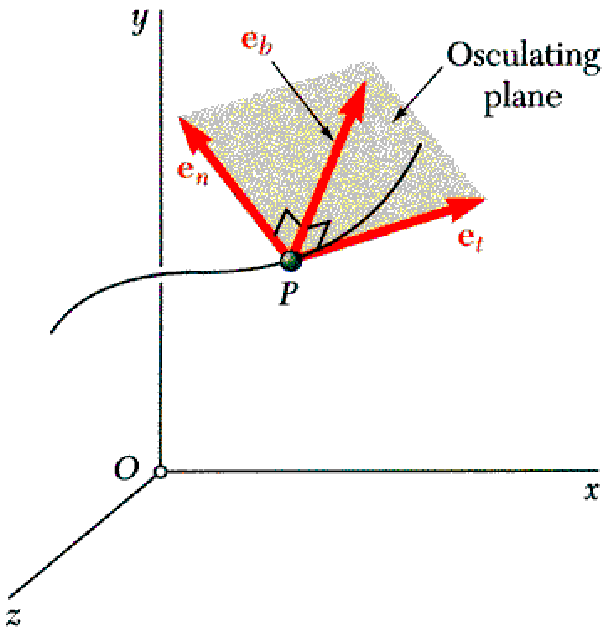
$$\frac{d\vec{e}_t}{d\theta} = \vec{e}_n \qquad \rho\, d\theta = ds \qquad \frac{ds}{dt} = v$$

After substituting,

$$\vec{a} = \frac{dv}{dt}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n \qquad a_t = \frac{dv}{dt} \quad a_n = \frac{v^2}{\rho}$$



- The tangential component of acceleration reflects change of speed and the normal component reflects change of direction.
- The tangential component may be positive or negative. Normal component always points toward center of path curvature.



- Relations for tangential and normal acceleration also apply for particle moving along a space curve.

$$\vec{a} = \frac{dv}{dt}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n \qquad a_t = \frac{dv}{dt} \quad a_n = \frac{v^2}{\rho}$$

- The plane containing tangential and normal unit vectors is called the *osculating plane*.

- The normal to the osculating plane is found from

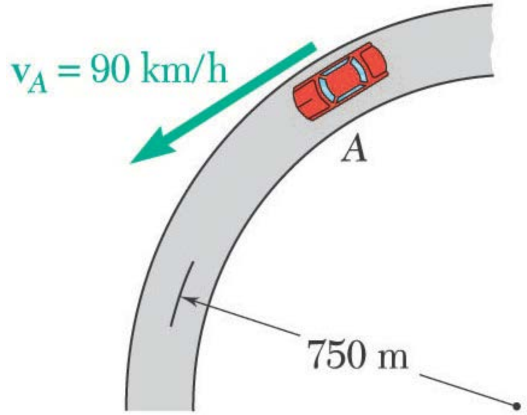
$$\vec{e}_b = \vec{e}_t \times \vec{e}_n$$

$$\vec{e}_n = principal\ normal$$

$$\vec{e}_b = binormal$$

- Acceleration has no component along the binormal.

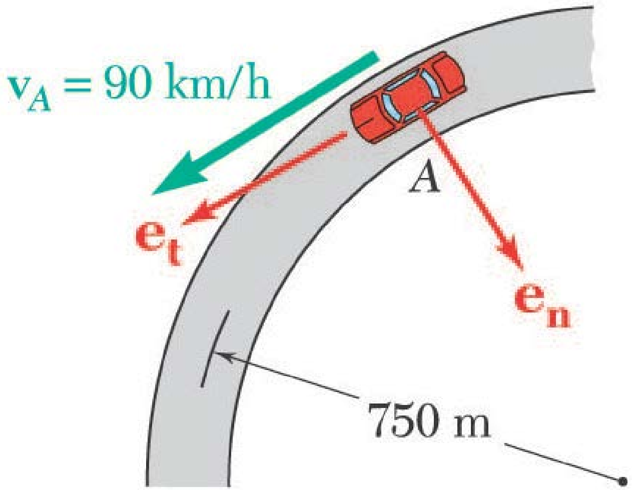
Sample Problem 11.16



A motorist is traveling on a curved section of highway of radius 750 m at the speed of 90 km/h. The motorist suddenly applies the brakes, causing the automobile to slow down at a constant rate. Knowing that after 8 s the speed has been reduced to 72 km/h, determine the acceleration of the automobile immediately after the brakes have been applied.

STRATEGY:

- Define your coordinate system
- Calculate the tangential velocity and tangential acceleration
- Calculate the normal acceleration
- Determine overall acceleration magnitude after the brakes have been applied



MODELING and ANALYSIS:

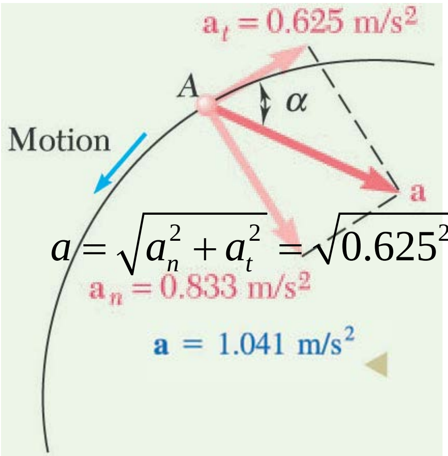
- Define your coordinate system
- Determine velocity and acceleration in the tangential direction

$$90\ \text{km/h} = \left(90\ \frac{\text{km}}{\text{h}}\right)\left(\frac{1000\ \text{m}}{1\ \text{km}}\right)\left(\frac{1\ \text{h}}{3600\ \text{s}}\right) = 25\ \text{m/s}$$

$$72\ \text{km/h} = 20\ \text{m/s}$$

- The deceleration constant, therefore

$$a_t = \text{average } a_t = \frac{\Delta v}{\Delta t} = \frac{20\ \text{m/s} - 25\ \text{m/s}}{8\ \text{s}} = -0.625\ \text{m/s}^2$$



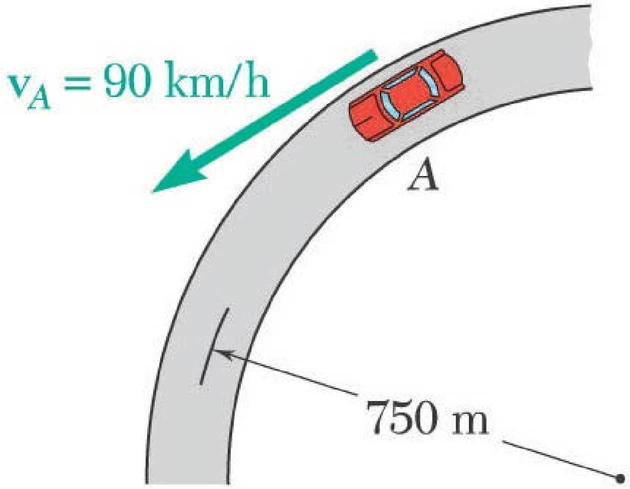
$$a = \sqrt{a_n^2 + a_t^2} = \sqrt{0.625^2 + 0.833^2}$$

- Immediately after the brakes are applied, the speed is still 25 m/s

$$a_n = \frac{v^2}{\rho} = \frac{(25\ \text{m/s})^2}{750\ \text{m}} = 0.833\ \text{m/s}^2$$

$$\tan\alpha = \frac{a_n}{a_t} = \frac{0.833\ \text{m/s}^2}{0.625\ \text{m/s}^2}$$

$$\alpha = 53.1°$$



REFLECT and THINK:

The tangential component of acceleration is opposite the direction of motion, and the normal component of acceleration points to the center of curvature, which is what you would expect for slowing down on a curved path. Attempting to do the problem in Cartesian coordinates is quite difficult.

11.5B Radial and Transverse Components

The foot pedal on an elliptical machine rotates about and extends from a central pivot point. This motion can be analyzed using radial and transverse components

Fire truck ladders can rotate as well as extend; the motion of the end of the ladder can be analyzed using radial and transverse components.

©Syracuse Newspapers/M Greenlar/The Image Works

©Fuse/Getty Images RF

- The position of a particle P is expressed as a distance r from the origin O to P – this defines the radial direction $\mathbf{e_r}$. The transverse direction $\vec{e}_\theta$ is perpendicular to $\mathbf{e_r}$

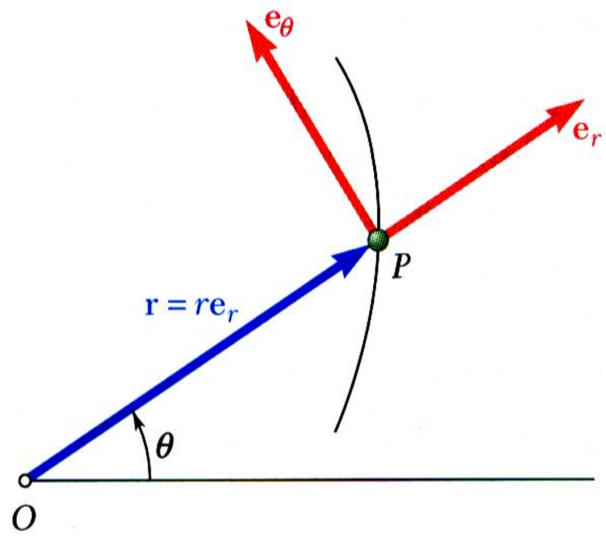
$$\vec{r} = r\vec{e}_r$$

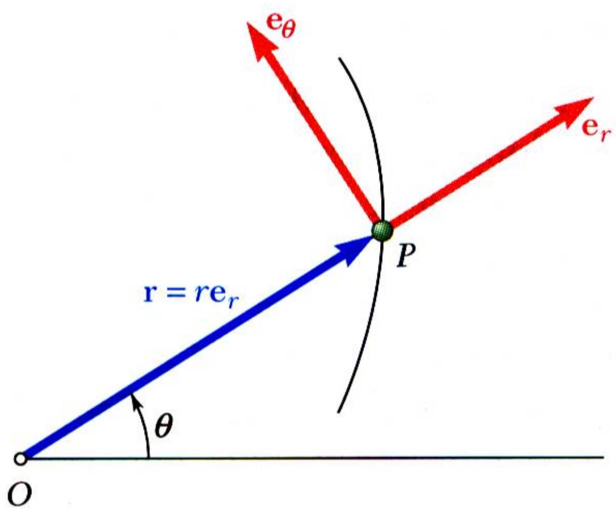
- The particle velocity vector is

$$\vec{v} = \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta$$

- The particle acceleration vector is

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta$$





$$\vec{r} = r\vec{e}_r$$

$$\frac{d\vec{e}_r}{d\theta} = \vec{e}_\theta \qquad \frac{d\vec{e}_\theta}{d\theta} = -\vec{e}_r$$

$$\frac{d\vec{e}_r}{dt} = \frac{d\vec{e}_r}{d\theta}\frac{d\theta}{dt} = \vec{e}_\theta \frac{d\theta}{dt}$$

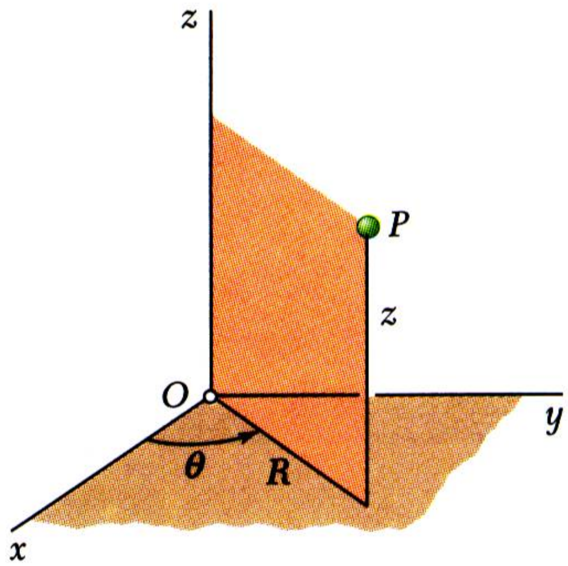
$$\frac{d\vec{e}_\theta}{dt} = \frac{d\vec{e}_\theta}{d\theta}\frac{d\theta}{dt} = -\vec{e}_r \frac{d\theta}{dt}$$

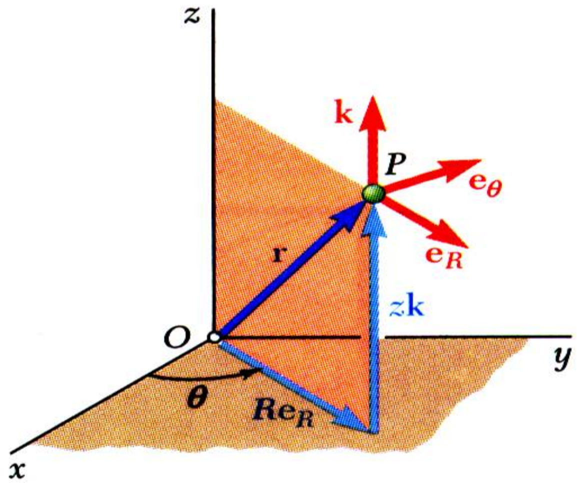
- We can derive the velocity and acceleration relationships by recognizing that the unit vectors change direction.
- The particle velocity vector is

$$\vec{v} = \frac{d}{dt}(r\vec{e}_r) = \frac{dr}{dt}\vec{e}_r + r\frac{d\vec{e}_r}{dt} = \frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta$$
$$= \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta$$

Similarly, the particle acceleration vector is

$$\vec{a} = \frac{d}{dt}\left(\frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta\right)$$
$$= \frac{d^2r}{dt^2}\vec{e}_r + \frac{dr}{dt}\frac{d\vec{e}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\vec{e}_\theta + r\frac{d^2\theta}{dt^2}\vec{e}_\theta + r\frac{d\theta}{dt}\frac{d\vec{e}_\theta}{dt}$$
$$= \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta$$





- When particle position is given in cylindrical coordinates, it is convenient to express the velocity and acceleration vectors using the unit vectors $\vec{e}_R, \vec{e}_\theta$, and $\vec{k}$.
- Position vector,

$$\vec{r} = R\vec{e}_R + z\vec{k}$$

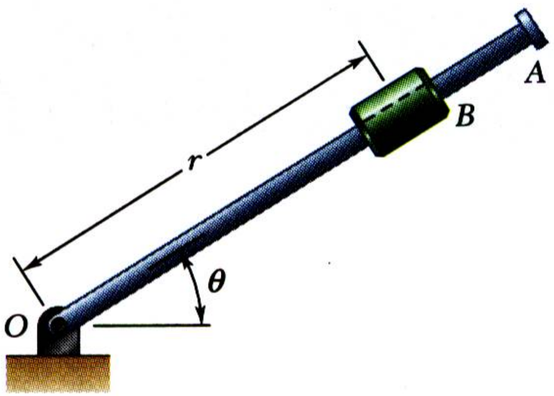
- Velocity vector,

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{R}\vec{e}_R + R\dot{\theta}\vec{e}_\theta + \dot{z}\vec{k}$$

Acceleration vector,

$$\vec{a} = \frac{d\vec{v}}{dt} = \left(\ddot{R} - R\dot{\theta}^2\right)\vec{e}_R + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\vec{e}_\theta + \ddot{z}\vec{k}$$

Sample Problem 11.18



Rotation of the arm about O is defined by $q = 0.15t^2$ where q is in radians and t in seconds. Collar B slides along the arm such that $r = 0.9 - 0.12t^2$ where r is in meters.

After the arm has rotated through 30°, determine *(a)* the total velocity of the collar, *(b)* the total acceleration of the collar, and *(c)* the relative acceleration of the collar with respect to the arm.

STRATEGY:

- Evaluate time t for q = 30°.
- Evaluate radial and angular positions, and first and second derivatives at time t.
- Calculate velocity and acceleration in cylindrical coordinates.
- Evaluate acceleration with respect to arm.

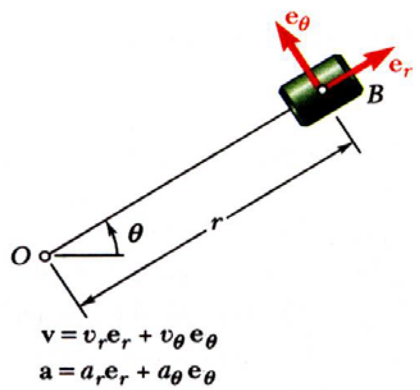
MODELING and ANALYSIS

- Evaluate time t for q = 30°.

$$\theta = 0.15t^2$$
$$= 30° = 0.524\,\mathrm{rad} \qquad t = 1.869\,\mathrm{s}$$

- Evaluate radial and angular positions, and first and second derivatives at time t.

$$r = 0.9 - 0.12t^2 = 0.481\,\mathrm{m}$$
$$\dot{r} = -0.24t = -0.449\,\mathrm{m/s}$$
$$\ddot{r} = -0.24\,\mathrm{m/s^2}$$
$$\theta = 0.15t^2 = 0.524\,\mathrm{rad}$$
$$\dot{\theta} = 0.30t = 0.561\,\mathrm{rad/s}$$
$$\ddot{\theta} = 0.30\,\mathrm{rad/s^2}$$



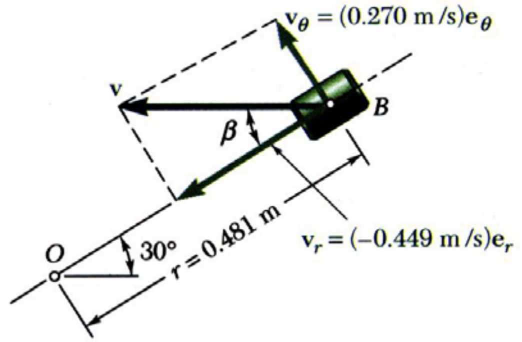
- Calculate velocity and acceleration.

$$v_r = \dot{r} = -0.449\,\mathrm{m/s}$$

$$v_\theta = r\dot{\theta} = (0.481\,\mathrm{m})(0.561\,\mathrm{rad/s}) = 0.270\,\mathrm{m/s}$$

$$v = \sqrt{v_r^2 + v_\theta^2} \qquad \beta = \tan^{-1}\frac{v_\theta}{v_r}$$

$$\boxed{v = 0.524\,\mathrm{m/s} \qquad \beta = 31.0^\circ}$$



$$a_r = \ddot{r} - r\dot{\theta}^2$$

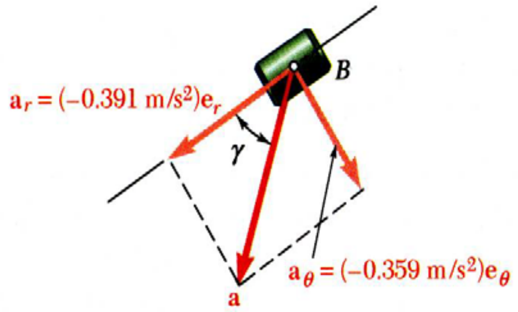
$$= -0.240\,\mathrm{m/s^2} - (0.481\,\mathrm{m})(0.561\,\mathrm{rad/s})^2$$

$$= -0.391\,\mathrm{m/s^2}$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

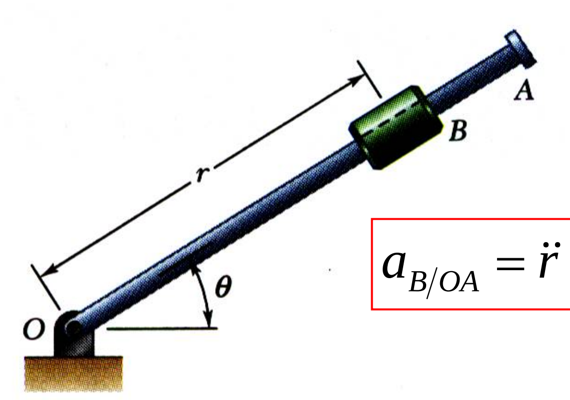
$$= (0.481\,\mathrm{m})\left(0.3\,\mathrm{rad/s^2}\right) + 2(-0.449\,\mathrm{m/s})(0.561\,\mathrm{rad/s})$$

$$= -0.359\,\mathrm{m/s^2}$$



$$a = \sqrt{a_r^2 + a_\theta^2} \qquad \gamma = \tan^{-1}\frac{a_\theta}{a_r}$$

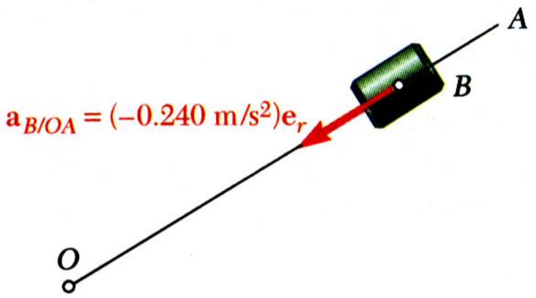
$$\boxed{a = 0.531\,\mathrm{m/s} \qquad \gamma = 42.6^\circ}$$



- Evaluate acceleration with respect to arm.

  Motion of collar with respect to arm is rectilinear and defined by coordinate *r*.

$$\boxed{a_{B/OA} = \ddot{r} = -0.240\,\mathrm{m/s^2}}$$



REFLECT and THINK:

You should consider polar coordinates for any kind of rotational motion. They turn this problem into a straightforward solution, whereas any other coordinate system would make this problem much more difficult. One way to make this problem harder would be to ask you to find the radius of curvature in addition to the velocity and acceleration. To do this, you would have to find the normal component of the acceleration; that is, the component of acceleration that is perpendicular to the tangential direction defined by the velocity vector.