Chapter 13. Kinetics of Particles: Energy and Momentum Methods

Energy and Momentum Methods

The potential energy of the roller coaster car is converted into kinetic energy as it descends the track.

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Impact tests are often analyzed by using momentum methods

Introduction

- **Previously, problems dealing with the motion of particles were solved through the fundamental equation of motion,** $\Sigma\vec{F} = m\vec{a}$**.**

- **The current chapter introduces two additional methods of analysis.**

- ***Method of work and energy*: directly relates force, mass, velocity and displacement.**

- ***Method of impulse and momentum*: directly relates force, mass, velocity, and time.**

Approaches to Kinetics Problems

Forces and Accelerations

Newton's Second Law

$$\sum \vec{F} = m\vec{a}_G$$

Velocities and Displacements

Work-Energy

$$T_1 + U_{1\to 2} = T_2$$

Velocities and Time

Impulse-Momentum

$$m\vec{v}_1 + \int_{t_1}^{t_2} \vec{F}\, dt = m\vec{v}_2$$

13.1A Work of a Force

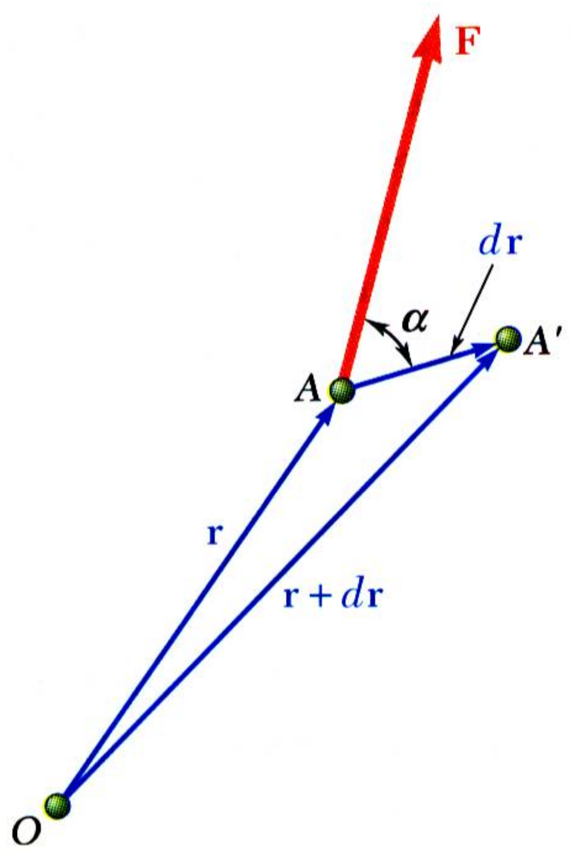


Fig.13.1

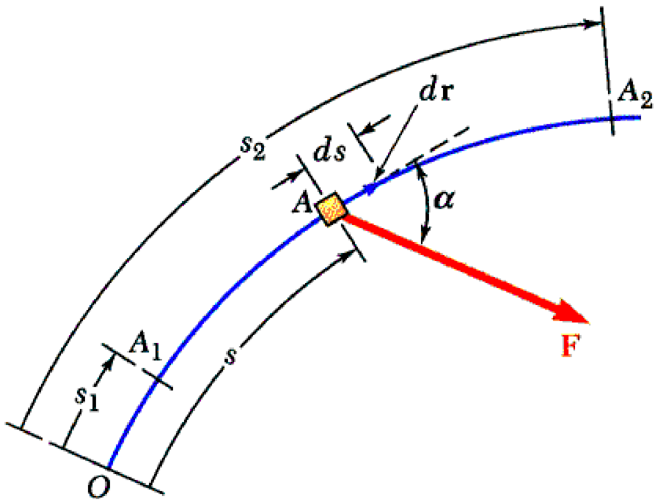
- **Differential vector is the *particle displacement*.**
- ***Work of the force* is**

$$dU = \vec{F} \bullet d\vec{r}$$
$$= F\,ds\ \cos\alpha$$
$$= F_x dx + F_y dy + F_z dz$$

- **Work is a *scalar* quantity, i.e., it has magnitude and sign but not direction.**
- **Dimensions of work are** $\text{length} \times \text{force.}$ **Units are**

$1\,\mathrm{J}\,(joule) = (1\,\mathrm{N})(1\,\mathrm{m}) \qquad 1\,\mathrm{ft}\cdot\mathrm{lb} = 1.356\,\mathrm{J}$

Work of a Force



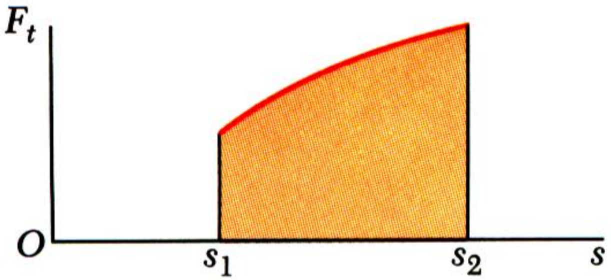


Fig.13.2

- **Work of a force during a finite displacement,**

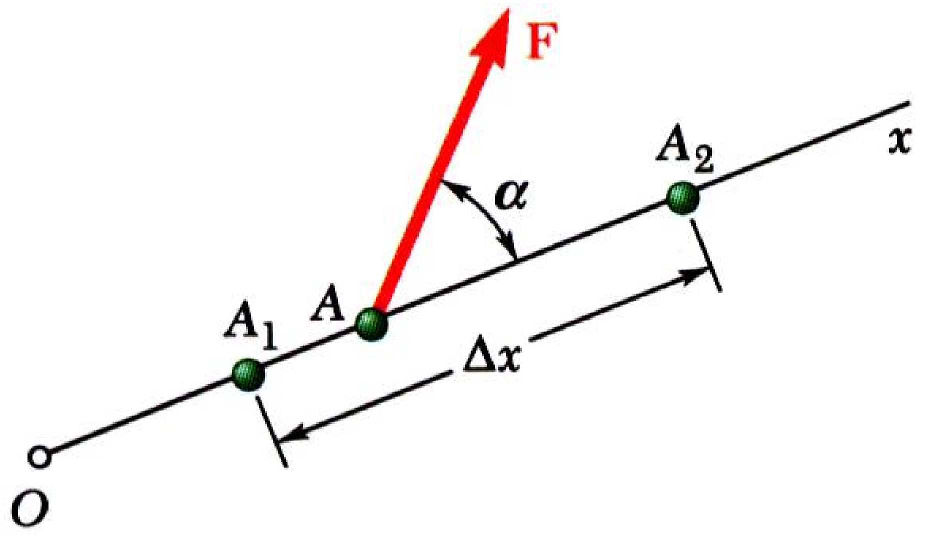
$$U_{1\to 2} = \int_{A_1}^{A_2} \vec{F} \bullet d\vec{r}$$

$$= \int_{s_1}^{s_2} (F\cos\alpha)ds = \int_{s_1}^{s_2} F_t \ ds$$

$$= \int_{A_1}^{A_2} \left(F_x dx + F_y dy + F_z dz\right)$$

- **Work is represented by the area under the curve of F_t plotted against s.**
- **F_t is the force in the direction of the displacement ds**

What is the work of a constant force in rectilinear motion?



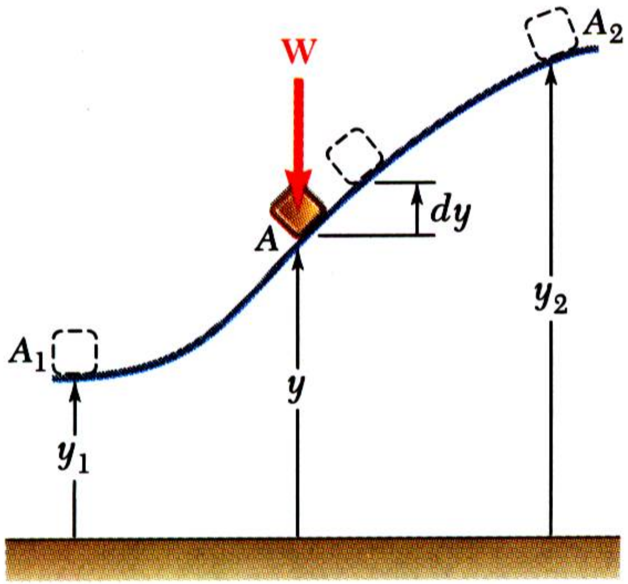
a) $U_{1\to 2} = F\ \Delta x$

b) $U_{1\to 2} = \left(F\ \cos\alpha\right)\Delta x$

c) $U_{1\to 2} = \left(F\ \sin\alpha\right)\Delta x$

d) $U_{1\to 2} = 0$

answer b)



Work of the force of gravity,

$$dU = F_x dx + F_y dy + F_z dz$$
$$= -W\ dy$$

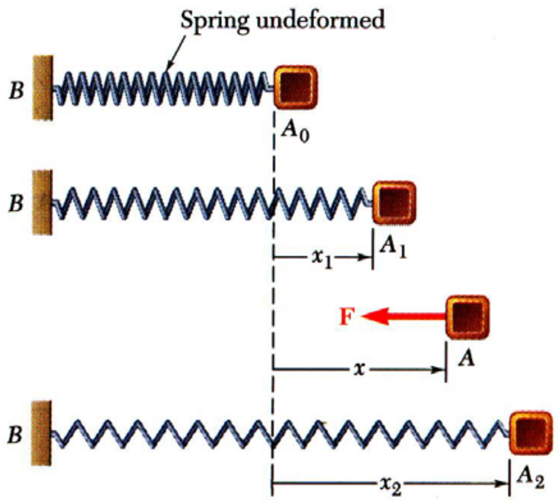
$$U_{1\to 2} = -\int_{y_1}^{y_2} W\ dy$$
$$= -W(y_2 - y_1) = -W\ \Delta y$$

- **Work *of the weight* is equal to product of weight *W* and vertical displacement Δy.**

- **In the figure above, when is the work done by the weight positive?**

a) Moving from y_1 to y_2 **b) Moving from y_2 to y_1** **c) Never**

answer b)



Magnitude of the force exerted by a spring is proportional to deflection,

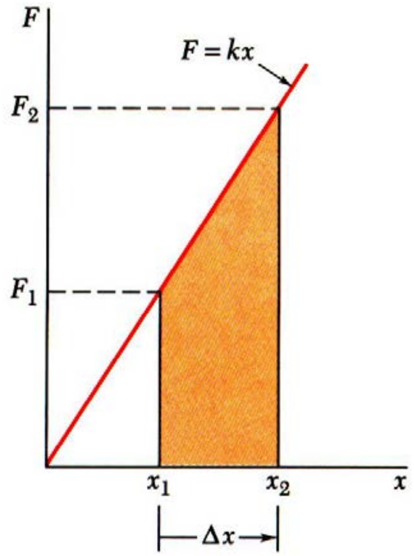
$$F = kx$$

$$k = \text{spring constant } (\text{N/m or lb/in.})$$

Work ***of the force exerted by spring,***

$$dU = -F\,dx = -kx\,dx$$

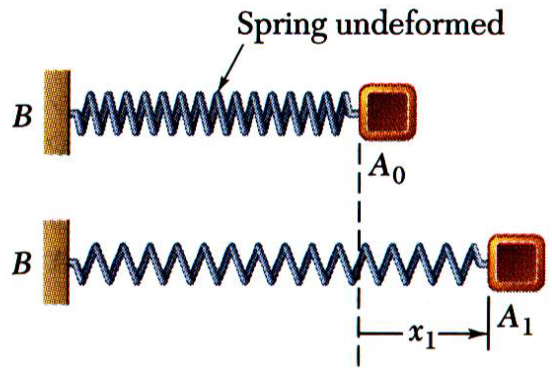
$$U_{1\to 2} = -\int_{x_1}^{x_2} kx\,dx = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2$$

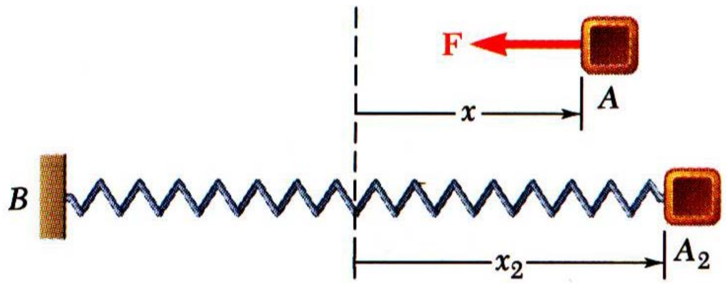


- Work *of the force exerted by spring* is positive when $x_2 < x_1$, i.e., when the spring is returning to its undeformed position.
- Work of the force exerted by the spring is equal to negative of area under curve of *F* plotted against *x*,

$$U_{1\to 2} = -\tfrac{1}{2}\left(F_1 + F_2\right)\Delta x$$

Work of a Force





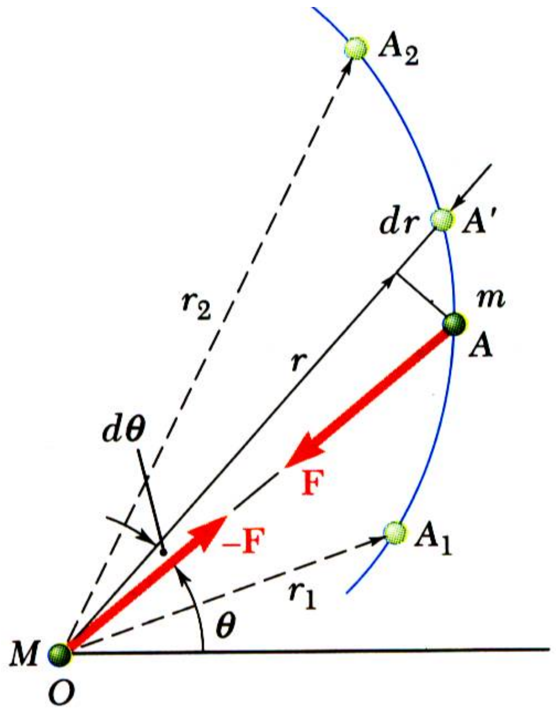
<u>As the block moves from A_0 to A_1, is the work positive or negative?</u>

Positive ?　Negative ?

<u>As the block moves from A_2 to A_o, is the work positive or negative?</u>

Positive ?　　　　　　Negative ?

answer ; negative, positive



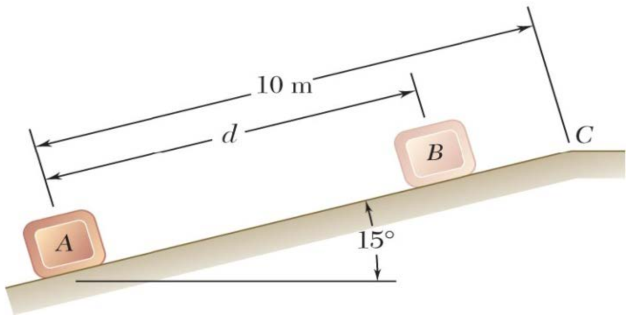
Work of a gravitational force (assume particle *M* occupies fixed position *O* while particle *m* follows path shown),

$$dU = -F dr = -G\frac{Mm}{r^2}dr$$

$$U_{1\to 2} = -\int_{r_1}^{r_2} G\frac{Mm}{r^2}dr = G\frac{Mm}{r_2} - G\frac{Mm}{r_1}$$

Does the normal force do work as the block slides from B to A?

Yes No



Does the weight do work as the block slides from B to A?

Yes No

Positive or Negative work?

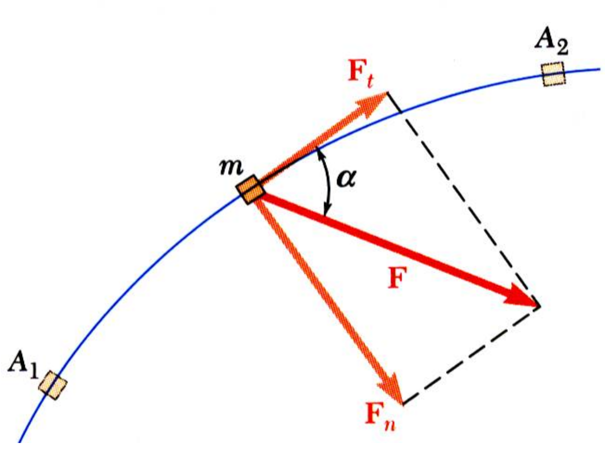
Answer ; No, Yes, Positive

Work of a Force

Forces which *do not* do work *(ds* = 0 or $\cos\alpha$ = 0):

- **Reaction at frictionless pin supporting rotating body,**
- **Reaction at frictionless surface when body in contact moves along surface,**
- **Reaction at a roller moving along its track, and**
- **Weight of a body when its center of gravity moves horizontally.**

13.1B Principle of Work & Energy



Consider a particle of mass *m* acted upon by force

$$F_t = ma_t = m\frac{dv}{dt}$$

$$= m\frac{dv}{ds}\frac{ds}{dt} = mv\frac{dv}{ds}$$

$$F_t\, ds = mv\, dv$$

- **Integrating from A_1 to A_2 ,**

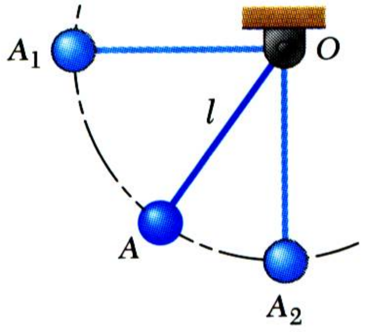
$$\int_{s_1}^{s_2} F_t ds = m\int_{v_1}^{v_2} v\, dv = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$

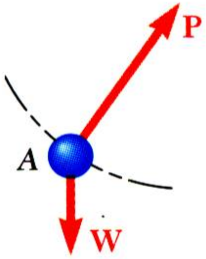
$$U_{1\to 2} = T_2 - T_1 \qquad T = \tfrac{1}{2}mv^2 = kinetic\ energy$$

- *The work of the force* $\vec{F}$ *is equal to the change in kinetic energy of the particle.*
- **Units of work and kinetic energy are the same:**

$$T = \tfrac{1}{2}mv^2 = \mathrm{kg}\left(\frac{\mathrm{m}}{\mathrm{s}}\right)^2 = \left(\mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}^2}\right)\mathrm{m} = \mathrm{N}\cdot\mathrm{m} = \mathrm{J}$$

13.1.C Applications of the Principle of Work and Energy





The bob is released from rest at position A_1. Determine the velocity of the pendulum bob at A_2 using work & kinetic energy.

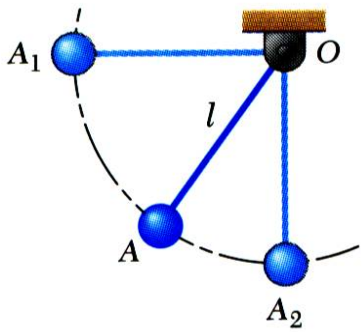
- **Force $\vec{P}$ acts normal to path and does no work.**

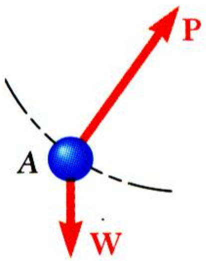
$$T_1 + U_{1\to 2} = T_2$$

$$0 + Wl = \frac{1}{2}\frac{W}{g}v_2^2$$

$$v_2 = \sqrt{2gl}$$

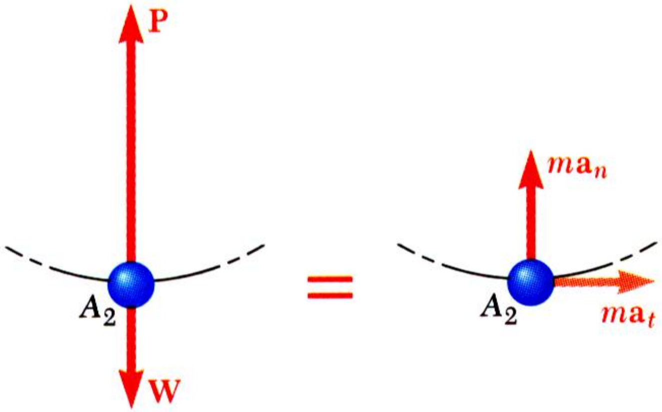
- **Velocity is found without determining expression for acceleration and integrating.**
- **All quantities are scalars and can be added directly.**
- **Forces which do no work are eliminated from the problem.**





Principle of work and energy cannot be applied to directly determine the acceleration of the pendulum bob.

- Calculating the tension in the cord requires supplementing the method of work and energy with an application of Newton's second law.



- As the bob passes through *A₂* ,

$$\sum F_n = m a_n$$

$$P - W = \frac{W}{g}\frac{v_2^2}{l}$$

$$P = W + \frac{W}{g}\frac{2gl}{l} = 3W$$

If you designed the rope to hold twice the weight of the bob, what would happen?

13.1.D Power and Efficiency

- $Power =$ **rate at which work is done.**

$$= \frac{dU}{dt} = \frac{\vec{F} \bullet d\vec{r}}{dt}$$

$$= \vec{F} \bullet \vec{v}$$

- **Dimensions of power are work/time or force*velocity. Units for power are**

$$1\,\mathrm{W}\ (\mathrm{watt}) = 1\frac{\mathrm{J}}{\mathrm{s}} = 1\,\mathrm{N}\cdot\frac{\mathrm{m}}{\mathrm{s}} \quad \text{or} \quad 1\,\mathrm{hp} = 550\frac{\mathrm{ft}\cdot\mathrm{lb}}{\mathrm{s}} = 746\,\mathrm{W}$$

$$\eta = \text{efficiency}$$

$$= \frac{\text{output work}}{\text{input work}}$$

$$= \frac{\text{power output}}{\text{power input}}$$

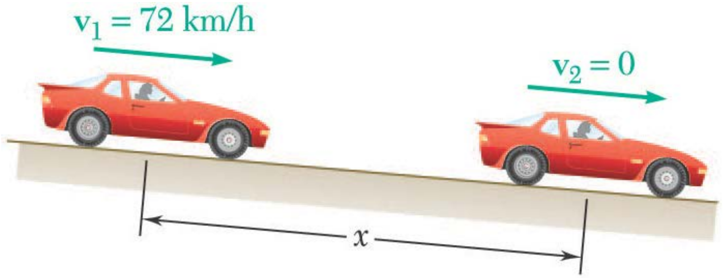
Sample Problem 13.1

An automobile of mass 1000 kg is driven down a 5° incline at a speed of 72 km/h when the brakes are applied causing a constant total breaking force of 5000 N.

Determine the distance traveled by the automobile as it comes to a stop.

STRATEGY:

- Evaluate the change in kinetic energy.
- Determine the distance required for the work to equal the kinetic energy change.



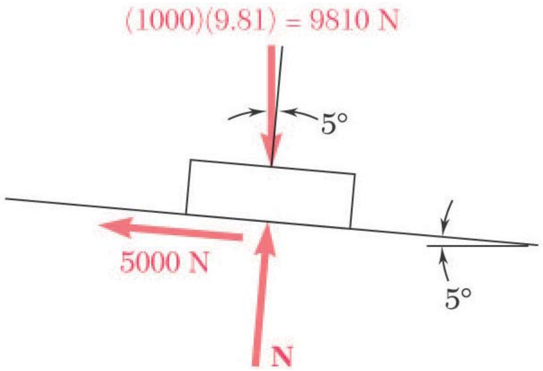
MODELING and ANALYSIS:

- **Evaluate the change in kinetic energy.**

$$v_1 = \left(72\frac{\text{km}}{\text{h}}\right)\left(\frac{1000\text{ m}}{1\text{ km}}\right)\left(\frac{1\text{h}}{3600\text{ s}}\right) = 20\text{ m/s}$$

$$T_1 = \tfrac{1}{2}mv_1^2 = \tfrac{1}{2}(1000\text{ kg})(20\text{ m/s})^2 = 200{,}000\text{ J}$$

$$v_2 = 0 \qquad T_2 = 0$$



- **Determine the distance required for the work to equal the kinetic energy change.**

$$U_{1\to2} = (-5000\text{ N})x + (1000\text{kg})(9.81\text{ m/s}^2)(\sin 5°)x$$
$$= -(4145\text{ N})x$$

$$T_1 + U_{1\to2} = T_2$$

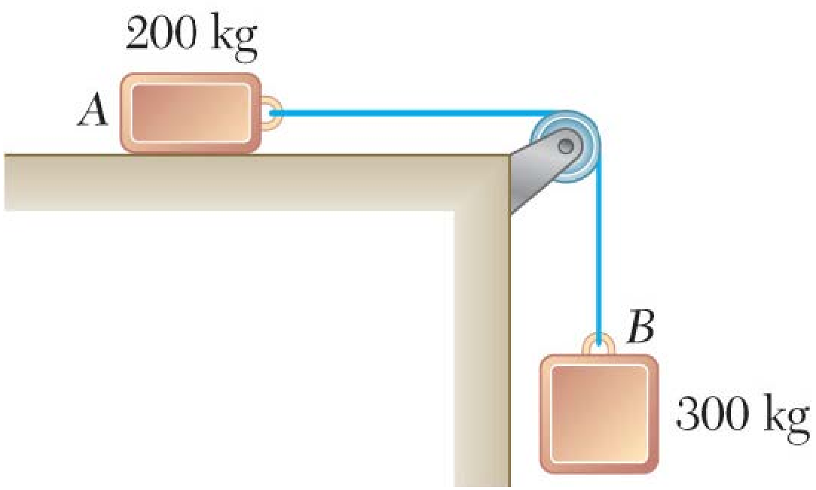
$$200{,}000\text{ J} - (4145\text{ N})x = 0$$

$$\boxed{x = 48.3\text{ m}}$$

REFLECT and THINK

- **Solving this problem using Newton's second law would require determining the car's deceleration from the free-body diagram and then integrating this to use the given velocity information.**
- **Using the principle of work and energy allows you to avoid that calculation.**

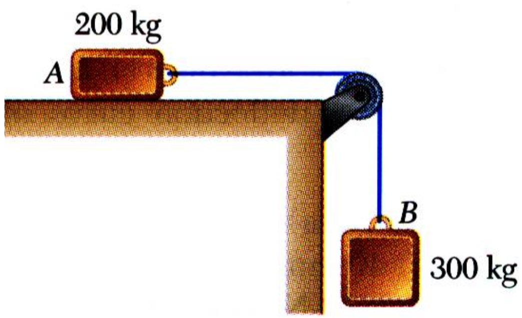
Sample Problem 13.2



Two blocks are joined by an inextensible cable as shown. If the system is released from rest, determine the velocity of block *A* after it has moved 2 m. Assume that the coefficient of friction between block *A* and the plane is m_k = 0.25 and that the pulley is weightless and frictionless.

STRATEGY:

- **Apply the principle of work and energy separately to blocks *A* and *B*.**
- **When the two relations are combined, the work of the cable forces cancel. Solve for the velocity.**



MODELING and ANALYSIS

- Apply the principle of work and energy separately to blocks *A* and *B*.

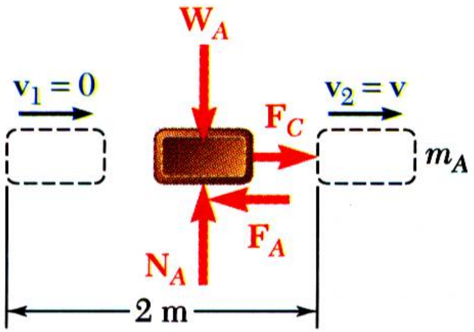
$$W_A = (200\,\text{kg})(9.81\,\text{m/s}^2) = 1962\,\text{N}$$

$$F_A = \mu_k N_A = \mu_k W_A = 0.25(1962\,\text{N}) = 490\,\text{N}$$

$$T_1 + U_{1\to 2} = T_2 :$$

$$0 + F_C(2\,\text{m}) - F_A(2\,\text{m}) = \tfrac{1}{2} m_A v^2$$

$$F_C(2\,\text{m}) - (490\,\text{N})(2\,\text{m}) = \tfrac{1}{2}(200\,\text{kg})v^2$$

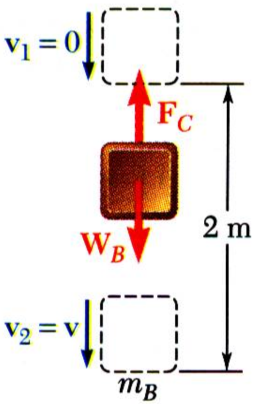


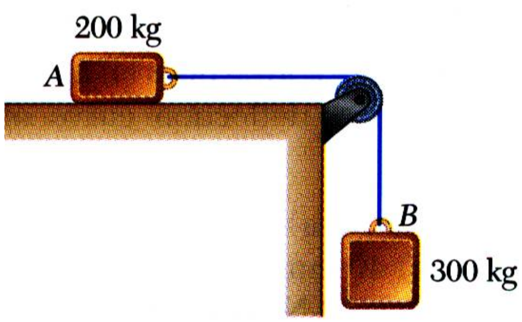
$$W_B = (300\,\text{kg})(9.81\,\text{m/s}^2) = 2940\,\text{N}$$

$$T_1 + U_{1\to 2} = T_2 :$$

$$0 - F_c(2\,\text{m}) + W_B(2\,\text{m}) = \tfrac{1}{2} m_B v^2$$

$$-F_c(2\,\text{m}) + (2940\,\text{N})(2\,\text{m}) = \tfrac{1}{2}(300\,\text{kg})v^2$$





- When the two relations are combined, the work of the cable forces cancel. Solve for the velocity.

$$F_C(2\,\mathrm{m}) - (490\,\mathrm{N})(2\,\mathrm{m}) = \tfrac{1}{2}(200\,\mathrm{kg})v^2$$

$$-F_c(2\,\mathrm{m}) + (2940\,\mathrm{N})(2\,\mathrm{m}) = \tfrac{1}{2}(300\,\mathrm{kg})v^2$$

$$(2940\,\mathrm{N})(2\,\mathrm{m}) - (490\,\mathrm{N})(2\,\mathrm{m}) = \tfrac{1}{2}(200\,\mathrm{kg} + 300\,\mathrm{kg})v^2$$

$$4900\,\mathrm{J} = \tfrac{1}{2}(500\,\mathrm{kg})v^2$$

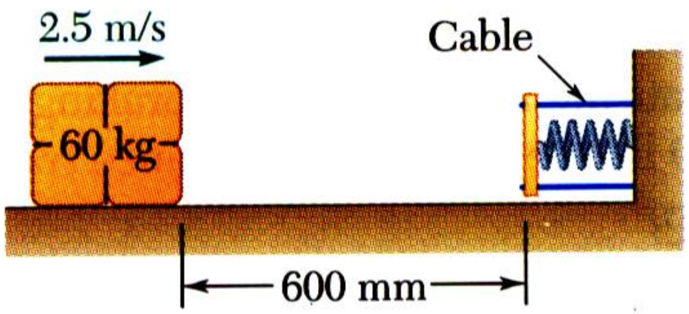
$$\boxed{v = 4.43\,\mathrm{m/s}}$$

REFLECT and THINK:

This problem can also be solved by applying the principle of work and energy to the combined system of blocks.

When using the principle of work and energy, it usually saves time to choose your system to be everything that moves.

Sample Problem 13.3

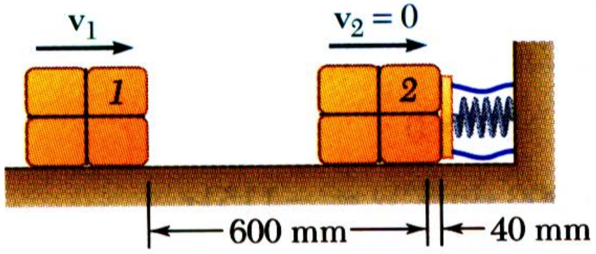


A spring is used to stop a 60 kg package which is sliding on a horizontal surface. The spring has a constant k = 20 kN/m and is held by cables so that it is initially compressed 120 mm. The package has a velocity of 2.5 m/s in the position shown and the maximum deflection of the spring is 40 mm.

Determine *(a)* the coefficient of kinetic friction between the package and surface and *(b)* the velocity of the package as it passes again through the position shown.

STRATEGY:

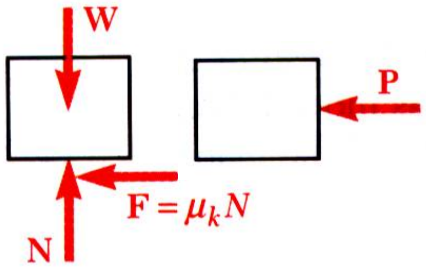
- Apply the principle of work and energy between the initial position and the point at which the spring is fully compressed and the velocity is zero. The only unknown in the relation is the friction coefficient.
- Apply the principle of work and energy for the rebound of the package. The only unknown in the relation is the velocity at the final position.



MODELING and ANALYSIS:

- Apply principle of work and energy between initial position and the point at which spring is fully compressed.

$$T_1 = \tfrac{1}{2} m v_1^2 = \tfrac{1}{2}(60\,\text{kg})(2.5\,\text{m/s})^2 = 187.5\,\text{J} \qquad T_2 = 0$$

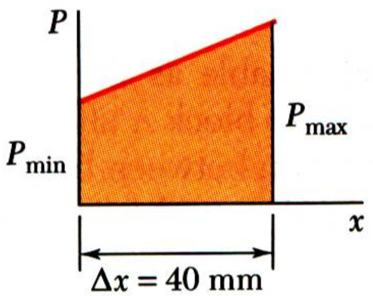


$$(U_{1\to 2})_f = -\mu_k W x$$

$$= -\mu_k (60\,\text{kg})\left(9.81\,\text{m/s}^2\right)(0.640\,\text{m}) = -(377\,\text{J})\mu_k$$

$$P_{\min} = k x_0 = (20\,\text{kN/m})(0.120\,\text{m}) = 2400\,\text{N}$$

$$P_{\max} = k(x_0 + \Delta x) = (20\,\text{kN/m})(0.160\,\text{m}) = 3200\,\text{N}$$



$$(U_{1\to 2})_e = -\tfrac{1}{2}(P_{\min} + P_{\max})\Delta x$$

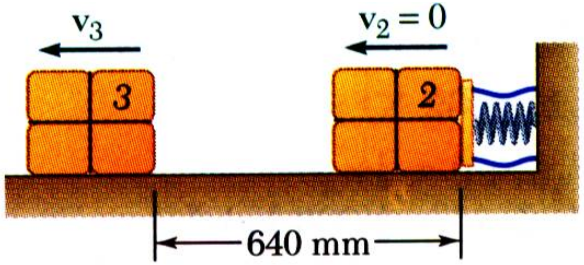
$$= -\tfrac{1}{2}(2400\,\text{N} + 3200\,\text{N})(0.040\,\text{m}) = -112.0\,\text{J}$$

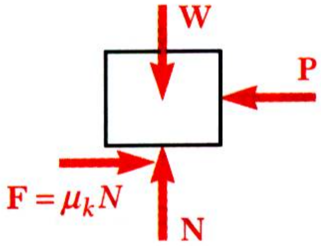
$$U_{1\to 2} = (U_{1\to 2})_f + (U_{1\to 2})_e = -(377\,\text{J})\,\mu_k - 112\,\text{J}$$

$$T_1 + U_{1\to 2} = T_2 :$$

$$187.5\,\text{J-}(377\text{ J})\,\mu_k - 112\,\text{J} = 0$$

$$\boxed{\mu_k = 0.20}$$





***Apply the principle of work and energy for the rebound of the package.**

$$T_2 = 0 \qquad T_3 = \tfrac{1}{2} m v_3^2 = \tfrac{1}{2}(60\,\text{kg})v_3^2$$

$$U_{2\to3} = (U_{2\to3})_f + (U_{2\to3})_e = -(377\,\text{J})\mu_k + 112\,\text{J}$$

$$= +36.5\,\text{J}$$

$$T_2 + U_{2\to3} = T_3 :$$

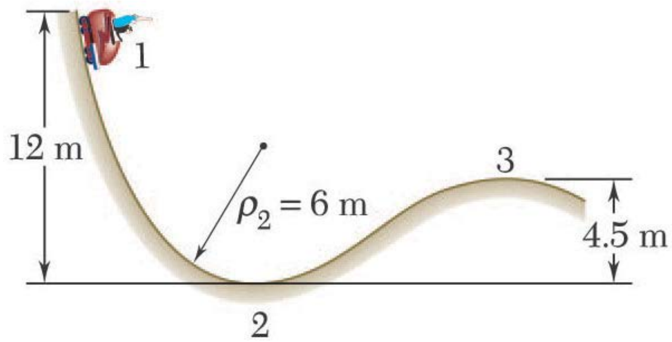
$$0 + 36.5\,\text{J} = \tfrac{1}{2}(60\,\text{kg})v_3^2$$

$$\boxed{v_3 = 1.103\,\text{m/s}}$$

REFLECT and THINK:

You needed to break this problem into two segments. From the first segment you were able to determine the coefficient of friction. Then you could use the principle of work and energy to determine the velocity of the package at any other location. Note that the system does not lose any energy due to the spring; it returns all of its energy back to the package. You would need to design something that could absorb the kinetic energy of the package in order to bring it to rest.

Sample Problem 13.6



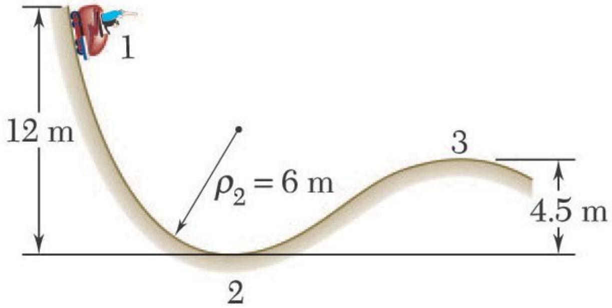
A 1000 kg car starts from rest at point 1 and moves without friction down the track shown.

Determine:

a) the force exerted by the track on the car at point 2, and b) the minimum safe value of the radius of curvature at point 3.

STRATEGY:

- Apply principle of work and energy to determine velocity at point 2.
- Apply Newton's second law to find normal force by the track at point 2.
- Apply principle of work and energy to determine velocity at point 3.
- Apply Newton's second law to find minimum radius of curvature at point 3 such that a positive normal force is exerted by the track.



MODELING and ANALYSIS:

a. Apply principle of work and energy to determine velocity at point 2.

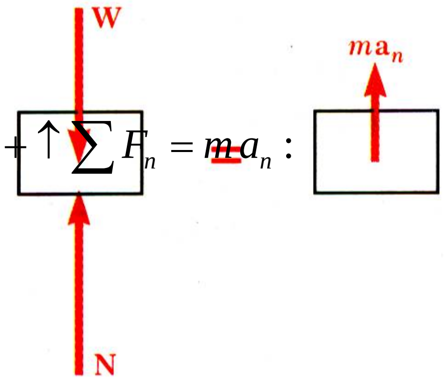
$$T_1 = 0 \qquad T_2 = \tfrac{1}{2} m v_2^2$$

$$U_{1\to 2} = +\text{mg}(12\,\text{m})$$

$$T_1 + U_{1\to 2} = T_2 : \qquad 0 + \text{mg}(12\,\text{m}) = \frac{1}{2} m v_2^2$$

$$v_2^2 = 2(12\,\text{m})g = 2(12\,\text{m})\left(9.81\,\text{m}/s^2\right) \quad v_2 = 15.34\,\text{m/s}$$

- **Apply Newton's second law to find normal force by the track at point 2.**

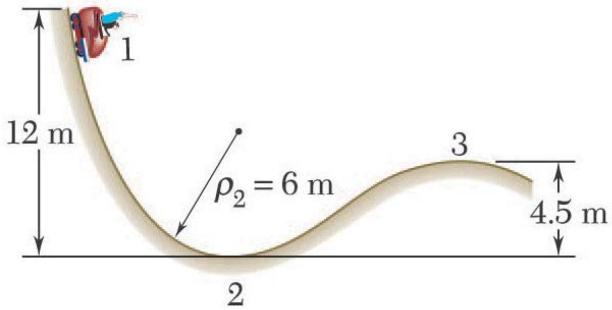


$$+\uparrow \sum F_n = m a_n :$$

$$-W + N = m\,a_n = m\frac{v_2^2}{\rho_2} = m\frac{(24\,g)}{6\text{ m}} = 4\text{ mg} = 4W$$

$$N = 5W = 5(1000\text{ kg})\left(9.81\text{ m/s}^2\right)$$

$$\boxed{N = 49.05\text{ kN}}$$

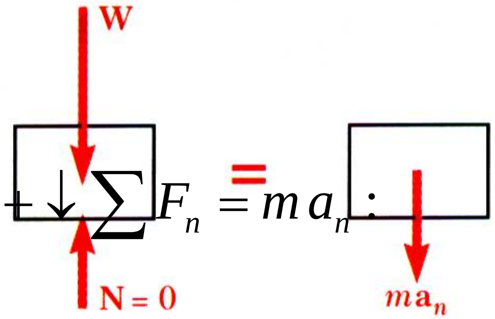


b. **Apply principle of work and energy to determine velocity at point 3.**

$$T_1 + U_{1\to 3} = T_3 \qquad 0 + mg(7.5\,\text{m}) = \frac{1}{2}mv_3^2$$

$$v_3^2 = 2(7.5\,\text{m})g = 2(7.5\,\text{m})\left(9.81\,\text{m/s}^2\right) \qquad v_3 = 12.13\,\text{m/s}$$

- **Apply Newton's second law to find minimum radius of curvature at point 3 such that a positive normal force is exerted by the track.**

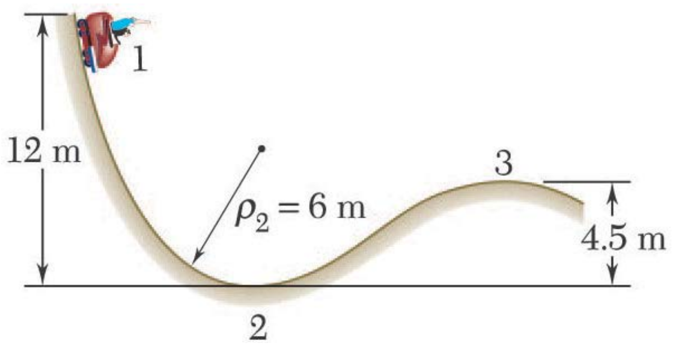


$$+\downarrow \sum F_n = m a_n :$$

$$W = m a_n$$

$$= m \frac{v_3^2}{\rho_3} = m \frac{2(7.5\,\mathrm{m})g}{\rho_3}$$

$$\boxed{\rho_3 = 15\,\mathrm{m}}$$



REFLECT and THINK

This is an example where you need both Newton's second law and the principle of work and energy.

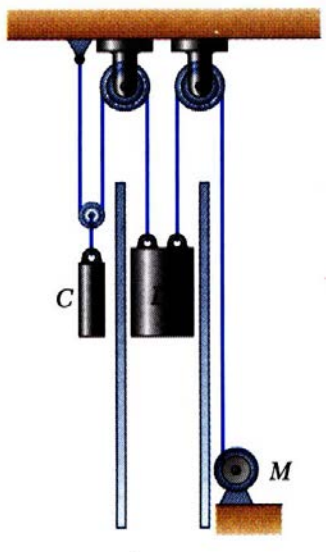
Work–energy is used to determine the speed of the car, and Newton's second law is used to determine the normal force.

A normal force of 5W is equivalent to a fighter pilot pulling 5g's and should only be

experienced for a very short time.

For safety, you would also want to make sure your radius of curvature was quite a bit larger than 15 m.

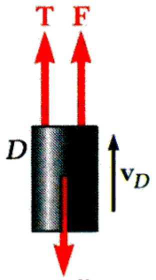
Sample Problem 13.7



The dumbwaiter *D* and its load have a combined mass of 300 kg, while the counterweight *C* has a mass of 400 kg.

Determine the power delivered by the electric motor *M* when the dumbwaiter *(a)* is moving up at a constant speed of 2.5 m/s and *(b)* has an instantaneous velocity of 2.5 m/s and an acceleration of 1 m/s^2, both directed upwards.

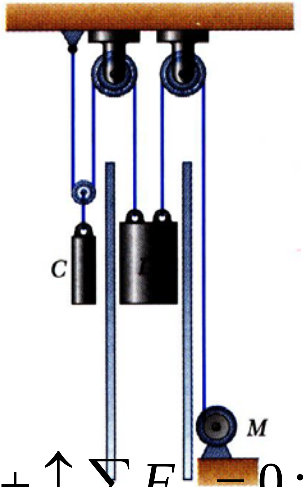
STRATEGY:



Force exerted by the motor cable has same direction as the dumbwaiter velocity. Power delivered by motor is equal to Fv_D, v_D = 2.5 m/s.

• In the first case, bodies are in uniform motion. Determine force exerted by motor cable from conditions for static equilibrium.

• In the second case, both bodies are accelerating. Apply Newton's second law to each body to determine the required motor cable force.



MODELING and ANALYSIS:

- **In the first case, bodies are in uniform motion. Determine force exerted by motor cable from conditions for static equilibrium.**

- **Free-body C:**

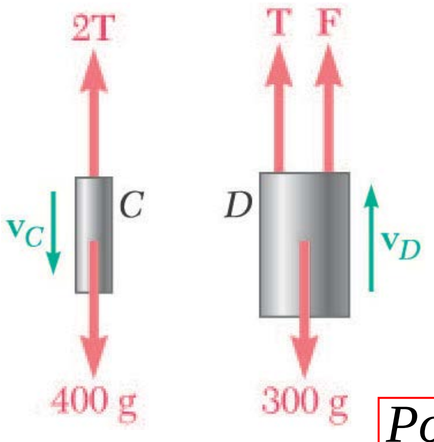
$$+\uparrow \sum F_y = 0: \quad 2T - 400\,g = 0 \qquad T = 200\,g = 1962\ \text{N}$$

Free-body D:

$$+\uparrow \sum F_y = 0: \quad F + T - 300\,g = 0$$

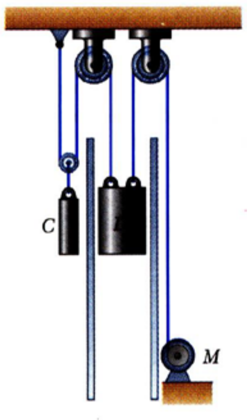
$$F = 300\,g - T = 300\,g - 200\,g = 100\,g$$

$$= 981\ \text{N}$$



$$Power = Fv_D = (981\ \text{N})(2.5\,\text{m/s})$$
$$= 2452\ \text{W}$$

$$\boxed{Power = 2450\ \text{W}}$$

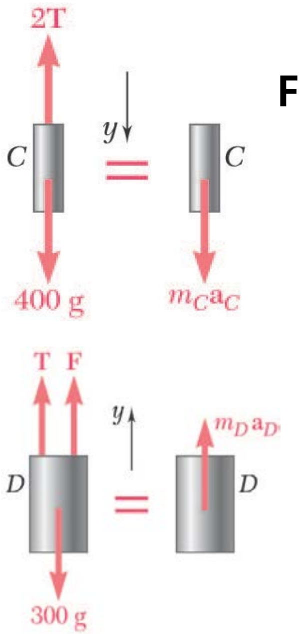


In the second case, both bodies are accelerating. Apply Newton's second law to each body to determine the required motor cable force.

$$a_D = 1\,\text{m/s}^2 \uparrow \qquad a_C = -\tfrac{1}{2}a_D = 0.5\,\text{m/s}^2 \downarrow$$

Free-body C:

$$+\downarrow \sum F_y = m_C a_C: \;\; 400\text{g} \square 2T \square 400 \boxed{0}.5 \square \qquad T \square 1862\ \text{N}$$



Free-body D:

$$+\uparrow \sum F_y = m_D a_D :$$

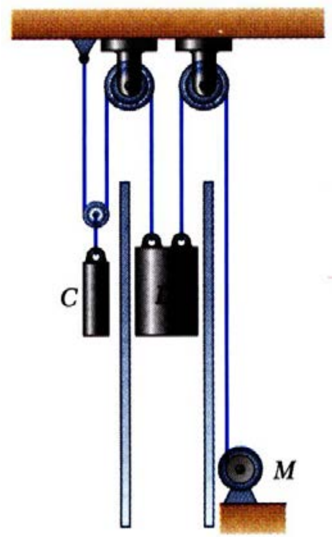
$$F + T - 300g = 300(1)$$

$$F = -1862 + (300)(9.81) + 300$$

$$F = 1381\ \text{N}$$

$$Power = Fv_D = (1381\ \text{N})(2.5\ \text{m/s}) = 3452\ \text{W}$$

$$\boxed{Power = 3450\ \text{W}}$$

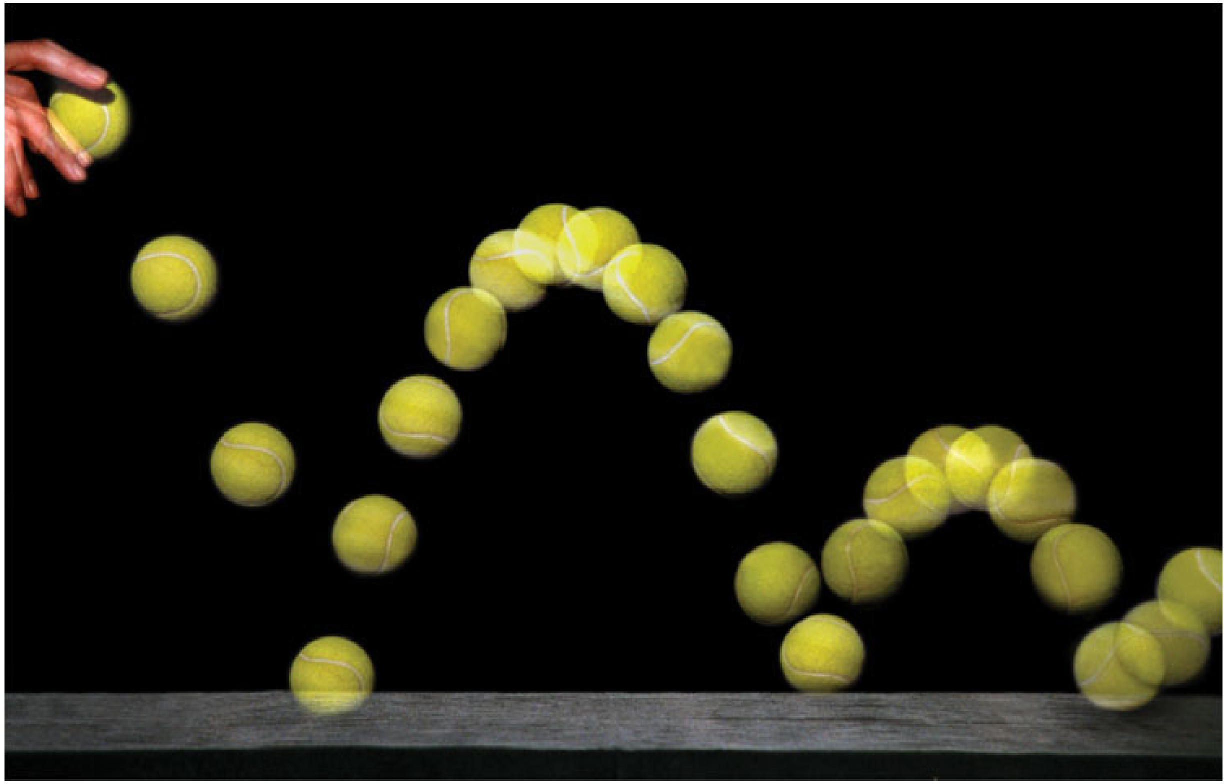


REFLECT and THINK

As you might expect, the motor needs to deliver more power to produce accelerated motion than to produce motion at constant velocity.

13.2 Conservation of Energy

The potential energy stored at the top of the ball's path is transferred to kinetic energy as the ball meets the ground. Why is the ball's height reducing?



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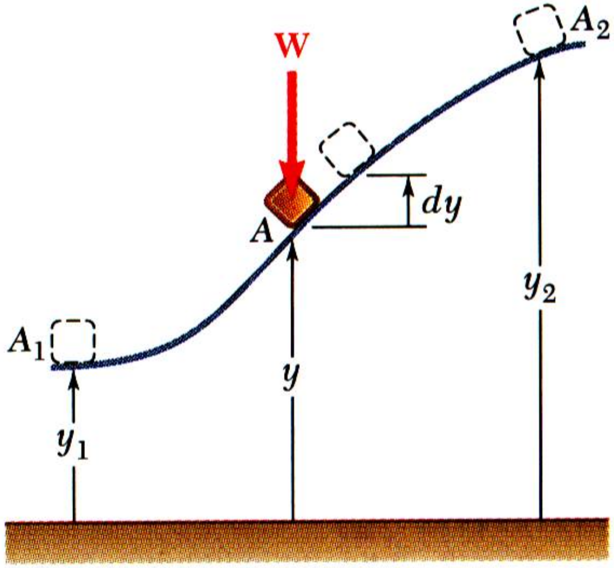
13.2A Potential Energy

If the work of a force only depends on differences in position, we can express this work as potential energy.

<u>Can the work done by the following forces be expressed as potential energy?</u>

Weight	**YES**	**NO**
Friction	**YES**	**NO**
Normal force	**YES**	**NO**
Spring force	**YES**	**NO**

YES NO NO YES



- **Work of the force of gravity $\vec{W}$,**

$$U_{1\to 2} = W\,y_1 - W\,y_2$$

- **Work is independent of path followed; depends only on the initial and final values of** ***Wy.***

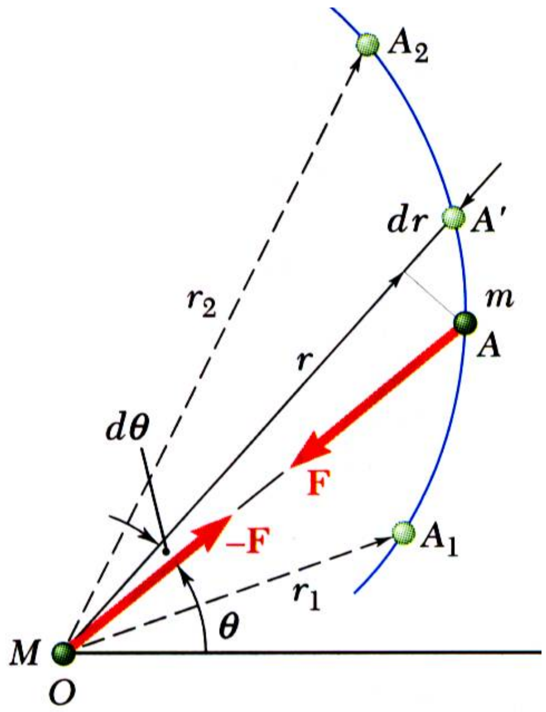
$$V_g = Wy$$

$=$ ***potential energy*** **of the body with respect to** ***force of gravity.***

$$U_{1\to 2} = \left(V_g\right)_1 - \left(V_g\right)_2$$

- **Choice of datum from which the elevation** ***y*** **is measured is arbitrary.**
- **Units of work and potential energy are the same:**

$$V_g = Wy = \mathrm{N}\cdot\mathrm{m} = \mathrm{J}$$

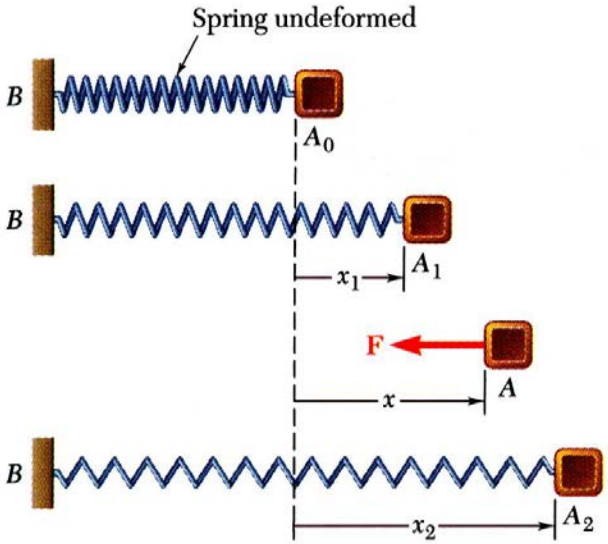


- Previous expression for potential energy of a body with respect to gravity is only valid when the weight of the body can be assumed constant.
- For a space vehicle, the variation of the force of gravity with distance from the center of the earth should be considered.
- Work of a gravitational force,

$$U_{1\to 2} = \frac{GMm}{r_2} - \frac{GMm}{r_1}$$

- Potential energy V_g when the variation in the force of gravity can not be neglected,

$$V_g = -\frac{GMm}{r} = -\frac{WR^2}{r}$$



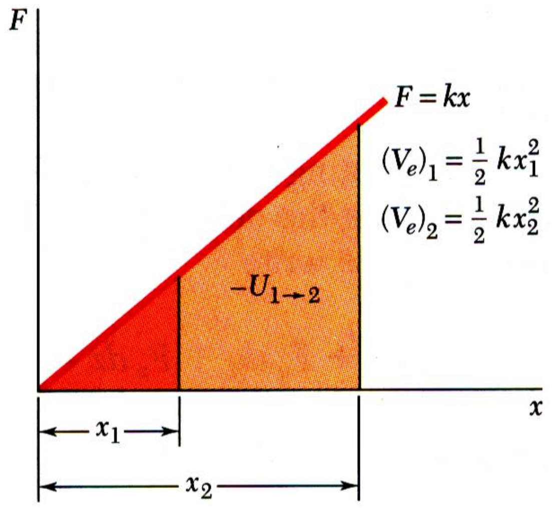
- **Work of the force exerted by a spring depends only on the initial and final deflections of the spring,**

$$U_{1\to 2} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2$$

- **The potential energy of the body with respect to the elastic force,**

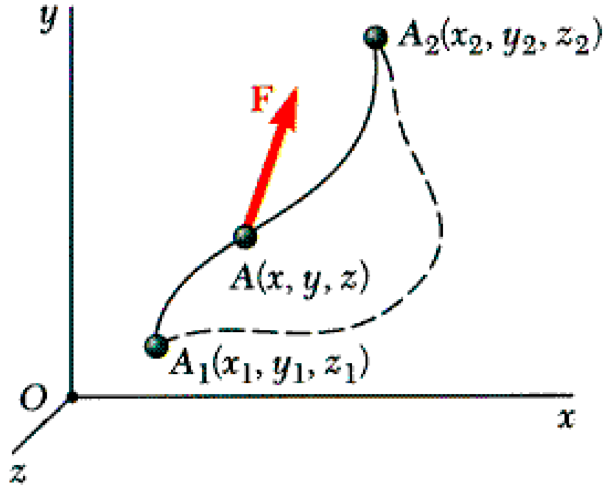
$$V_e = \frac{1}{2}kx^2$$

$$U_{1\to 2} = (V_e)_1 - (V_e)_2$$



- Note that the preceding expression for V_e is valid only if the deflection of the spring is measured from its undeformed position.

13.2B Conservative Forces

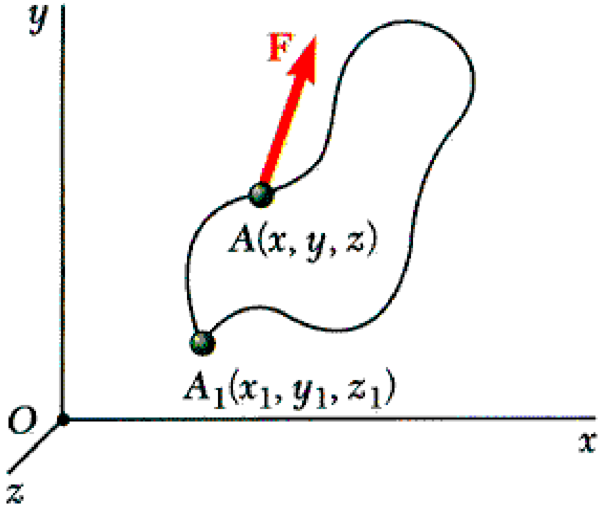


Concept of potential energy can be applied if the work of the force is independent of the path followed by its point of application.

$$U_{1\to 2} = V(x_1, y_1, z_1) - V(x_2, y_2, z_2)$$

Such forces are described as *conservative forces*.

- For any conservative force applied on a closed path,



$$\oint \vec{F} \bullet d\vec{r} = 0$$

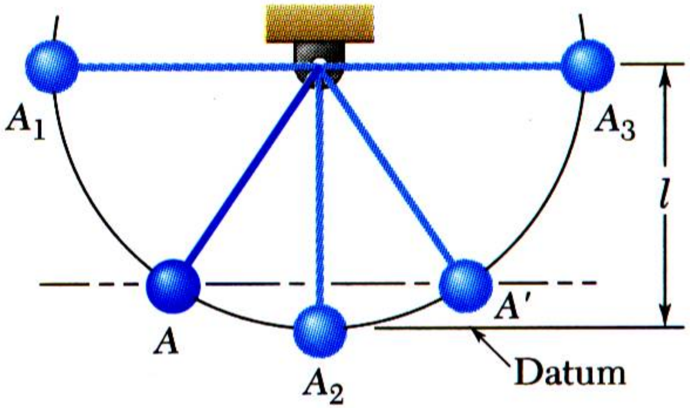
• **Elementary work corresponding to displacement between two neighboring points,**

$$dU = V(x, y, z) - V(x + dx, y + dy, z + dz)$$
$$= -dV(x, y, z)$$

$$F_x dx + F_y dy + F_z dz = -\left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz\right)$$

$$\vec{F} = -\left(\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial V}{\partial z}\right) = -\mathbf{grad}\, V$$

13.2C The Principle of Conservation of Energy



$$T_1 = 0 \quad V_1 = W\ell$$

$$T_1 + V_1 = W\ell$$

$$T_2 = \tfrac{1}{2} m v_2^2 = \frac{1}{2}\frac{W}{g}(2g\ell) = W\ell \quad V_2 = 0$$

$$T_2 + V_2 = W\ell$$

- **Work of a conservative force,**

$$U_{1\to 2} = V_1 - V_2$$

- **Concept of work and energy,**

$$U_{1\to 2} = T_2 - T_1$$

- **Follows that**

$$T_1 + V_1 = T_2 + V_2$$

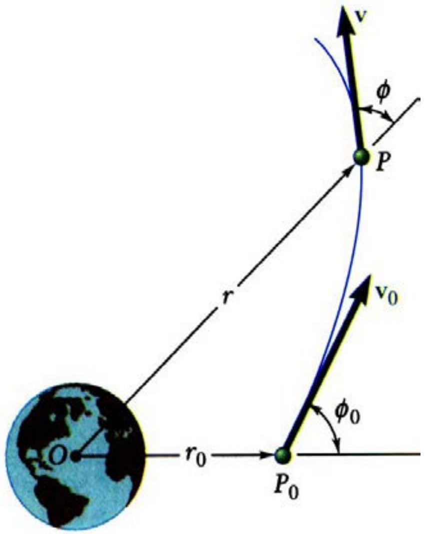
$$E = T + V = \text{constant}$$

When a particle moves under the action of conservative forces, the total mechanical energy is constant.

Friction forces are not conservative. Total mechanical energy of a system involving friction decreases.

Mechanical energy is dissipated by friction into thermal energy. Total energy is constant.

13.2D Motion Under a Conservative Central Force



- **When a particle moves under a conservative central force, both the principle of conservation of angular momentum**

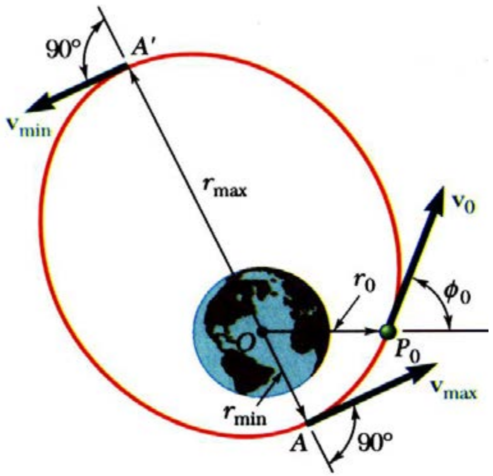
$$r_0 m v_0 \sin\phi_0 = r m v \sin\phi$$

and the principle of conservation of energy

$$T_0 + V_0 = T + V$$

$$\frac{1}{2} m v_0^2 - \frac{GMm}{r_0} = \frac{1}{2} m v^2 - \frac{GMm}{r}$$

Given *r*, the equations may be solved for *v*.

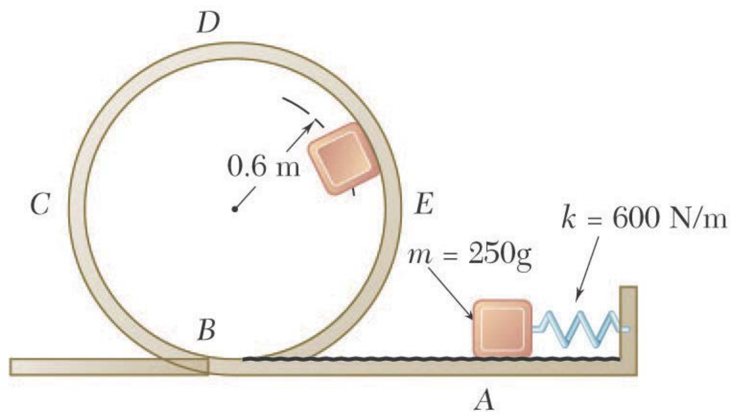


Also, using Eqns (25),(26)

- At minimum and maximum r, $\phi = 90^o$. Given the launch conditions, the equations may be solved for r_{min}, r_{max}, v_{min}, and v_{max}.

Sample Problem 13.8

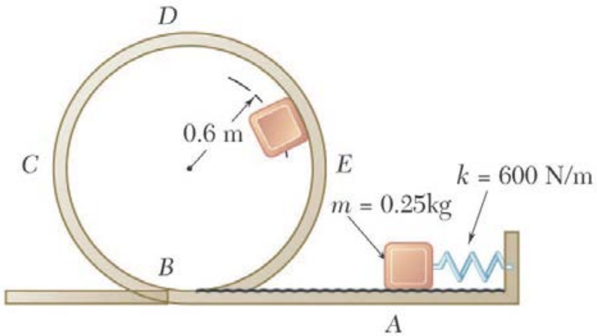
Sample Problem 13.10



The 250 g pellet is pushed against the spring and released from rest at A. Neglecting friction, determine the smallest deflection of the spring for which the pellet will travel around the loop and remain in contact with the loop at all times.

STRATEGY:

- **Since the pellet must remain in contact with the loop, the force exerted on the pellet must be greater than or equal to zero. Setting the force exerted by the loop to zero, solve for the minimum velocity at *D*.**
- **Apply the principle of conservation of energy between points *A* and *D*. Solve for the spring deflection required to produce the required velocity and kinetic energy at *D*.**

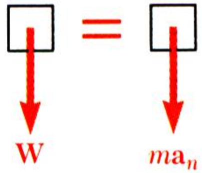


MODELING and ANALYSIS:

- **Setting the force exerted by the loop to zero, solve for the minimum velocity at *D*.**

$$+\downarrow \sum F_n = ma_n : \quad W = ma_n \qquad mg = m v_D^2/r$$

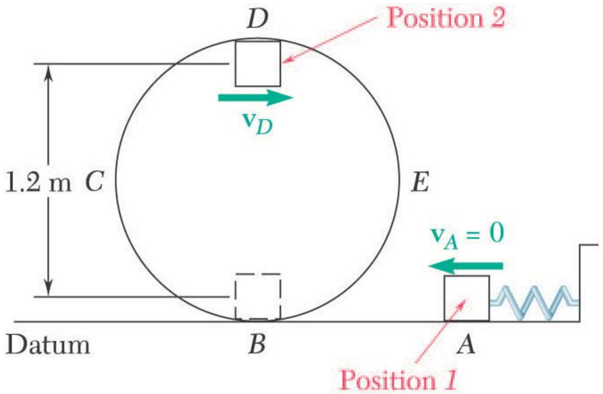
$$v_D^2 = rg = (0.6\ \mathrm{m})(9.81\ \mathrm{m/s^2}) = 5.886\ \mathrm{m^2/s^2}$$



- **Apply the principle of conservation of energy between points *A* and *D*.**

$$V_1 = V_e + V_g = \tfrac{1}{2}kx^2 + 0 = \tfrac{1}{2}(600\ \mathrm{N/m})x^2 = 300x^2$$

$$T_1 = 0$$



$$V_2 = V_e + V_g = 0 + Wy = (0.25\ \mathrm{kg})\left(9.81\ \mathrm{m/s^2}\right)(1.2\ \mathrm{m}) = 2.943\ \mathrm{N\cdot m}$$

$$T_2 = \tfrac{1}{2}mv_D^2 = \frac{1}{2}(0.25\ \mathrm{kg})\left(5.886\ \mathrm{m^2/s^2}\right) = 0.73575\ \mathrm{N\cdot m}$$

$$T_1 + V_1 = T_2 + V_2$$

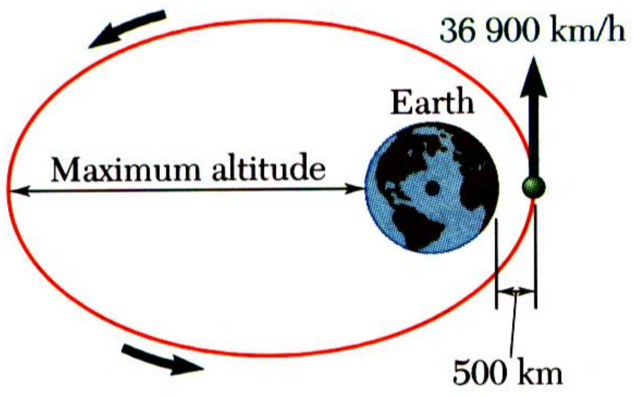
$$0 + 300x^2 = 0.73575 + 2.943$$

$$\boxed{x = 0.1107\ \mathrm{m}}$$

REFLECT and THINK

A common misconception in problems like this is assuming that the speed of the particle is zero at the top of the loop, rather than that the normal force is equal to or greater than zero. If the pellet had a speed of zero at the top, it would clearly fall straight down, which is impossible.

Sample Problem 13.12

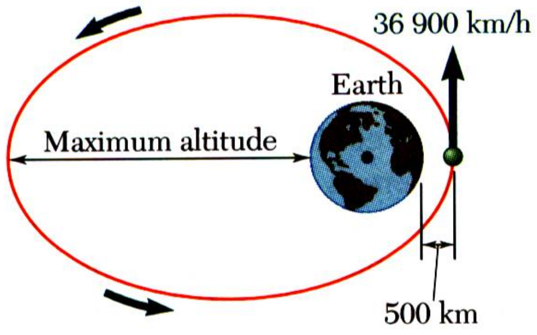


A satellite is launched in a direction parallel to the surface of the earth with a velocity of 36900 km/h from an altitude of 500 km.

Determine *(a)* the maximum altitude reached by the satellite, and *(b)* the maximum allowable error in the direction of launching if the satellite is to come no closer than 200 km to the surface of the earth

STRATEGY:

- For motion under a conservative central force, the principles of conservation of energy and conservation of angular momentum may be applied simultaneously.
- Apply the principles to the points of minimum and maximum altitude to determine the maximum altitude.
- Apply the principles to the orbit insertion point and the point of minimum altitude to determine maximum allowable orbit insertion angle error.

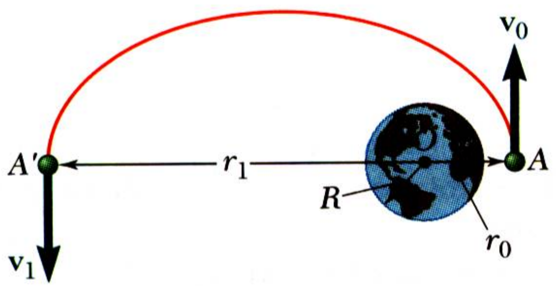


MODELING and ANALYSIS:

- Apply the principles of conservation of energy and conservation of angular momentum to the points of minimum and maximum altitude to determine the maximum altitude.

Conservation of energy:

$$T_A + V_A = T_{A'} + V_{A'} \qquad \tfrac{1}{2}mv_0^2 - \frac{GMm}{r_0} = \tfrac{1}{2}mv_1^2 - \frac{GMm}{r_1}$$



Conservation of angular momentum:

$$r_0 m v_0 = r_1 m v_1 \qquad v_1 = v_0 \frac{r_0}{r_1}$$

Combining,

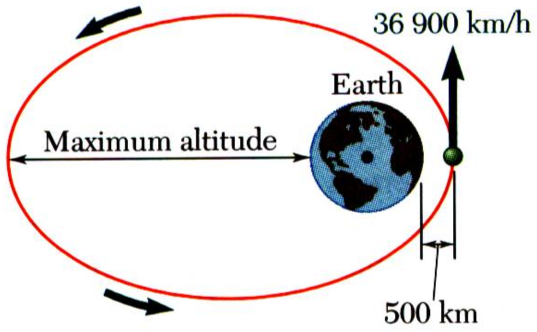
$$\tfrac{1}{2}v_0^2\left(1 - \frac{r_0^2}{r_1^2}\right) = \frac{GM}{r_0}\left(1 - \frac{r_0}{r_1}\right) \qquad 1 + \frac{r_0}{r_1} = \frac{2GM}{r_0 v_0^2}$$

$$r_0 = 6370\,\text{km} + 500\,\text{km} = 6870\,\text{km}$$

$$v_0 = 36900\,\text{km/h} = 10.25\times10^6\,\text{m/s}$$

$$GM = gR^2 = \left(9.81\,\text{m/s}^2\right)\left(6.37\times10^6\,\text{m}\right)^2 = 398\times10^{12}\,\text{m}^3/\text{s}^2$$

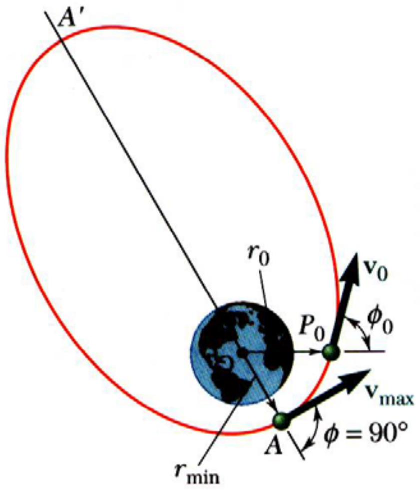
$$\boxed{r_1 = 60.4\times10^6\,\text{m} = 60400\,\text{km}}$$



Apply the principles to the orbit insertion point and the point of minimum altitude to determine maximum allowable orbit insertion angle error.

Conservation of energy:

$$T_0 + V_0 = T_A + V_A \qquad \frac{1}{2}mv_0^2 - \frac{GMm}{r_0} = \frac{1}{2}mv_{max}^2 - \frac{GMm}{r_{min}}$$



Conservation of angular momentum:

$$r_0 m v_0 \sin\phi_0 = r_{min} m v_{max} \qquad v_{max} = v_0 \sin\phi_0 \frac{r_0}{r_{min}}$$

Combining and solving for sin j_0,

$$\sin\phi_0 = 0.9801$$

$$\varphi_0 = 90° \pm 11.5° \qquad \boxed{\text{allowable error} = \pm 11.5°}$$

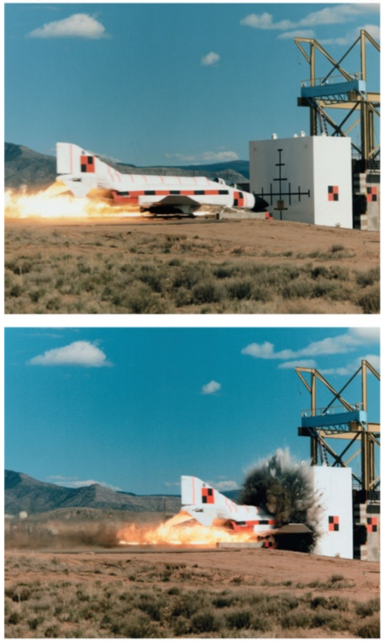
REFLECT and THINK:

- **Space probes and other long-distance vehicles are designed with small rockets to allow for mid-course corrections. Satellites launched from the Space Station usually do not need this kind of fine-tuning.**

Impulsive Motion

NASA.

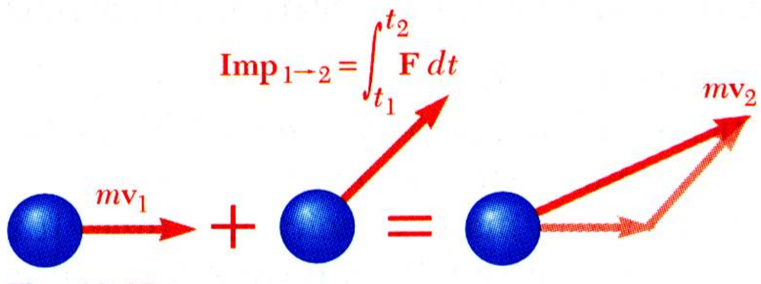
The thrust of a rocket acts over a specific time period to give the rocket linear momentum.



©Sandia National Laboratories/Getty Images RF

The impulse applied to the railcar by the wall brings its momentum to zero. Crash tests are often performed to help improve safety in different vehicles.

13.3A Principle of Impulse and Momentum



- From Newton's second law, linear momentum (mv)

$$\vec{F} = \frac{d}{dt}(m\vec{v}) \quad \textbf{(13.27)}$$

$$\vec{F}dt = d\left(m\vec{v}\right)$$

$$\int_{t_1}^{t_2} \vec{F}dt = m\vec{v}_2 - m\vec{v}_1$$

- **Dimensions of the impulse of a force are *force*time*.**
- **Units for the impulse of a force are** $\mathrm{N}\cdot\mathrm{s} = \left(\mathrm{kg}\cdot\mathrm{m}/\mathrm{s}^2\right)\cdot\mathrm{s} = \mathrm{kg}\cdot\mathrm{m}/\mathrm{s}$
- $$\int_{t_1}^{t_2} \vec{F}dt = \mathbf{Imp}_{1\to 2} = \text{ impulse of the force } \vec{F}$$

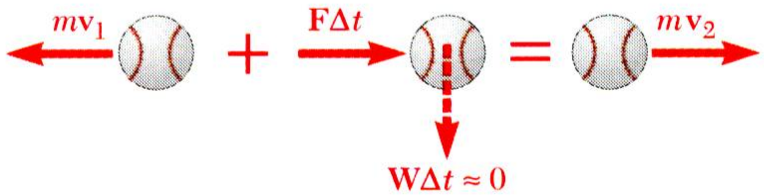
$$m\vec{v}_1 + \mathbf{Imp}_{1\to 2} = m\vec{v}_2 ....(13.28)$$

The final momentum of the particle can be obtained by adding vectorially its initial momentum and the impulse of the force during the time interval.

Fig.13.17

Fig.13.18

13.3B Impulsive Motion



Force acting on a particle during a very short time interval that is large enough to cause a significant change in momentum is called an *impulsive force*.

- **When impulsive forces act on a particle,**

$$m\vec{v}_1 + \sum \vec{F}\,\Delta t = m\vec{v}_2 \quad \textbf{(13.35)}$$

- **When a baseball is struck by a bat, contact occurs over a short time interval but force is large enough to change sense of ball motion.**
- ***Nonimpulsive forces* are forces for which $\vec{F}\Delta t$ is small and therefore, may be neglected – an example of this is the weight of the baseball.**

Sample Problem 13.13

An automobile weighing 1800 kg is driven down a 5° incline at a speed of 100 km/h when the brakes are applied, causing a constant total braking force of 7000 N.

Determine the time required for the automobile to come to a stop.

STRATEGY:

- Apply the principle of impulse and momentum. The impulse is equal to the product of the constant forces and the time interval.

MODELING and ANALYSIS:

- Apply the principle of impulse and momentum.

$$m\vec{v}_1 + \sum \mathbf{Imp}_{1\to 2} = m\vec{v}_2$$

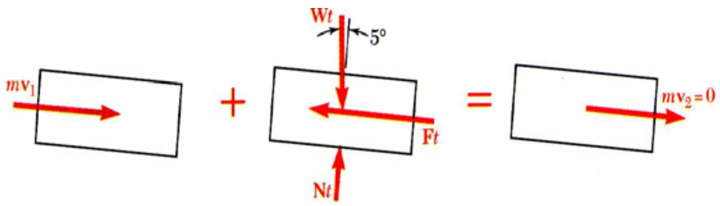
Taking components parallel to the incline,

$$mv_1 + (mg\sin 5°)t - Ft = 0$$

$$100\ \text{km/h} = 100\left(\frac{1000\ \text{m}}{1\ \text{km}}\right)\left(\frac{1\ \text{h}}{3600\ \text{s}}\right) = 27.78\ \text{ms}$$

$$(1800\ \text{kg})(27.78\ \text{m/s}) + (1800\ \text{kg})$$

$$(9.81\ \text{m/s}^2)\sin 5° t - (7000\ \text{N})t = 0$$

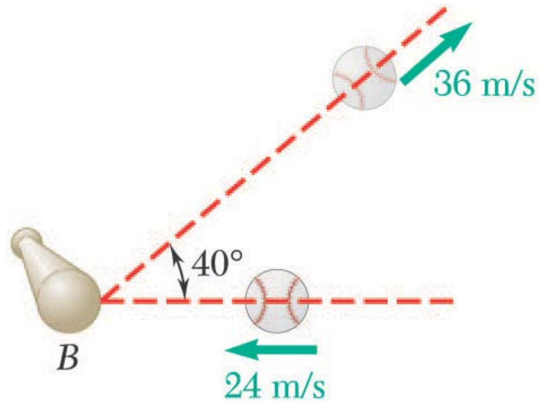


$$\boxed{t = 9.16\text{s}}$$

REFLECT and THINK

- You could use Newton's second law to solve this problem. First, you would determine the car's deceleration, separate variables, and then integrate a = dv/dt to relate the velocity, deceleration, and time. You could not use conservation of energy to solve this problem, because this principle does not involve time.

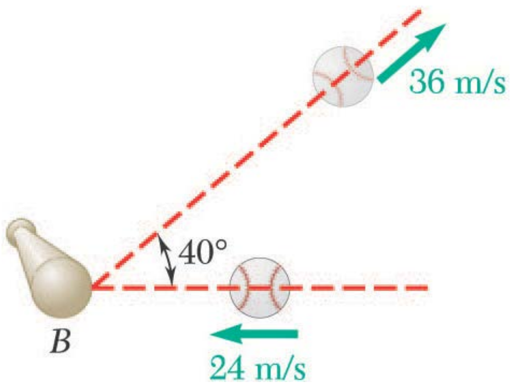
Sample Problem 13.16



A 120 g baseball is pitched with a velocity of 24 m/s. After the ball is hit by the bat, it has a velocity of 36 m/s in the direction shown. If the bat and ball are in contact for 0.015 s, determine the average impulsive force exerted on the ball during the impact.

STRATEGY:

- Apply the principle of impulse and momentum in terms of horizontal and vertical component equations.

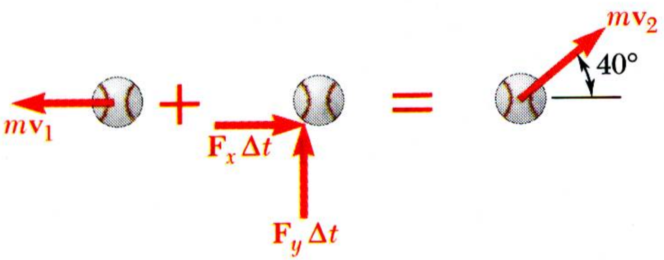


MODELING and ANALYSIS:

- **Apply the principle of impulse and momentum in terms of horizontal and vertical component equations.**

$$m\vec{v}_1 + \mathbf{Imp}_{1\to2} = m\vec{v}_2$$

***x* component equation**:



$$-mv_1 + F_x\Delta t = mv_2\cos 40^\circ$$

$$-(0.12)(24) + F_x(0.015) = (0.12)(36\cos 40^\circ)$$

$$F_x = +412.6\ \mathrm{N}$$

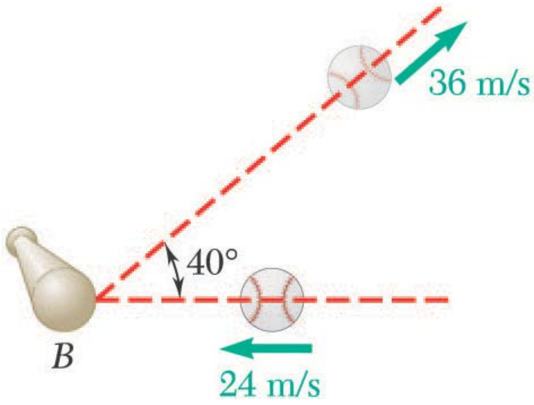
***y* component equation**:

$$0 + F_y\Delta t = mv_2\sin 40^\circ$$

$$F_y(0.015) = (0.12)(36\sin 40^\circ)$$

$$F_y = +185.1\ \mathrm{N}$$

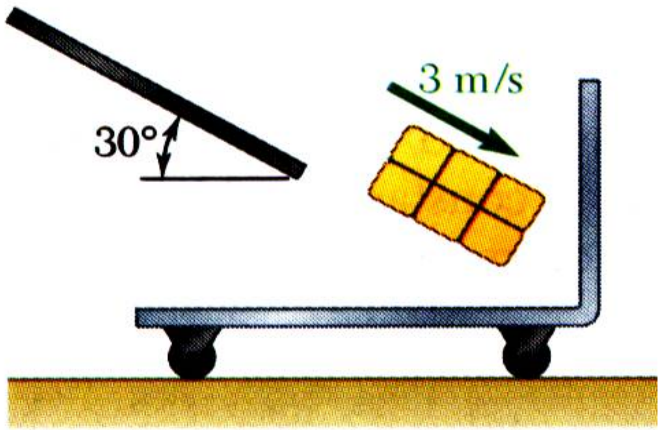
$$\boxed{\vec{F} = (413\,\mathrm{N})\vec{i} + (185.1\,\mathrm{N})\vec{j},\quad F = 452\,\mathrm{N}}$$



REFLECT and THINK:

In this problem, we neglected the impulse due to the weight. This would have had a magnitude of $(0.12\text{ kg})\boxed{\left(9.81\dfrac{m}{S^2}\right)}(0.015\text{s})$ $= 0.01766\text{ N}\cdot\text{S}$. This indeed is much smaller than the impulse exerted on the ball by the bat, which is (452 N)(0.015 s) $= 6.78\text{ N}\cdot\text{S}$.

Sample Problem 13.17



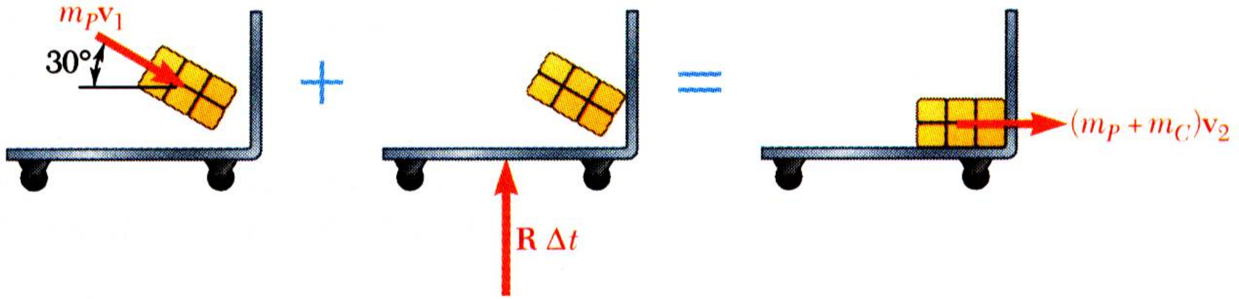
A 10 kg package drops from a chute into a 24 kg cart with a velocity of 3 m/s. Knowing that the cart is initially at rest and can roll freely, determine *(a)* the final velocity of the cart, *(b)* the impulse exerted by the cart on the package, and *(c)* the fraction of the initial energy lost in the impact.

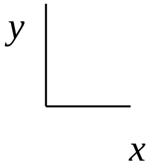
STRATEGY:

- Apply the principle of impulse and momentum to the package-cart system to determine the final velocity.
- Apply the same principle to the package alone to determine the impulse exerted on it from the change in its momentum.

MODELING and ANALYSIS

- **Apply the principle of impulse and momentum to the package-cart system to determine the final velocity.**





$$m_p\vec{v}_1 + \sum \mathbf{Imp}_{1\to 2} = \left(m_p + m_c\right)\vec{v}_2$$

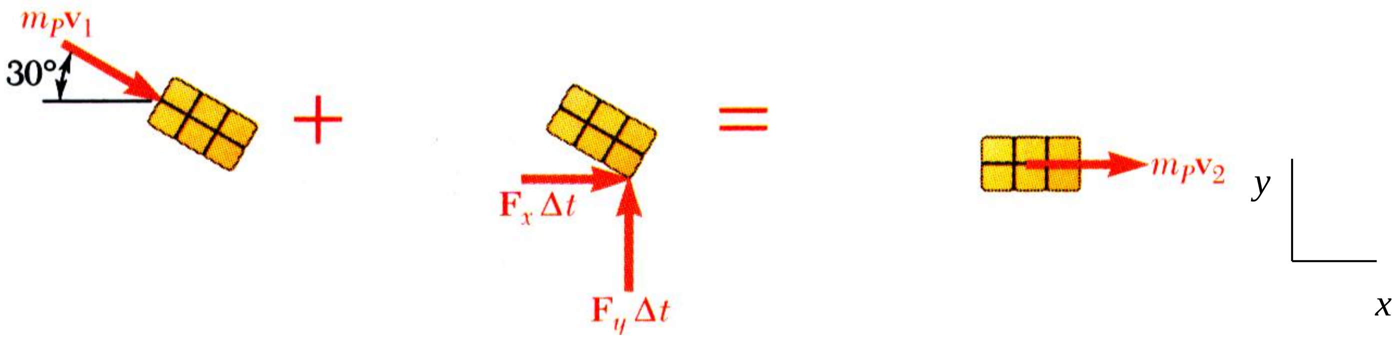
x **components:**

$$m_p v_1 \cos 30° + 0 = \left(m_p + m_c\right)v_2$$

$$(10\text{ kg})(3\text{ m/s})\cos 30° = (10\text{ kg} + 25\text{ kg})v_2$$

$$\boxed{v_2 = 0.742\text{ m/s}}$$

- **Apply the same principle to the package alone to determine the impulse exerted on it from the change in its momentum.**



$$m_p \vec{v}_1 + \sum \mathbf{Imp}_{1\to 2} = m_p \vec{v}_2$$

x components:

$$m_p v_1 \cos 30° + F_x \Delta t = m_p v_2$$

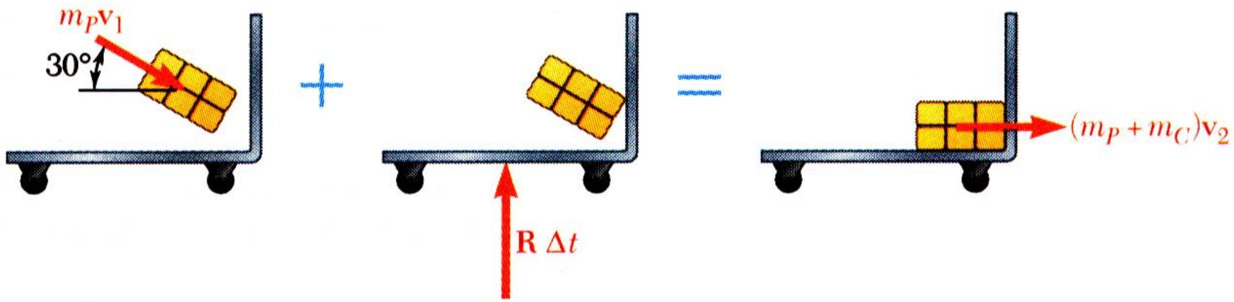
$$(10\text{ kg})(3\text{ m/s})\cos 30° + F_x \Delta t = (10\text{ kg})v_2 \qquad F_x \Delta t = -18.56\text{ N}\cdot\text{s}$$

y components:

$$-m_p v_1 \sin 30° + F_y \Delta t = 0$$

$$-(10\text{ kg})(3\text{ m/s})\sin 30° + F_y \Delta t = 0 \qquad F_y \Delta t = 15\text{ N}\cdot\text{s}$$

$$\boxed{\sum \mathbf{Imp}_{1\to 2} = \vec{F}\Delta t = (-18.56\text{ N}\cdot\text{s})\,\vec{i} + (15\text{ N}\cdot\text{s})\,\vec{j} \qquad F\Delta t = 23.9\text{ N}\cdot\text{s}}$$



To determine the fraction of energy lost,

$$T_1 = \tfrac{1}{2} m_p v_1^2 = \tfrac{1}{2}(10\text{ kg})(3\,\text{m/s})^2 = 45\text{ J}$$

$$T_2 = \tfrac{1}{2}\left(m_p + m_c\right)v_2^2 = \tfrac{1}{2}(10\text{ kg} + 25\text{ kg})(0.742\,\text{m/s})^2 = 9.63\text{ J}$$

$$\boxed{\frac{T_1 - T_2}{T_1} = \frac{45\text{ J} - 9.63\text{ J}}{45\text{ J}} = 0.786}$$

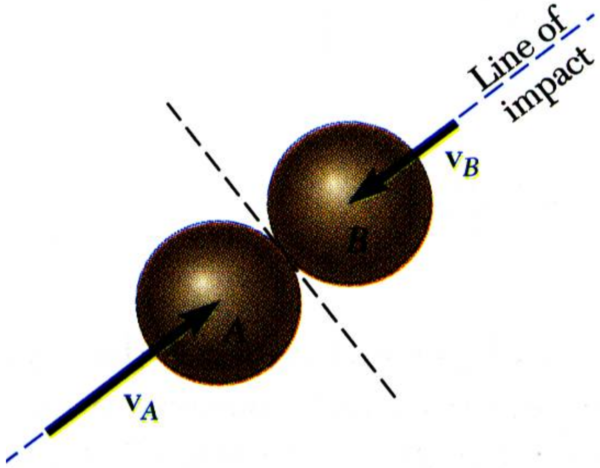
REFLECT and THINK:

Except in the purely theoretical case of a “perfectly elastic” collision, mechanical energy is never conserved in a collision between two objects, even though linear momentum may be conserved. Note that, in this problem, momentum was conserved in the x direction but was not conserved in the y direction because of the vertical impulse on the wheels of the cart. Whenever you deal with an impact, you need to use impulse-momentum methods.

13.4 Impact

The coefficient of restitution is used to characterize the “bounciness” of different sports equipment. The U.S. Golf Association limits the COR of golf balls at 0.83

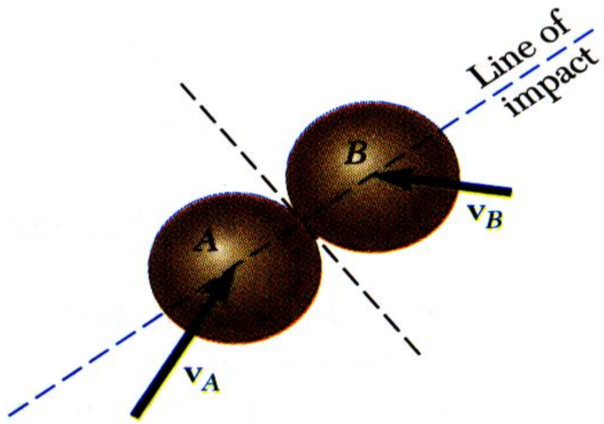
©Tom Miles



- *Impact:* **Collision between two bodies which occurs during a small time interval and during which the bodies exert large forces on each other.**
- *Line of Impact:* **Common normal to the surfaces in contact during impact.**

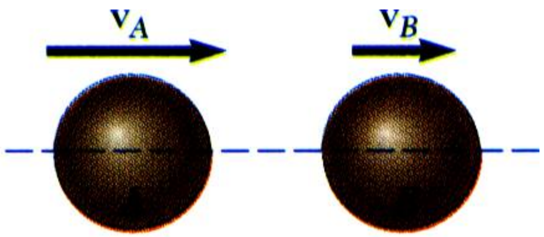
Central Impact: **Impact for which the mass centers of the two bodies lie on the line of impact; otherwise, it is an** ***eccentric impact..***

Direct Central Impact

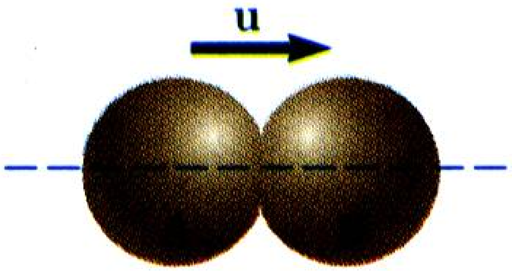


- *Direct Impact:* **Impact for which the velocities of the two bodies are directed along the line of impact.**
- *Oblique Impact:* **Impact for which one or both of the bodies move along a line other than the line of impact.**

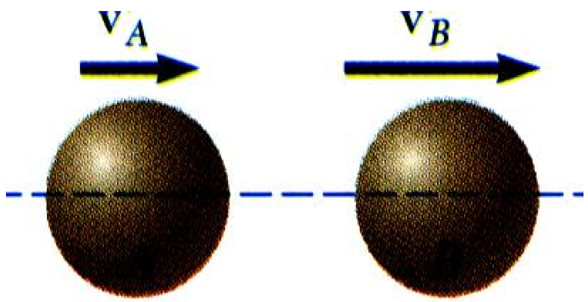
13.4 A Direct Central Impact



- **Bodies moving in the same straight line,** ***$v_A > v_B$.***
- **Upon impact the bodies undergo a** ***period of deformation,*** **at the end of which, they are in contact and moving at a common velocity.**



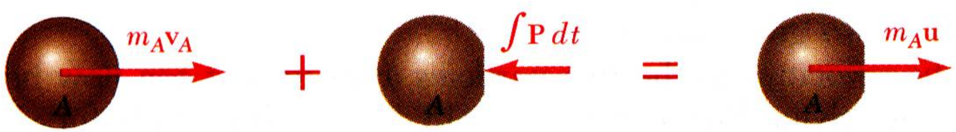
- **A** ***period of restitution*** **follows during which the bodies either regain their original shape or remain permanently deformed.**



- **Wish to determine the final velocities of the two bodies. The total momentum of the two body system is preserved,**

$$m_A v_A + m_B v_B = m_B v'_B + m_B v'_B$$

- **A second relation between the final velocities is required.**

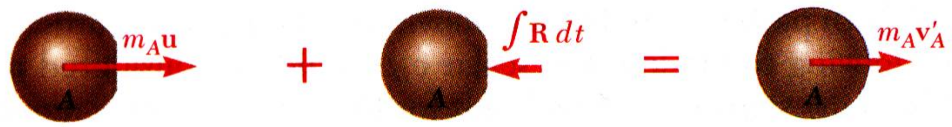


$$e = coefficient\ of\ restitution$$

$$= \frac{\int R dt}{\int P dt} = \frac{u - v'_A}{v_A - u}$$

$$0 \le e \le 1$$

- **Period of deformation:** $$m_A v_A - \int P dt = m_A u$$



- **Period of restitution:** $$m_A u - \int R dt = m_A v'_A$$

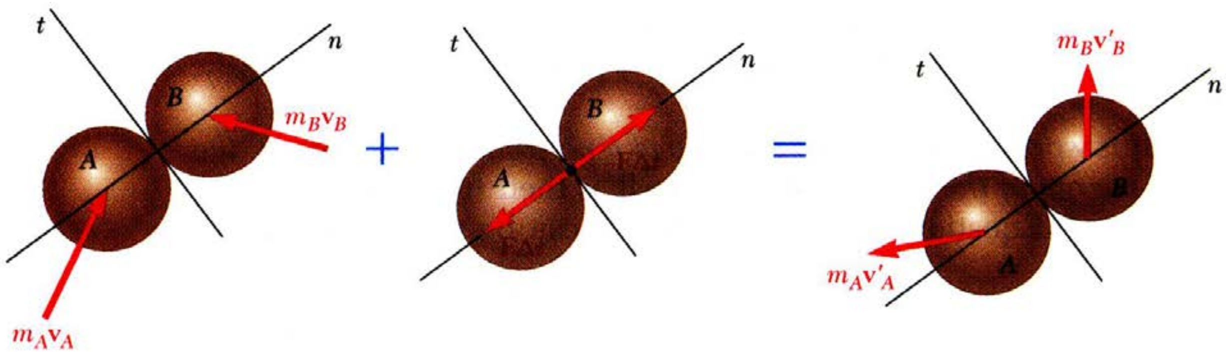
$$e = \frac{v'_B - u}{u - v_B}$$

- **A similar analysis of particle *B* yields**

- **Combining the relations leads to the desired second relation between the final velocities.** $$v'_B - v'_A = e(v_A - v_B)$$
- ***Perfectly plastic impact, e* = 0:** $v'_B = v'_A = v'$ $\quad m_A v_A + m_B v_B = (m_A + m_B)v'$
- ***Perfectly elastic impact, e* = 1:** $$v'_B - v'_A = v_A - v_B$$

  Total energy and total momentum conserved.

13.4B Oblique Central Impact



- Final velocities are unknown in magnitude and direction. Four equations are required.

- No tangential impulse component; tangential component of momentum for each particle is conserved.

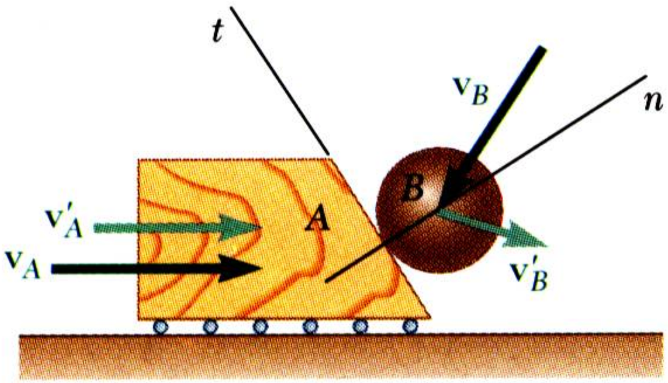
$$(v_A)_t = (v'_A)_t \qquad (v_B)_t = (v'_B)_t$$

- Normal component of total momentum of the two particles is conserved.

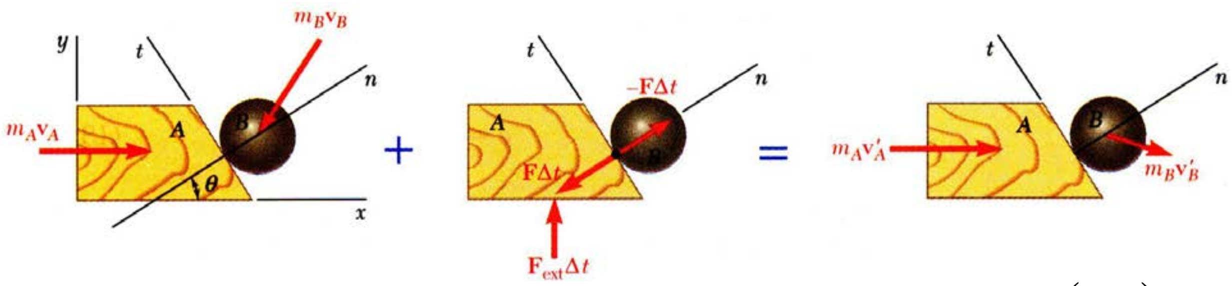
$$m_A(v_A)_n + m_B(v_B)_n = m_A(v'_A)_n + m_B(v'_B)_n$$

- Normal components of relative velocities before and after impact are related by the coefficient of restitution.

$$(v'_B)_n - (v'_A)_n = e\left[(v_A)_n - (v_B)_n\right]$$



- **Block constrained to move along horizontal surface.**
- **Impulses from internal forces** $\vec{F}$ and $-\vec{F}$ **along the *n* axis and from external force** $\vec{F}_{ext}$ **exerted by horizontal surface and directed along the vertical to the surface.**
- **Final velocity of ball unknown in direction and magnitude and unknown final block velocity magnitude.   Three equations required.**



- **Tangential momentum of ball is conserved.** $$(v_B)_t = (v'_B)_t$$

- **Total horizontal momentum of block and ball is conserved.**

$$m_A(v_A) + m_B(v_B)_x = m_A(v'_A) + m_B(v'_B)_x$$

- **Normal component of relative velocities of block and ball are related by coefficient of restitution.**

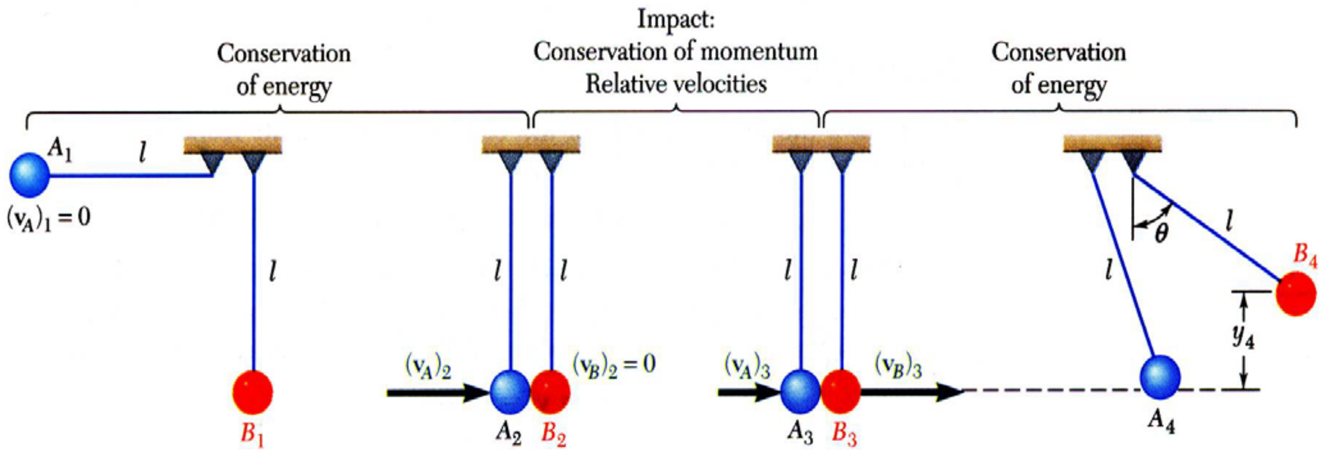
$$(v'_B)_n - (v'_A)_n = e\left[(v_A)_n - (v_B)_n\right]$$

- **Note: Validity of last expression does not follow from previous relation for the coefficient of restitution. A similar but separate derivation is required.**

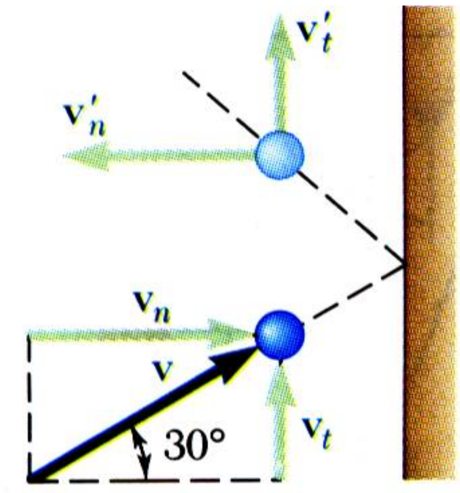
13.4C Problems Involving Multiple Principles

- Three methods for the analysis of kinetics problems:
  - Direct application of Newton's second law
  - Method of work and energy
  - Method of impulse and momentum

- Select the method best suited for the problem or part of a problem under consideration.



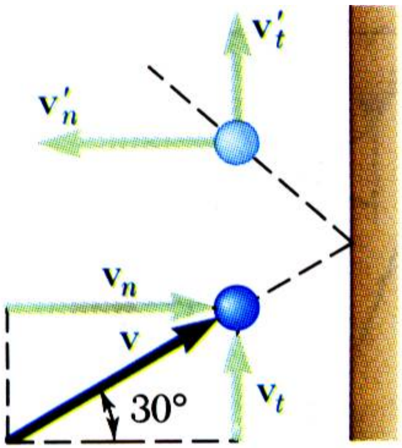
Sample Problem 13.19



A ball is thrown against a frictionless, vertical wall. Immediately before the ball strikes the wall, its velocity has a magnitude *v* and forms angle of 30° with the horizontal. Knowing that *e* = 0.90, determine the magnitude and direction of the velocity of the ball as it rebounds from the wall.

STRATEGY:

- Resolve ball velocity into components normal and tangential to wall.
- Impulse exerted by the wall is normal to the wall. Component of ball momentum tangential to wall is conserved.
- Assume that the wall has infinite mass so that wall velocity before and after impact is zero. Apply coefficient of restitution relation to find change in normal relative velocity between wall and ball, i.e., the normal ball velocity.



MODELING and ANALYSIS:

Resolve ball velocity into components parallel and perpendicular to wall.

$$v_n = v\cos 30° = 0.866v \qquad v_t = v\sin 30° = 0.500v$$

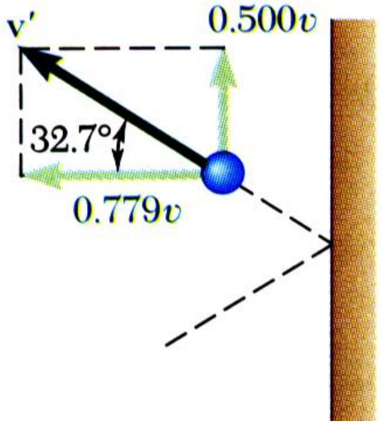
Component of ball momentum tangential to wall is conserved.

$$v'_t = v_t = 0.500v$$

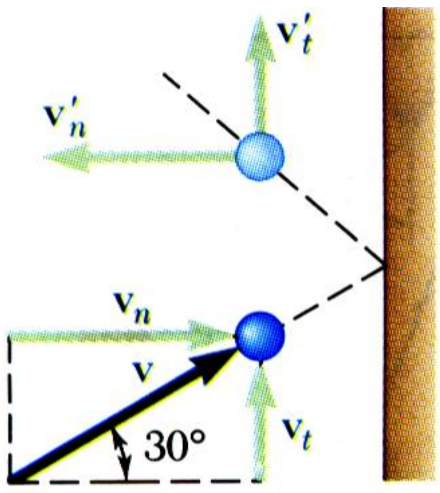
Apply coefficient of restitution relation with zero wall velocity.

$$0 - v'_n = e(v_n - 0)$$

$$v'_n = -0.9(0.866v) = -0.779v$$



$$\boxed{\begin{aligned} \vec{v}' &= -0.779v\,\vec{\lambda}_n + 0.500v\,\vec{\lambda}_t \\ v' &= 0.926v \quad \tan^{-1}\left(\frac{0.779}{0.500}\right) = 32.7° \end{aligned}}$$

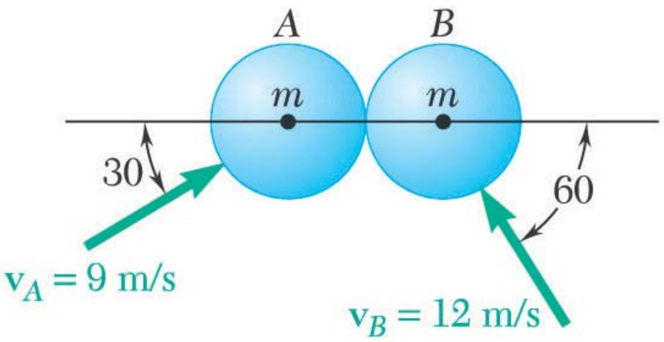


REFLECT and THINK:

Tests similar to this are done to make sure that sporting equipment—such as tennis balls, golf balls, and basketballs—are consistent and fall within certain specifications. Testing modern golf balls

and clubs shows that the coefficient of restitution actually decreases with increasing club speed (from about 0.84 at a speed of 145 kmph to about 0.80 at club speeds of 210 kmph).

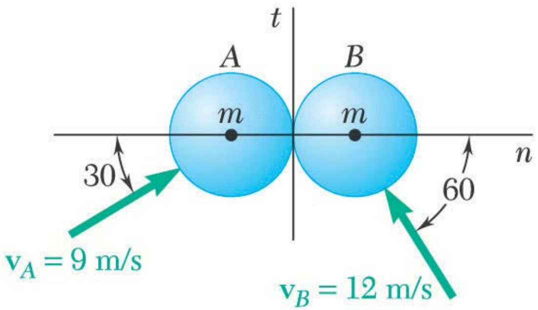
Sample Problem 13.20



The magnitude and direction of the velocities of two identical frictionless balls before they strike each other are as shown. Assuming *e* = 0.9, determine the magnitude and direction of the velocity of each ball after the impact.

STRATEGY:

- Resolve the ball velocities into components normal and tangential to the contact plane.
- Tangential component of momentum for each ball is conserved.
- Total normal component of the momentum of the two ball system is conserved.
- The normal relative velocities of the balls are related by the coefficient of restitution.
- Solve the last two equations simultaneously for the normal velocities of the balls after the impact.

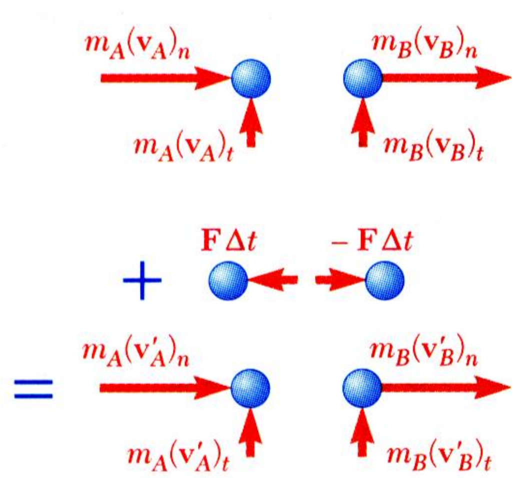


MODELING and ANALYSIS:

- **Resolve the ball velocities into components normal and tangential to the contact plane.**

$$\left(v_A\right)_n = v_A \cos 30° = 7.79\ \mathrm{m/s} \qquad \left(v_A\right)_t = v_A \sin 30° = 4.5\ \mathrm{m/s}$$

$$\left(v_B\right)_n = -v_B \cos 60° = -6\ \mathrm{m/s} \qquad \left(v_B\right)_t = v_B \sin 60° = 10.39\ \mathrm{m/s}$$



- **Tangential component of momentum for each ball is conserved.**

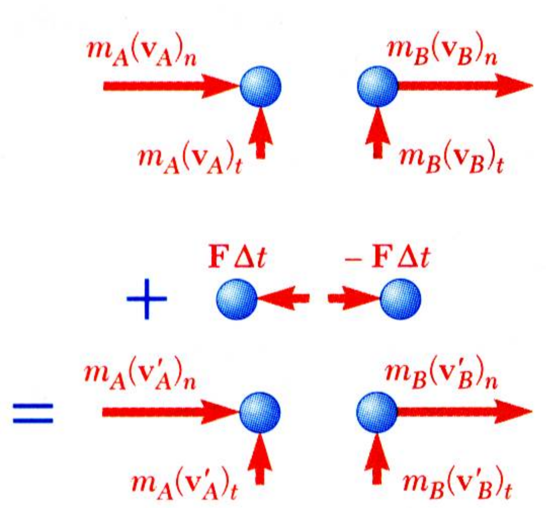
$$\left(v'_A\right)_t = \left(v_A\right)_t = 4.5\ \mathrm{m/s} \qquad \left(v'_B\right)_t = \left(v_B\right)_t = 10.39\ \mathrm{m/s}$$

- **Total normal component of the momentum of the two ball system is conserved.**

$$m_A\left(v_A\right)_n + m_B\left(v_B\right)_n = m_A\left(v'_A\right)_n + m_B\left(v'_B\right)_n$$

$$m(7.79) + m(-6) = m\left(v'_A\right)_n + m\left(v'_B\right)_n$$

$$\left(v'_A\right)_n + \left(v'_B\right)_n = 1.79$$



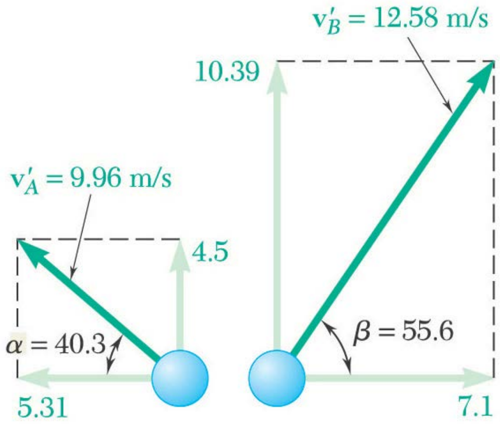
- **The normal relative velocities of the balls are related by the coefficient of restitution.**

$$\left(v'_A\right)_n - \left(v'_B\right)_n = e\left[\left(v_A\right)_n - \left(v_B\right)_n\right]$$

$$= 0.90\left[7.79 - (-6)\right] = 12.41$$

- **Solve the last two equations simultaneously for the normal velocities of the balls after the impact.**

$$\left(v'_A\right)_n = -5.31\ \mathrm{m/s} \qquad \left(v'_B\right)_n = +7.1\ \mathrm{m/s}$$

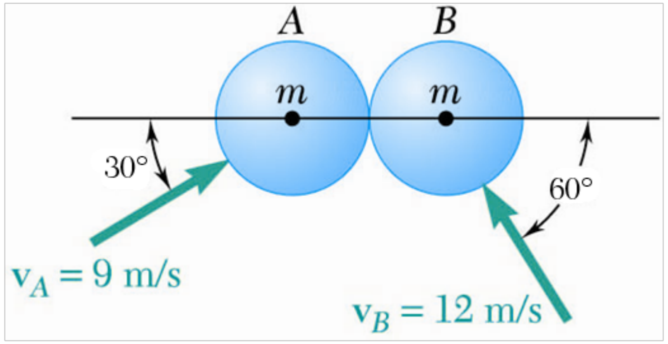


$$\vec{v}'_A = -5.31\vec{\lambda}_t + 4.5\vec{\lambda}_n$$

$$v'_A = 9.96\ \mathrm{m/s} \quad \tan^{-1}\left(\frac{4.5}{5.31}\right) = 40.3^\circ$$

$$\vec{v}'_B = 7.1\vec{\lambda}_t + 10.39\vec{\lambda}_n$$

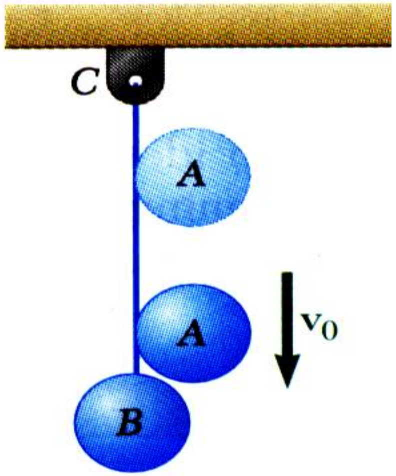
$$v'_B = 12.58\ \mathrm{m/s} \quad \tan^{-1}\left(\frac{10.39}{7.1}\right) = 55.6^\circ$$



REFLECT and THINK:

- **Rather than choosing your system to be both balls, you could have applied impulse-momentum along the line of impact for each ball individually.**
- **This would have resulted in two equations and one additional unknown, FΔt. To determine the impulsive force F, you would need to be given the time for the impact, Δt.**

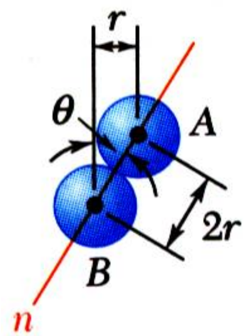
Sample Problem 13.21



Ball *B* is hanging from an inextensible cord. An identical ball *A* is released from rest when it is just touching the cord and acquires a velocity v_0 before striking ball *B*. Assuming perfectly elastic impact (e = 1) and no friction, determine the velocity of each ball immediately after impact.

STRATEGY:

- Determine orientation of impact line of action.
- The momentum component of ball *A* tangential to the contact plane is conserved.
- The total horizontal momentum of the two ball system is conserved.
- The relative velocities along the line of action before and after the impact are related by the coefficient of restitution.
- Solve the last two expressions for the velocity of ball *A* along the line of action and the velocity of ball *B* which is horizontal.

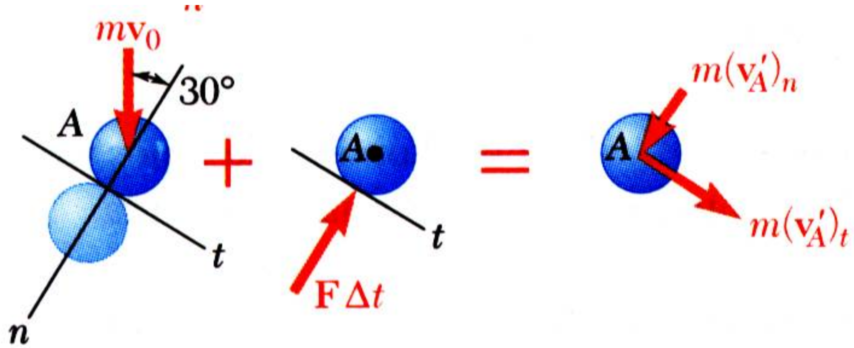


MODELING and ANALYSIS:

$$\sin\theta = \frac{r}{2r} = 0.5$$

$$\theta = 30^\circ$$

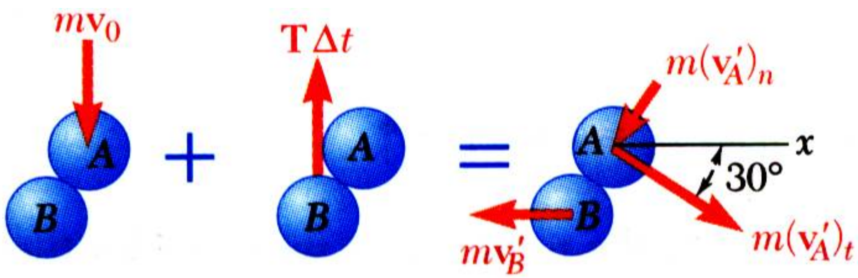
- **Determine orientation of impact line of action.**
- **The momentum component of ball *A* tangential to the contact plane is conserved.**



$$m\vec{v}_A + \vec{F}\Delta t = m\vec{v}'_A$$

$$mv_0 \sin 30^\circ + 0 = m(v'_A)_t$$

$$(v'_A)_t = 0.5v_0$$



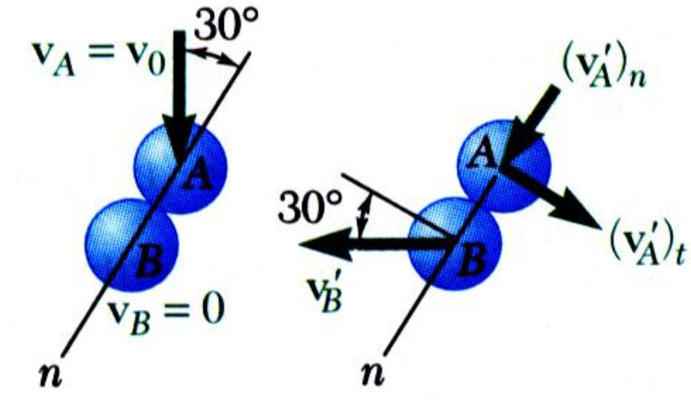
- **The total horizontal (*x* component) momentum of the two ball system is conserved.**

$$m\vec{v}_A + \vec{T}\Delta t = m\vec{v}'_A + m\vec{v}'_B$$

$$0 = m(v'_A)_t \cos 30^\circ - m(v'_A)_n \sin 30^\circ - mv'_B$$

$$0 = (0.5v_0)\cos 30^\circ - (v'_A)_n \sin 30^\circ - v'_B$$

$$0.5(v'_A)_n + v'_B = 0.433v_0$$



- The relative velocities along the line of action before and after the impact are related by the coefficient of restitution.

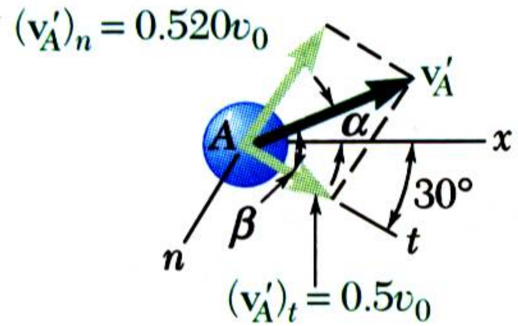
$$(v_B')_n - (v_A')_n = e\left[(v_A)_n - (v_B)_n\right]$$

$$v_B' \sin 30° - (v_A')_n = v_0 \cos 30° - 0$$

$$0.5 v_B' - (v_A')_n = 0.866 v_0$$

- Solve the last two expressions for the velocity of ball *A* along the line of action and the velocity of ball *B* which is horizontal.

$$(v_A')_n = -0.520 v_0 \qquad v_B' = 0.693 v_0$$

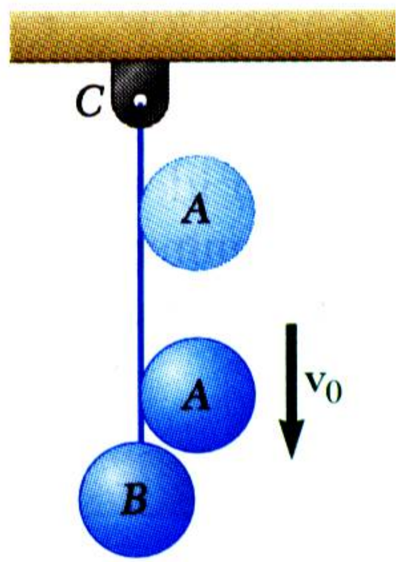


$$\vec{v}_A' = 0.5 v_0 \vec{\lambda}_t - 0.520 v_0 \vec{\lambda}_n$$

$$v_A' = 0.721 v_0 \qquad \beta = \tan^{-1}\left(\frac{0.52}{0.5}\right) = 46.1°$$

$$\alpha = 46.1° - 30° = 16.1°$$

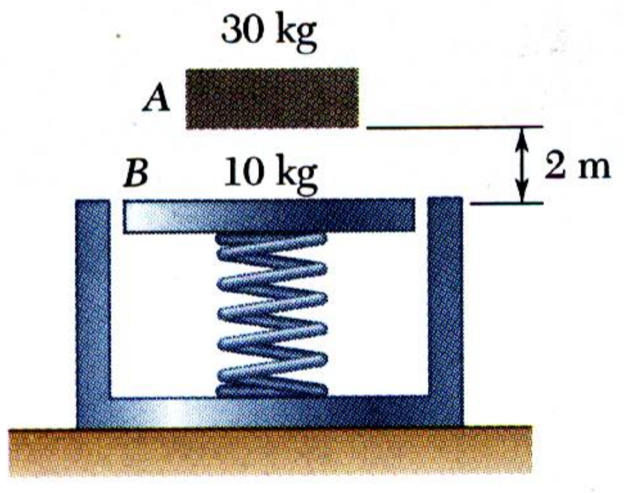
$$v_B' = 0.693 v_0 \leftarrow$$



REFLECT and THINK

Since e = 1, the impact between A and B is perfectly elastic. Therefore, rather than using the coefficient of restitution, you could have used the conservation of energy as your final equation.

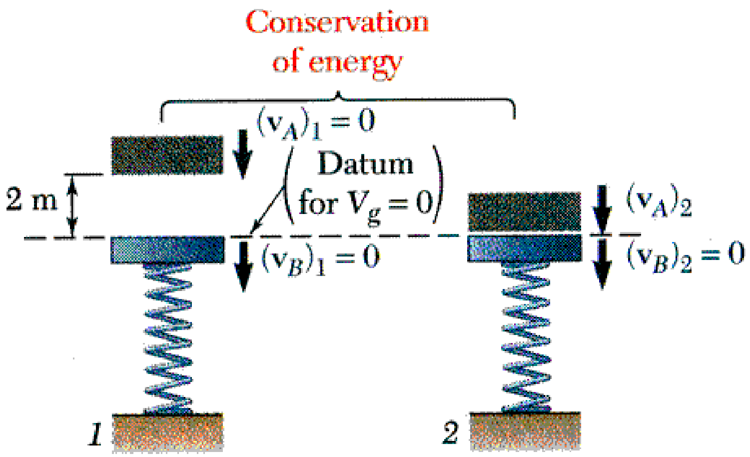
Sample Problem 13.22

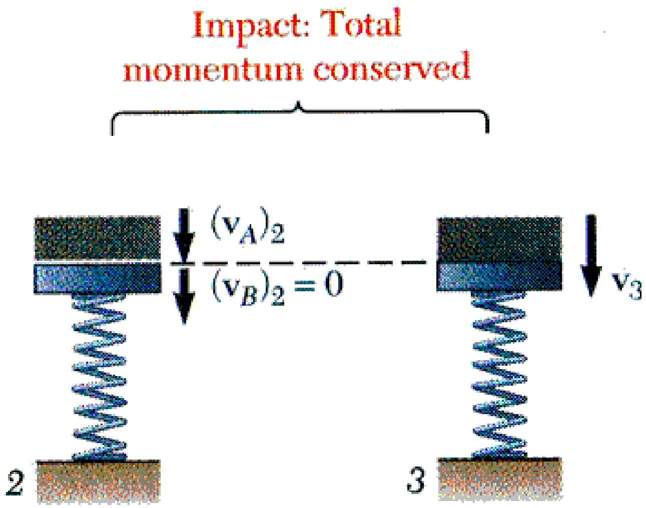


A 30 kg block is dropped from a height of 2 m onto the 10 kg pan of a spring scale. Assuming the impact to be perfectly plastic, determine the maximum deflection of the pan. The constant of the spring is *k* = 20 kN/m.

STRATEGY:

- Apply the principle of conservation of energy to determine the velocity of the block at the instant of impact.
- Since the impact is perfectly plastic, the block and pan move together at the same velocity after impact. Determine that velocity from the requirement that the total momentum of the block and pan is conserved.
- Apply the principle of conservation of energy to determine the maximum deflection of the spring.





MODELING and ANALYSIS:

- **Apply principle of conservation of energy to determine velocity of the block at instant of impact.**

$$T_1 = 0 \qquad V_1 = W_A y = (30)(9.81)(2) = 588\,\mathrm{J}$$

$$T_2 = \tfrac{1}{2} m_A (v_A)_2^2 = \tfrac{1}{2}(30)(v_A)_2^2 \qquad V_2 = 0$$

$$T_1 + V_1 = T_2 + V_2$$

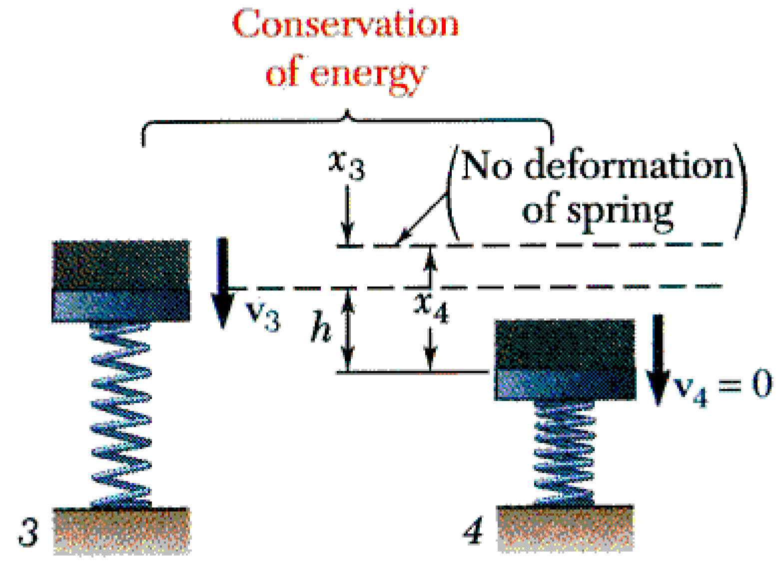
$$0 + 588\,\mathrm{J} = \tfrac{1}{2}(30)(v_A)_2^2 + 0 \qquad (v_A)_2 = 6.26\,\mathrm{m/s}$$

- **Determine velocity after impact from requirement that total momentum of the block and pan is conserved.**

$$m_A (v_A)_2 + m_B (v_B)_2 = (m_A + m_B) v_3$$

$$(30)(6.26) + 0 = (30 + 10) v_3 \qquad v_3 = 4.70\,\mathrm{m/s}$$

- **Apply the principle of conservation of energy to determine the maximum deflection of the spring.**



Initial spring deflection due to pan weight:

$$x_3 = \frac{W_B}{k} = \frac{(10)(9.81)}{20\times 10^3} = 4.91\times 10^{-3}\,\mathrm{m}$$

$$T_3 = \tfrac{1}{2}(m_A + m_B)v_3^2 = \tfrac{1}{2}(30+10)(4.7)^2 = 442\ \mathrm{J}$$

$$V_3 = V_g + V_e$$

$$= 0 + \tfrac{1}{2}kx_3^2 = \tfrac{1}{2}\left(20\times 10^3\right)\left(4.91\times 10^{-3}\right)^2 = 0.241\ \mathrm{J}$$

$$T_4 = 0$$

$$V_4 = V_g + V_e = (W_A + W_B)(-h) + \tfrac{1}{2}kx_4^2$$

$$= -392(x_4 - x_3) + \tfrac{1}{2}\left(20\times 10^3\right)x_4^2$$

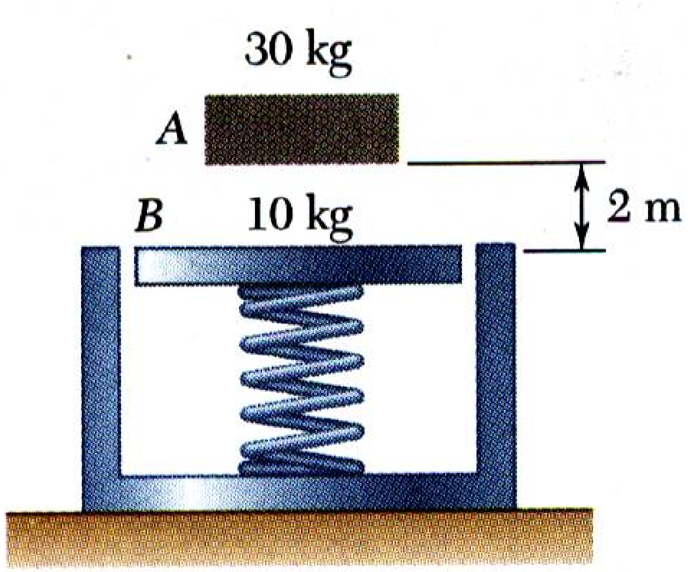
$$= -392\left(x_4 - 4.91\times 10^{-3}\right) + \tfrac{1}{2}\left(20\times 10^3\right)x_4^2$$

$$T_3 + V_3 = T_4 + V_4$$

$$442 + 0.241 = 0 - 392\left(x_4 - 4.91\times 10^{-3}\right) + \tfrac{1}{2}\left(20\times 10^3\right)x_4^2$$

$$x_4 = 0.230\ \mathrm{m}$$

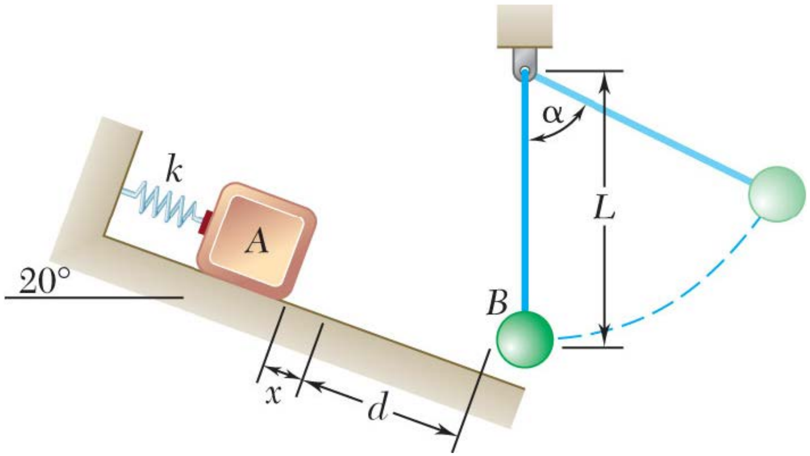
$$h = x_4 - x_3 = 0.230\ \mathrm{m} - 4.91\times 10^{-3}\ \mathrm{m} \qquad \boxed{h = 0.225\ \mathrm{m}}$$



REFLECT and THINK:

The spring constant for this scale is pretty large, but the block is fairly massive and is dropped from a height of 2 m. From this perspective, the deflection seems reasonable. We included the spring in the system so we could treat it as an energy term rather than finding the work of the spring force.

Sample Problem 13.23



A 2-kg block *A* is pushed up against a spring compressing it a distance *x*= 0.1 m. The block is then released from rest and slides down the 20° incline until it strikes a 1-kg sphere *B*, which is suspended from a 1 m inextensible rope. The spring constant *k*=800 N/m, the coefficient of friction between *A* and the ground is 0.2, the distance *A* slides from the unstretched length of the spring d=1.5 m, and the coefficient of restitution between *A* and *B* is 0.8. When α =40°, find (*a*) the speed of *B* (*b*) the tension in the rope.

STRATEGY:

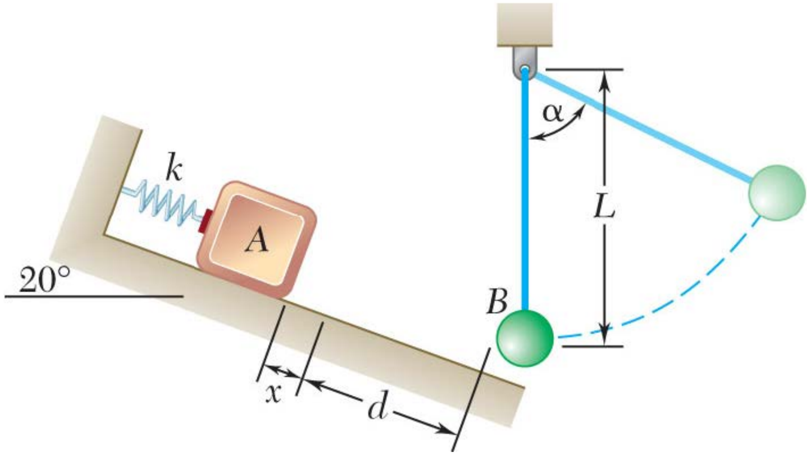
- This is a multiple step problem. Formulate your overall approach.
- Use work-energy to find the velocity of the block just before impact
- Use conservation of momentum to determine the speed of ball B after the impact
- Use work energy to find the velocity at a
- Use Newton's 2nd Law to find tension in the rope

MODELING and ANALYSIS:

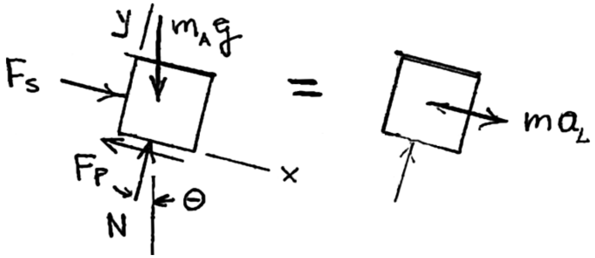
Given: m_A= 2-kg m_B= 1-kg, *k*= 800 N/m, m_A =0.2, e= 0.8

Find (*a*) v_B (*b*) T_{rope}

- **Use work-energy to find the velocity of the block just before impact**



Determine the friction force acting on the block A



Sum forces in the y-direction

$$\sum F_y = 0:$$

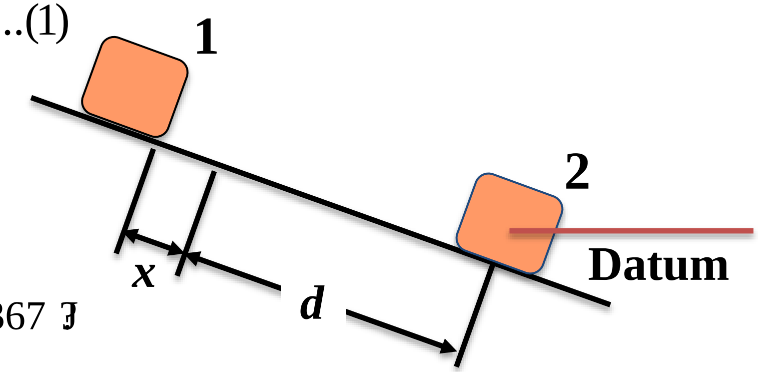
$$N - m_A g \cos\theta = 0$$

Solve for N

$$N = m_A g \cos\theta$$
$$= (2)(9.81)\cos 20°$$
$$= 18.4368 \text{ N}$$
$$F_f = \mu_k N = (0.2)(18.4368)$$
$$= 3.6874 \text{ N}$$

Set your datum, use work-energy to determine v_A at impact.

$$T_1 + (V_1)_e + (V_1)_g + U_{1\to 2} = T_2 + (V_2)_e + (V_2)_g \text{.....(1)}$$



Determine values for each term.

$$T_1 = 0, \quad (V_1)_e = \frac{1}{2} k x_1^2 = \frac{1}{2}(800)(0.1)^2 = 4.00 \text{ J}$$

$$(V_1)_g = m_A g h_1 = m_A g (x+d)\sin\theta = (2)(9.81)(1.6)\sin 20^\circ = 10.7367 \text{ J}$$

$$U_{1\to 2} = -F_f (x+d) = -(3.6874)(1.6) = -5.8998 \text{ J}$$

$$T_2 = \frac{1}{2} m_A v_A^2 = \frac{1}{2}(1)(v_A^2) = 1.000\, v_A^2 \qquad V_2 = 0$$

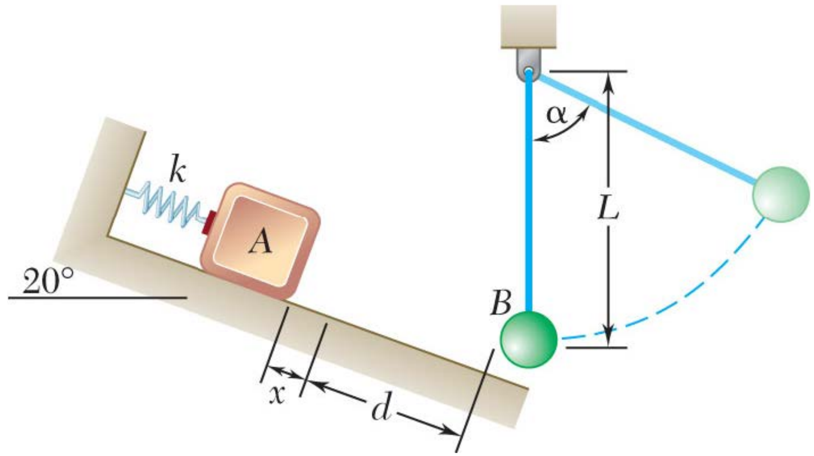
Substitute into the Work-Energy equation and solve for v_A

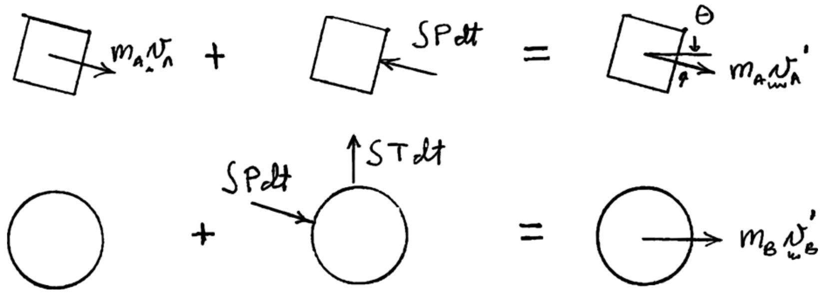
$$T_1 + V_1 + U_{1\to 2} = T_2 + V_2: \quad 0 + 4.00 + 10.7367 - 5.8998 = 1.000 v_A^2 + 0$$

$$v_A^2 = 8.8369 \text{ m}^2/\text{s}^2$$

$$\boxed{\mathbf{v}_A = 2.97 \text{ m/s}}$$

- **Use conservation of momentum to determine the speed of ball B after the impact**
- **Draw the impulse diagram**





Apply conservation of momentum in the x direction

$$m_A v_A \cos\theta + 0 = m_A v_A' \cos\theta + m_B v_B$$

$$(2)(2.9727)\cos 20° = 2v_A' \cos 20^{\circ} + (1.00)v_B \quad \textbf{(2)}$$

Use the relative velocity/coefficient of restitution equation

$$(v_B')_n - (v_A')_n = e[(v_B)_n - (v_A)_n]$$

$$v_B' \cos\theta - v_A' = e[v_A - 0]$$

$$v_B' \cos 20° - v_A' = (0.8)(2.9727) \quad \textbf{(3)}$$

Solve (2) and (3) simultaneously $v_A' = 1.0382 \text{ m/s}$ $v_B' = 3.6356 \text{ m/s}$

- **Use work energy to find the velocity at a**

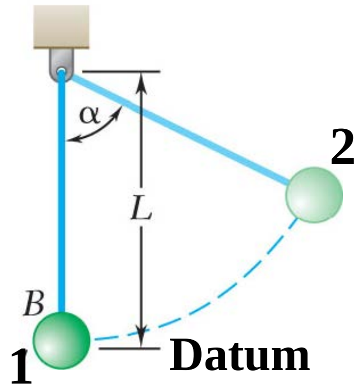
Set datum, use Work-Energy to determine v_B at a= 40°

$$T_1 + (V_1)_e + (V_1)_g + U_{1\to 2} = T_2 + (V_2)_e + (V_2)_g$$

Determine values for each term.

$$T_1 = \frac{1}{2} m_B (v'_B)^2 \quad V_1 = 0$$

$$T_2 = \frac{1}{2} m_B v_2^2 \qquad V_2 = m_B g h_2 = m_B g l (1 - \cos\alpha)$$



Substitute into the Work-Energy equation and solve for v_A

$$T_1 + V_1 = T_2 + V_2 : \quad \frac{1}{2} m_B (v'_B)^2 + 0 = \frac{1}{2} m_B v_2^2 + m_B g (1 - \cos\alpha)$$

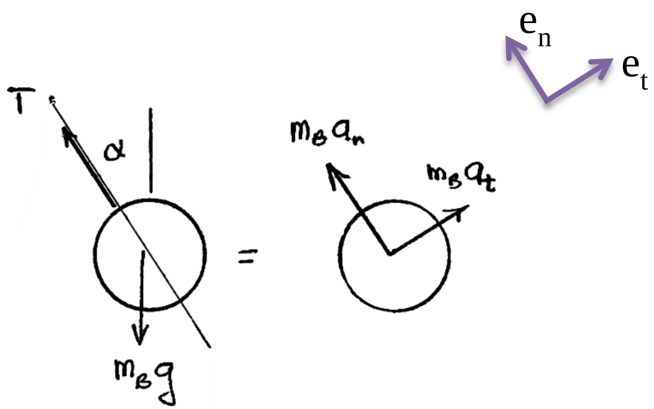
$$v_2^2 = (v'_B)^2 - 2gl(1 - \cos\alpha)$$

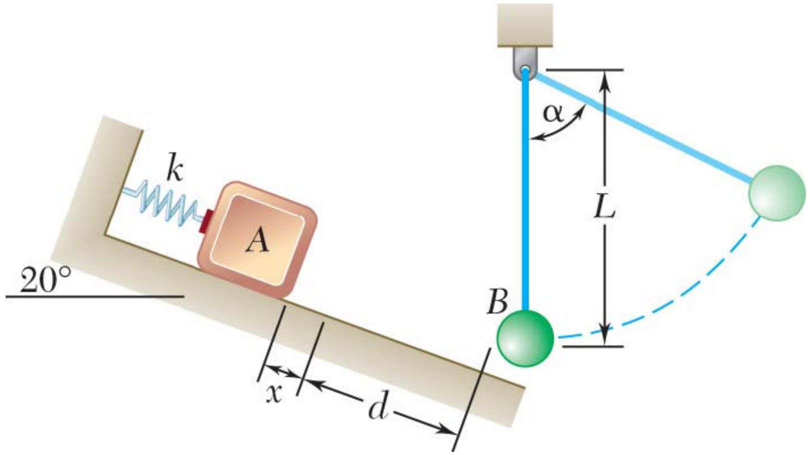
$$= (3.6356)^2 - (2)(9.81)(1 - \cos 40°)$$

$$= 8.6274 \ \text{m}^2/\text{s}^2$$

$$v_2 = 2.94 \ \text{m/s}$$

- **Use Newton's 2nd Law to find tension in the rope**
- **Draw your free-body and kinetic diagrams**





- **Sum forces in the normal direction**

$$\sum F_n = m_B\, a_n: \quad T - m_B g \cos\alpha = m_B a_n$$

$$T = m_B(a_n + g\cos\alpha)$$

- **Determine normal acceleration**

$$\rho = 1.00 \text{ m}$$

$$a_n = \frac{v_2^2}{\rho} = \frac{8.6274}{1.00} = 8.6274 \text{ m/s}^2$$

- **Substitute and solve**

$$T = (1.0)(8.6274 + 9.81\cos 40°)$$

$$\boxed{T = 16.14}$$

Summary

Approaches to Kinetics Problems

Forces and Accelerations -> Newton's Second Law $\sum \vec{F} = m\vec{a}_G$

Velocities and Displacements -> Work-Energy $T_1 + U_{1\to 2} = T_2$

Velocities and Time -> Impulse-Momentum $m\vec{v}_1 + \int_{t_1}^{t_2} \vec{F}\, dt = m\vec{v}_2$