Chapter 18. Kinetics of Rigid Bodies in Three Dimensions

Introduction

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Principle of Impulse and Momentum

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Motion About a Fixed Point or a Fixed Axis

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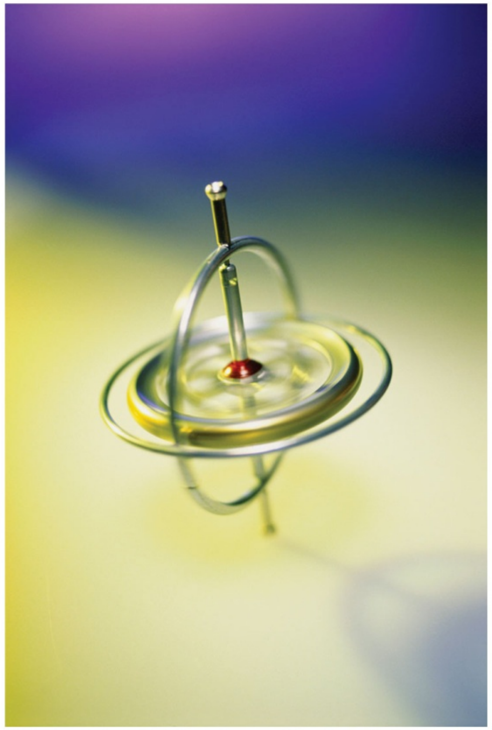
Three dimensional analyses are needed to determine the forces and moments on the gimbals of gyroscopes, the joints of robotic welders, and the supports of radio telescopes.



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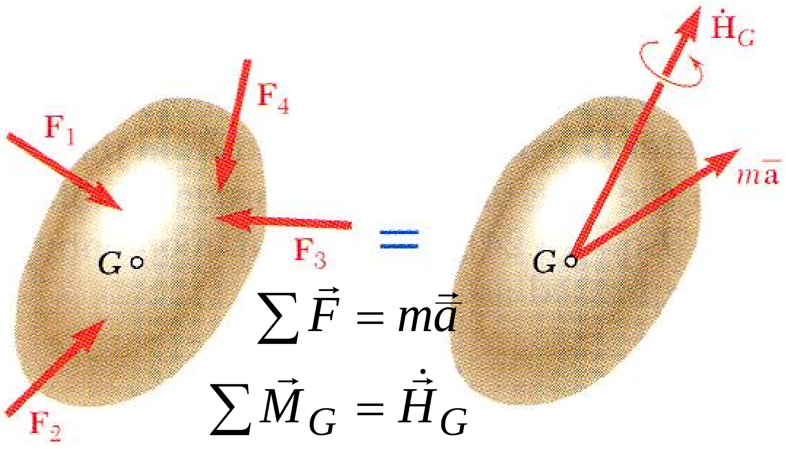
Introduction



$$\sum \vec{F} = m\vec{\bar{a}}$$

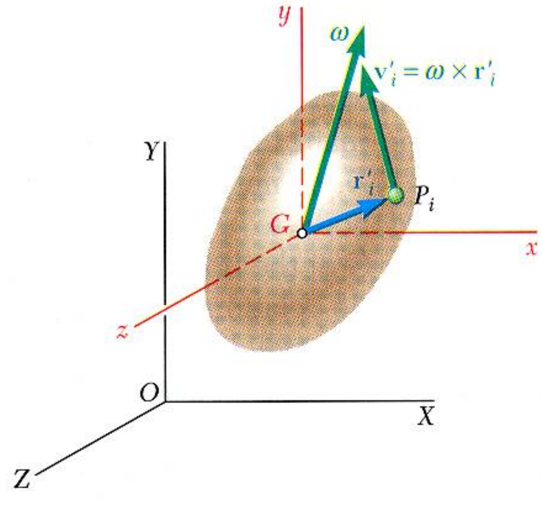
$$\sum \vec{M}_G = \dot{\vec{H}}_G$$

18.1

- The fundamental relations developed for the plane motion of rigid bodies may also be applied to the general motion of three dimensional bodies.
- The relation $\vec{H}_G = \bar{I}\vec{\omega}$ which was used to determine the angular momentum of a rigid slab is not valid for general three dimensional bodies and motion.

The current chapter is concerned with evaluation of the angular momentum and its rate of change for three dimensional motion and application to effective forces, the impulse-momentum and the work-energy principles.

18.1A Angular Momentum of a Rigid Body in Three Dimensions



Angular momentum of a body about its mass center,

$$\vec{H}_G = \sum_{i=1}^{n} (\vec{r}_i' \times \vec{v}_i \Delta m_i) = \sum_{i=1}^{n} [\vec{r}_i' \times (\vec{\omega} \times \vec{r}_i') \Delta m_i]$$

The x component of the angular momentum,

$$H_x = \sum_{i=1}^{n} \left[y_i (\vec{\omega} \times \vec{r}_i')_z - z_i (\vec{\omega} \times \vec{r}_i')_y \right] \Delta m_i$$

$$= \sum_{i=1}^{n} \left[y_i \left(\omega_x y_i - \omega_y x_i \right) - z_i (\omega_z x_i - \omega_x z_i) \right] \Delta m_i$$

$$= \omega_x \sum_{i=1}^{n} \left(y_i^2 + z_i^2 \right) \Delta m_i - \omega_y \sum_{i=1}^{n} x_i y_i \Delta m_i - \omega_z \sum_{i=1}^{n} z_i x_i \Delta m_i$$

$$H_x = \omega_x \int \left(y^2 + z^2 \right) dm - \omega_y \int xy \, dm - \omega_z \int zx \, dm$$

$$= +\bar{I}_x \omega_x - \bar{I}_{xy} \omega_y - \bar{I}_{xz} \omega_z$$

$$H_y = -\bar{I}_{yx} \omega_x + \bar{I}_y \omega_y - \bar{I}_{yz} \omega_z$$

$$H_z = -\bar{I}_{zx} \omega_x - \bar{I}_{zy} \omega_y + \bar{I}_z \omega_z$$

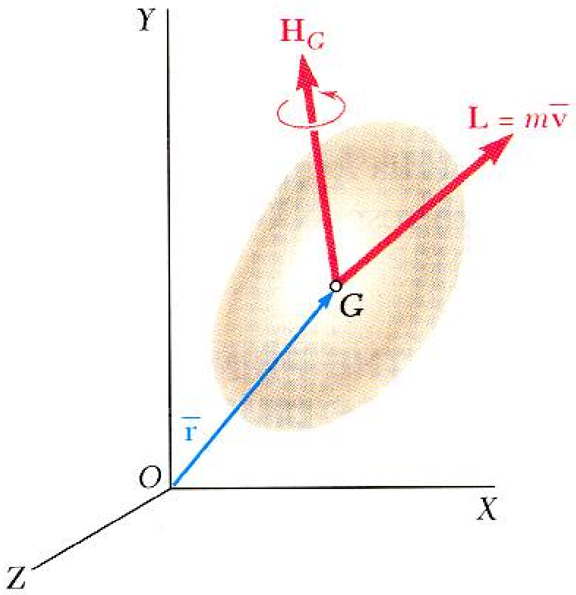
Transformation of $\vec{\omega}$ into is characterized by the inertia $\vec{H}_G$ tensor for the body,

$$\begin{pmatrix} +\bar{I}_x & -\bar{I}_{xy} & -\bar{I}_{xz} \\ -\bar{I}_{yx} & +\bar{I}_y & -\bar{I}_{yz} \\ -\bar{I}_{zx} & -\bar{I}_{zy} & +\bar{I}_z \end{pmatrix}$$

- With respect to the principal axes of inertia,

$$\begin{pmatrix} \bar{I}_{x'} & 0 & 0 \\ 0 & \bar{I}_{y'} & 0 \\ 0 & 0 & \bar{I}_{z'} \end{pmatrix} \qquad H_{x'} = \bar{I}_{x'}\omega_{x'} \quad H_{y'} = \bar{I}_{y'}\omega_{y'} \quad H_{z'} = \bar{I}_{z'}\omega_{z'}$$

The angular momentum $\vec{H}_G$ of a rigid body and its angular velocity $\vec{\omega}$ have the same direction if, and only if, $\vec{\omega}$ is directed along a principal axis of inertia.

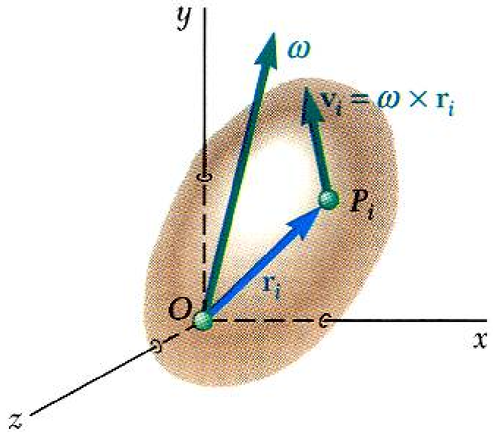


The momenta of the particles of a rigid body can be reduced to:

$$\vec{L} = \text{linear momentum}$$
$$= m\vec{\bar{v}}$$
$$\vec{H}_G = \text{angular momentum about } G$$
$$H_x = +\bar{I}_x\omega_x - \bar{I}_{xy}\omega_y - \bar{I}_{xz}\omega_z$$
$$H_y = -\bar{I}_{yx}\omega_x + \bar{I}_y\omega_y - \bar{I}_{yz}\omega_z$$
$$H_z = -\bar{I}_{zx}\omega_x - \bar{I}_{zy}\omega_y + \bar{I}_z\omega_z$$

- The angular momentum about any other given point *O* is

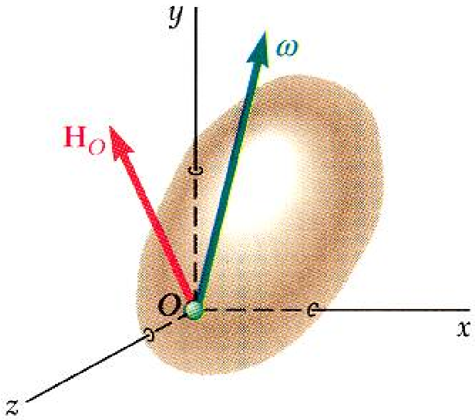
$$\vec{H}_O = \vec{\bar{r}} \times m\vec{\bar{v}} + \vec{H}_G$$



The angular momentum of a body constrained to rotate about a fixed point may be calculated from

$$\vec{H}_O = \vec{\bar{r}} \times m\vec{\bar{v}} + \vec{H}_G$$

- Or, the angular momentum may be computed directly from the moments and products of inertia with respect to the *Oxyz* frame.

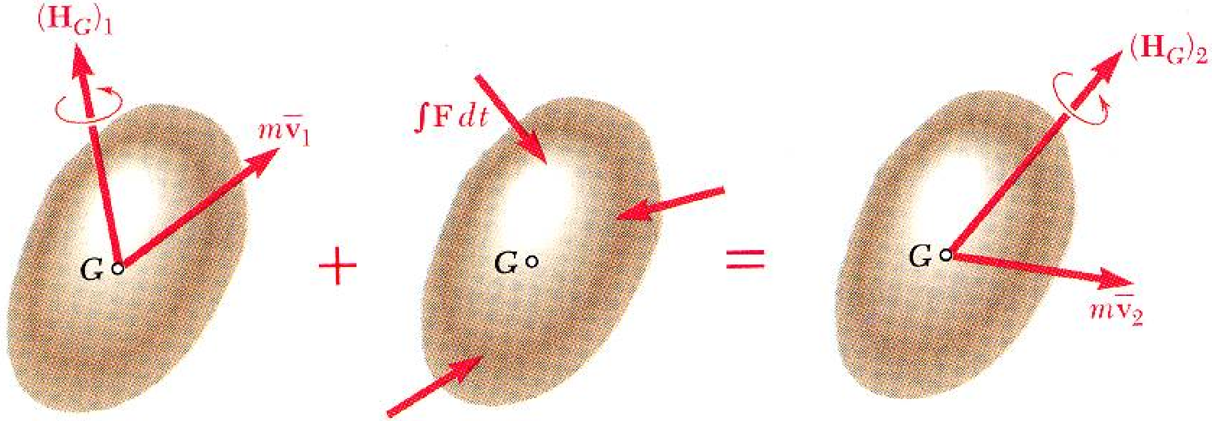


$$\vec{H}_O = \sum_{i=1}^{n} \left(\vec{r}_i \times \vec{v}_i \Delta m \right)$$

$$= \sum_{i=1}^{n} \left[\vec{r}_i \times \left(\vec{\omega} \times \vec{r}_i \right) \Delta m_i \right]$$

$$H_x = +I_x \omega_x - I_{xy}\omega_y - I_{xz}\omega_z$$
$$H_y = -I_{yx}\omega_x + I_y \omega_y - I_{yz}\omega_z$$
$$H_z = -I_{zx}\omega_x - I_{zy}\omega_y + I_z \omega_z$$

18.1B Principle of Impulse and Momentum

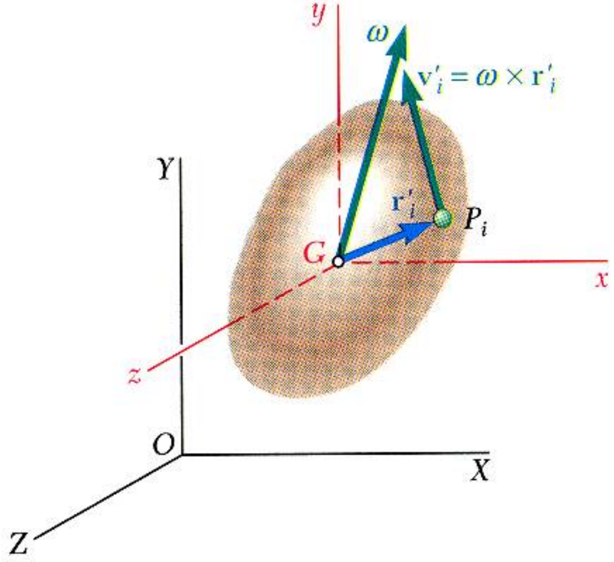


- The principle of impulse and momentum can be applied directly to the three-dimensional motion of a rigid body,

$$\textit{Syst Momenta}_1 + \textit{Syst Ext Imp}_{1\text{-}2} = \textit{Syst Momenta}_2$$

- The free-body diagram equation is used to develop component and moment equations.
- For bodies rotating about a fixed point, eliminate the impulse of the reactions at *O* by writing equation for moments of momenta and impulses about *O*.

18.1C Kinetic Energy



Kinetic energy of particles forming rigid body,

$$T = \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}\sum_{i=1}^{n} \Delta m_i \bar{v}_i'^2$$

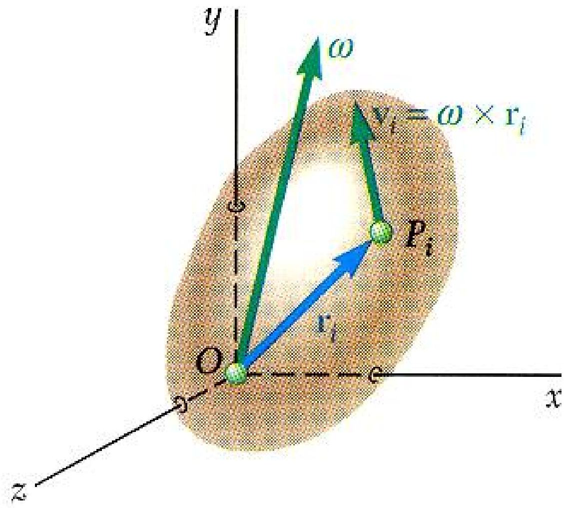
$$= \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}\sum_{i=1}^{n} \left|\vec{\omega} \times \vec{r}_i'\right|^2 \Delta m_i$$

$$= \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}(\bar{I}_x\omega_x^2 + \bar{I}_y\omega_y^2 + \bar{I}_z\omega_z^2 - 2\bar{I}_{xy}\omega_x\omega_y - 2\bar{I}_{yz}\omega_y\omega_z - 2\bar{I}_{zx}\omega_z\omega_x)$$

- If the axes correspond instantaneously with the principle axes,

$$T = \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}(\bar{I}_{x'}\omega_{x'}^2 + \bar{I}_{y'}\omega_{y'}^2 + \bar{I}_{z'}\omega_{z'}^2)$$

- With these results, the principles of work and energy and conservation of energy may be applied to the three-dimensional motion of a rigid body.



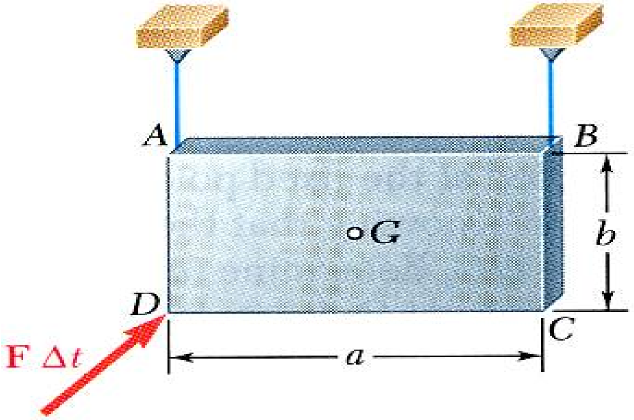
- Kinetic energy of a rigid body with a fixed point,

$$T = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 - 2I_{xy}\omega_x\omega_y - 2I_{yz}\omega_y\omega_z - 2I_{zx}\omega_z\omega_x)$$

- If the axes *Oxyz* correspond instantaneously with the principle axes *Ox'y'z'*,

$$T = \frac{1}{2}(I_{x'}\omega_{x'}^2 + I_{y'}\omega_{y'}^2 + I_{z'}\omega_{z'}^2)$$

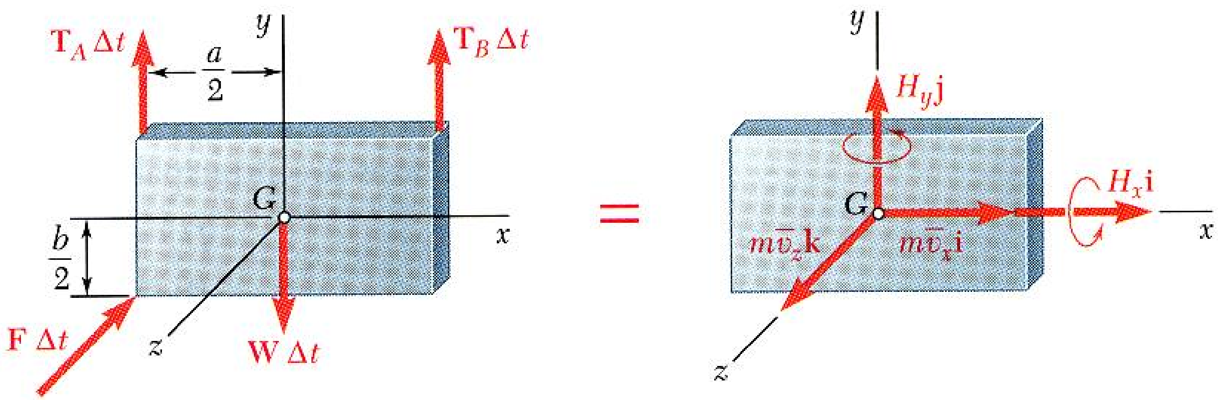
Sample Problem 18.1



Rectangular plate of mass *m* that is suspended from two wires is hit at *D* in a direction perpendicular to the plate. Immediately after the impact, determine *a*) the velocity of the mass center *G*, and *b*) the angular velocity of the plate.

STRATEGY:

- Apply the principle of impulse and momentum. Since the initial momenta is zero, the system of impulses must be equivalent to the final system of momenta.
- Assume that the supporting cables remain taut such that the vertical velocity and the rotation about an axis normal to the plate is zero.
- Principle of impulse and momentum yields to two equations for linear momentum and two equations for angular momentum.
- Solve for the two horizontal components of the linear and angular velocity vectors.



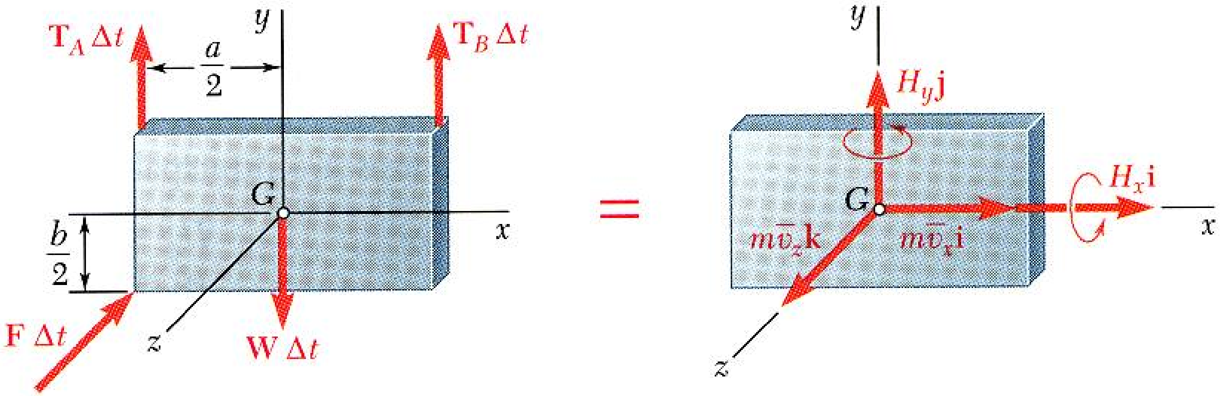
MODELING and ANALYSIS:

- Apply the principle of impulse and momentum. Since the initial momenta is zero, the system of impulses must be equivalent to the final system of momenta.
- Assume that the supporting cables remain taut such that the vertical velocity and the rotation about an axis normal to the plate is zero.

$$\vec{\bar{v}} = \bar{v}_x\vec{i} + v_z\vec{k} \qquad\qquad \vec{\omega} = \omega_x\vec{i} + \omega_y\vec{j}$$

Since the *x*, *y*, and *z* axes are principal axes of inertia,

$$\vec{H}_G = \bar{I}_x\omega_x\vec{i} + \bar{I}_y\omega_y\vec{j} = \tfrac{1}{12}mb^2\omega_x\vec{i} + \tfrac{1}{12}ma^2\omega_y\vec{j}$$



- Principle of impulse and momentum yields two equations for linear momentum and two equations for angular momentum.
- Solve for the two horizontal components of the linear and angular velocity vectors.

$$0 = mv_x \qquad -F\Delta t = m\bar{v}_z$$

$$v_x = 0 \qquad \bar{v}_z = -F\Delta t/m$$

$$\boxed{\vec{\bar{v}} = -\left(F\Delta t/m\right)\vec{k}}$$

$$\tfrac{1}{2}bF\Delta t = H_x = \tfrac{1}{12}mb^2\omega_x$$

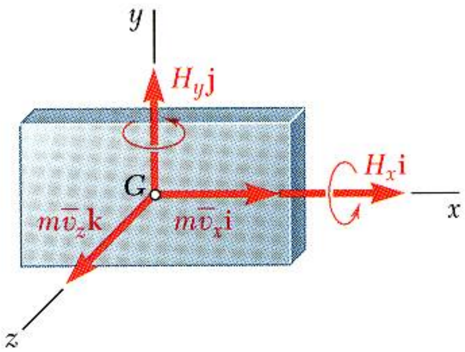
$$-\tfrac{1}{2}aF\Delta t = H_y = \tfrac{1}{12}ma^2\omega_y$$

$$\omega_x = 6F\Delta t/mb \qquad \omega_y = -\left(6F\Delta t/ma\right)$$

$$\boxed{\vec{\omega} = \frac{6F\Delta t}{mab}\left(a\vec{i} + b\vec{j}\right)}$$

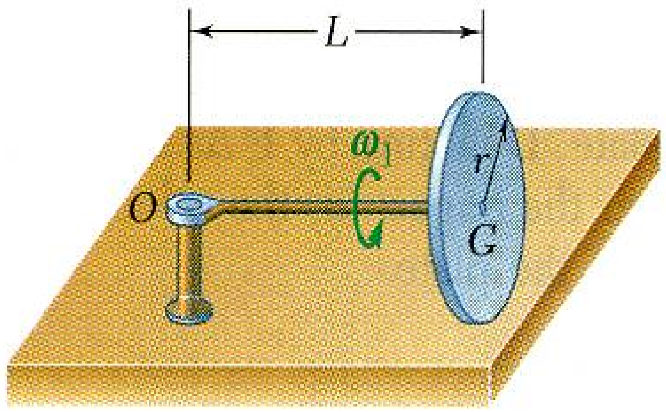
REFLECT and THINK:

- Equating the y components of the impulses and momenta and their moments about the z axis, you can obtain two additional equations that yield $T_A = T_B = ½W$.
- This verifies that the wires remain taut and that the initial assumption was correct. If the impulse were at G, this would reduce to a two-dimensional problem.



$)\vec{k}$

Sample Problem 18.2

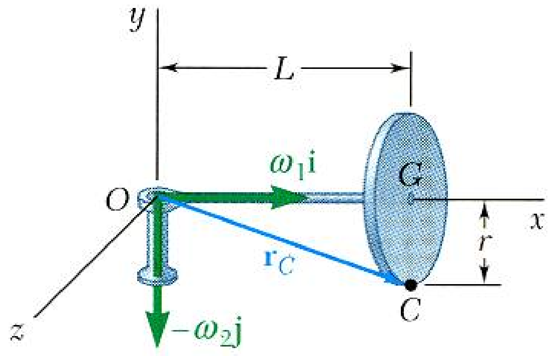


A homogeneous disk of mass *m* is mounted on an axle *OG* of negligible mass. The disk rotates counter-clockwise at the rate w_1 about *OG*.

Determine: *a*) the angular velocity of the disk, *b*) its angular momentum about *O*, *c*) its kinetic energy, and d) the vector and couple at *G* equivalent to the momenta of the particles of the disk.

STRATEGY:

- The disk rotates about the vertical axis through *O* as well as about *OG*. Combine the rotation components for the angular velocity of the disk.
- Compute the angular momentum of the disk using principle axes of inertia and noting that *O* is a fixed point.
- The kinetic energy is computed from the angular velocity and moments of inertia.
- The vector and couple at *G* are also computed from the angular velocity and moments of inertia.



MODELING and ANALYSIS:

- The disk rotates about the vertical axis through *O* as well as about *OG*. Combine the rotation components for the angular velocity of the disk.

$$\vec{\omega} = \omega_1 \vec{i} + \omega_2 \vec{j}$$

Noting that the velocity at *C* is zero,

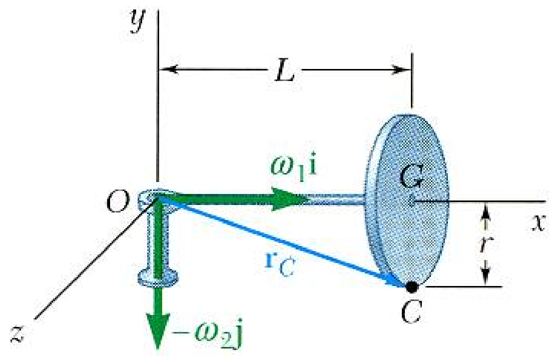
$$\vec{v}_C = \vec{\omega} \times \vec{r}_C = 0$$

$$0 = \left(\omega_1 \vec{i} + \omega_2 \vec{j}\right) \times \left(L\vec{i} - r\vec{j}\right)$$

$$= \left(L\omega_2 - r\omega_1\right)\vec{k}$$

$$\omega_2 = r\omega_1 / L$$

$$\boxed{\vec{\omega} = \omega_1 \vec{i} - \left(r\omega_1 / L\right)\vec{j}}$$



$$\vec{\omega} = \omega_1 \vec{i} - (r\omega_1/L)\vec{j}$$

- Compute the angular momentum of the disk using principle axes of inertia and noting that O is a fixed point.

$$\vec{H}_O = I_x\omega_x\vec{i} + I_y\omega_y\vec{j} + I_z\omega_z\vec{k}$$

$$H_x = I_x\omega_x = \left(\tfrac{1}{2}mr^2\right)\omega_1$$

$$H_y = I_y\omega_y = \left(mL^2 + \tfrac{1}{4}mr^2\right)\left(-r\omega_1/L\right)$$

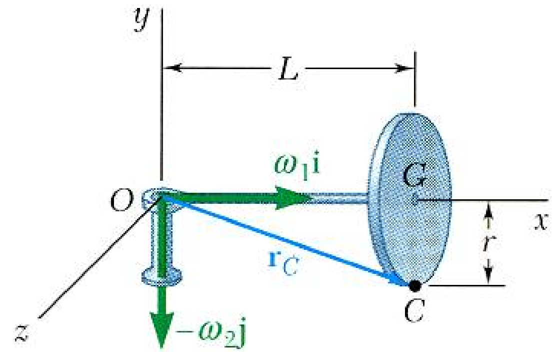
$$H_z = I_z\omega_z = \left(mL^2 + \tfrac{1}{4}mr^2\right)0 = 0$$

$$\boxed{\vec{H}_O = \tfrac{1}{2}mr^2\omega_1\vec{i} - m\left(L^2 + \tfrac{1}{4}r^2\right)\left(r\omega_1/L\right)\vec{j}}$$

- The kinetic energy is computed from the angular velocity and moments of inertia.

$$\begin{aligned} T &= \tfrac{1}{2}\left(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2\right) \\ &= \tfrac{1}{2}\left[mr^2\omega_1^2 + m\left(L^2 + \tfrac{1}{4}r^2\right)\left(-r\omega_1/L\right)^2\right] \end{aligned}$$

$$\boxed{T = \tfrac{1}{8}mr^2\left(6 + \frac{r^2}{L^2}\right)\omega_1^2}$$



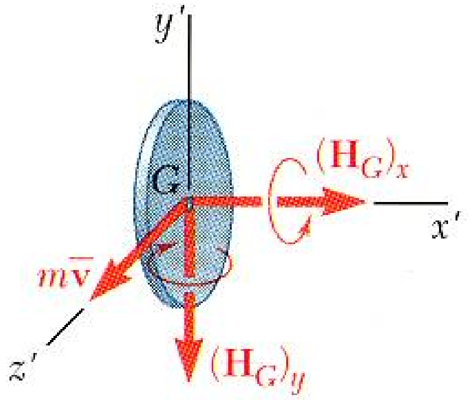
$$\vec{\omega} = \omega_1 \vec{i} - (r\omega_1/L)\vec{j}$$

The vector and couple at *G* are also computed from the angular velocity and moments of inertia.

$$\boxed{m\vec{\bar{v}} = mr\omega_1\vec{k}}$$

$$\vec{H}_G = \bar{I}_{x'}\omega_x\vec{i} + \bar{I}_{y'}\omega_y\vec{j} + \bar{I}_{z'}\omega_z\vec{k}$$
$$= \tfrac{1}{2}mr^2\omega_1\vec{i} + \tfrac{1}{4}mr^2\left(-r\omega/L\right)\vec{j}$$

$$\boxed{\vec{H}_G = \tfrac{1}{2}mr^2\omega_1\left(\vec{i} - \frac{r}{2L}\vec{j}\right)}$$



REFLECT and THINK:

- If the mass of the axle were not negligible and it was instead modeled as a slender rod with a mass M_{axle}, it would also contribute to the kinetic energy

T_{axle} = ½(1/3 $M_{axle}L^2$) ω_2^2 and the momenta

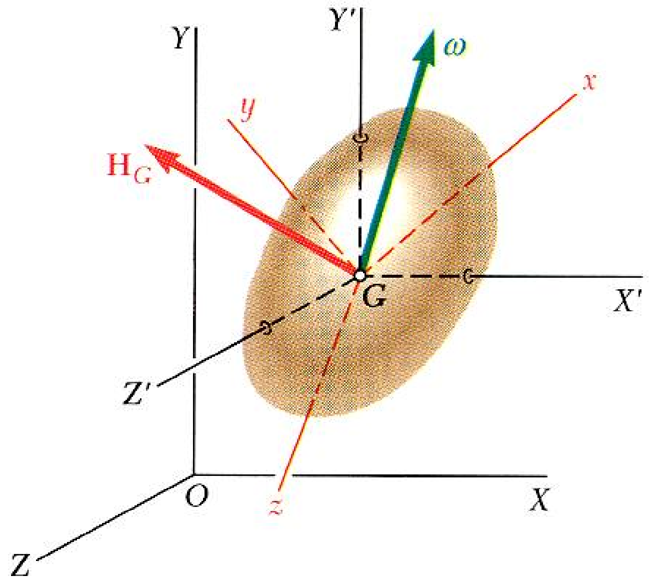
H_{axle} = ½(1/3 $M_{axle}L^2$) ω_2 of the system.

18.2 Motion of a Rigid Body in Three Dimensions

$$\sum \vec{F} = m\vec{a}$$

$$\sum \vec{M} = \dot{\vec{H}}_G$$

18.2A Rate of Change of Angular Momentum



Angular momentum and its rate of change are taken with respect to centroidal axes *GX′Y′Z′* of fixed orientation.

Transformation of $\vec{\omega}$ into $\vec{H}_G$ is independent of the system of coordinate axes.

Convenient to use body fixed axes *Gxyz* where moments and products of inertia are not time dependent.

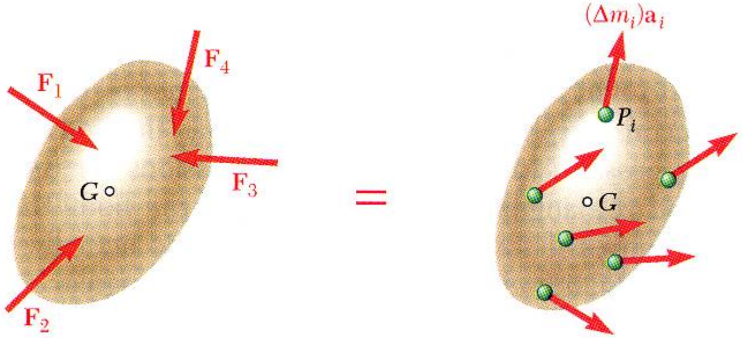
- Define rate of change of change of $\vec{H}_G$ with respect to the rotating frame,

$$\left(\dot{\vec{H}}_G\right)_{Gxyz} = \dot{H}_x\vec{i} + \dot{H}_y\vec{j} + \dot{H}_z\vec{k}$$

Then,

$$\dot{\vec{H}}_G = \left(\dot{\vec{H}}_G\right)_{Gxyz} + \vec{\Omega}\times\vec{H}_G \qquad \vec{\Omega} = \vec{\omega}$$

18.2B Euler's Eqs of Motion

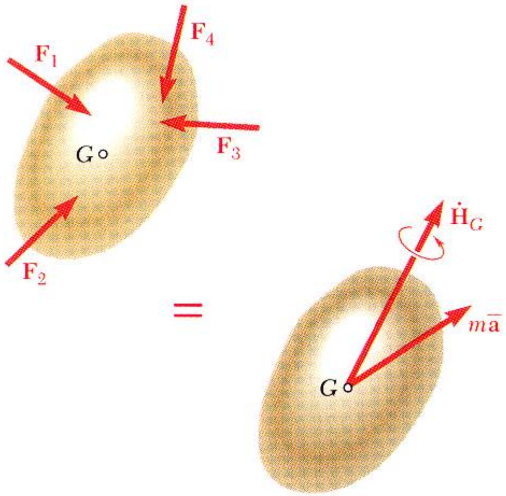


With $\vec{\Omega} = \vec{\omega}$ and *Gxyz* chosen to correspond to the principal axes of inertia,

$$\sum \vec{M}_G = \left(\dot{\vec{H}}_G\right)_{Gxyz} + \vec{\Omega}\times\vec{H}_G$$

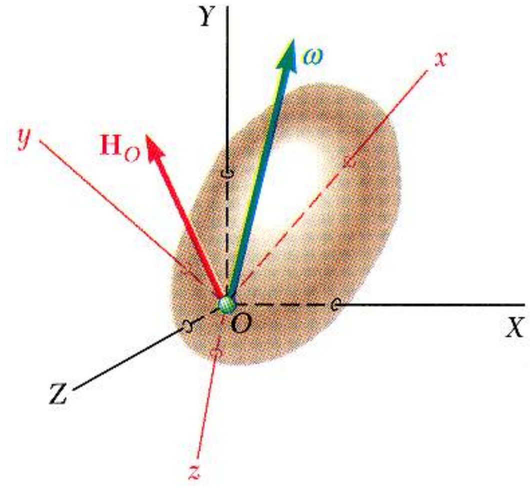
Euler's Equations:

$$\sum M_x = \bar{I}_x \dot{\omega}_x - \left(\bar{I}_y - \bar{I}_z\right)\omega_y \omega_z$$
$$\sum M_y = \bar{I}_y \dot{\omega}_y - \left(\bar{I}_z - \bar{I}_x\right)\omega_z \omega_x$$
$$\sum M_z = \bar{I}_z \dot{\omega}_z - \left(\bar{I}_x - \bar{I}_y\right)\omega_x \omega_y$$



- System of external forces and effective forces are equivalent for general three dimensional motion.
- System of external forces are equivalent to the vector and couple, $m\vec{a}$ and $\dot{\vec{H}}_G$.

18.2C,D Motion About a Fixed Point or a Fixed Axis

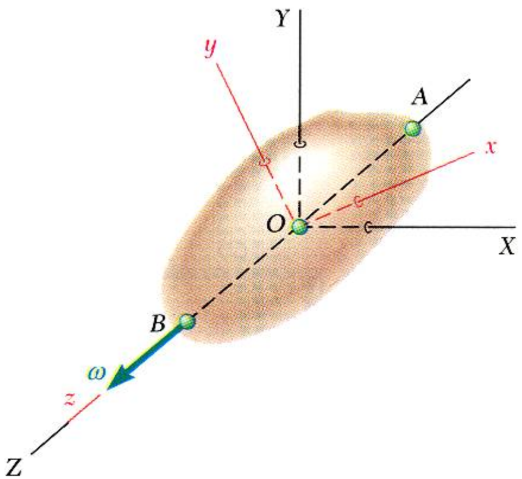


For a rigid body rotation around a fixed point,

$$\sum \vec{M}_O = \dot{\vec{H}}_O$$
$$= \left(\dot{\vec{H}}_O\right)_{Oxyz} + \vec{\Omega} \times \vec{H}_O$$

For a rigid body rotation around a fixed axis,

$$H_x = -I_{xz}\omega \qquad H_y = -I_{yz}\omega \qquad H_z = I_z\omega$$

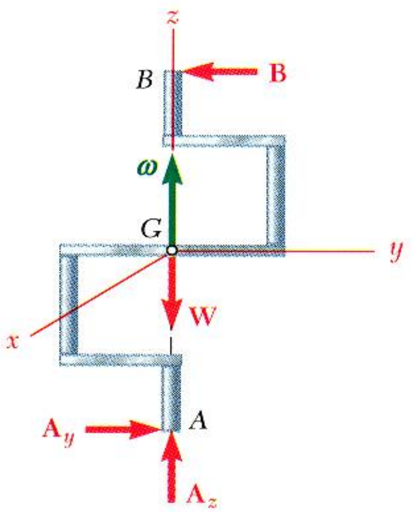


$$\sum \vec{M}_O = \left(\dot{\vec{H}}_O\right)_{Oxyz} + \vec{\omega} \times \vec{H}_O$$
$$= \left(-I_{xz}\vec{i} - I_{yz}\vec{j} + I_z\vec{k}\right)\dot{\omega}$$
$$+ \omega\vec{k} \times \left(-I_{xz}\vec{i} - I_{yz}\vec{j} + I_z\vec{k}\right)\omega$$
$$= \left(-I_{xz}\vec{i} - I_{yz}\vec{j} + I_z\vec{k}\right)\alpha + \left(-I_{xz}\vec{j} + I_{yz}\vec{i}\right)\omega^2$$

$$\sum M_x = -I_{xz}\alpha + I_{yz}\omega^2$$
$$\sum M_y = -I_{yz}\alpha + I_{xz}\omega^2$$
$$\sum M_z = I_z\alpha$$

If symmetrical with respect to the xy plane,

$$\sum M_x = 0 \quad \sum M_y = 0 \quad \sum M_z = I_z \alpha$$

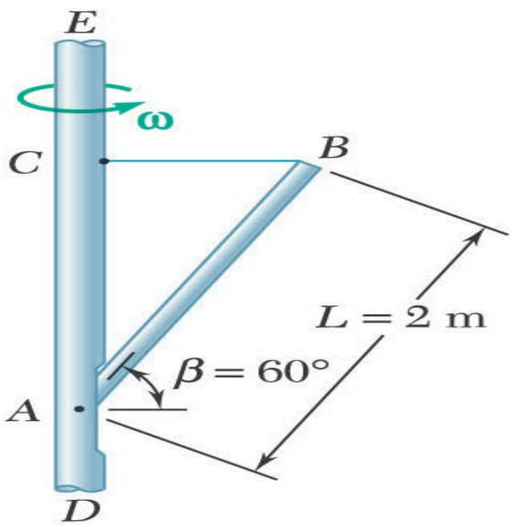


- If not symmetrical, the sum of external moments will not be zero, even if α = 0,

$$\sum M_x = I_{yz}\omega^2 \quad \sum M_y = I_{xz}\omega^2 \quad \sum M_z = 0$$

- A rotating shaft requires both static $(\omega = 0)$ and dynamic $(\omega \neq 0)$ balancing to avoid excessive vibration and bearing reactions.

Sample Problem 18.3

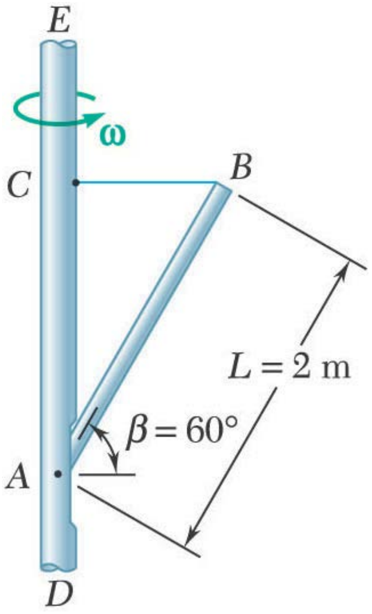


Rod *AB* with mass m = 20 kg is pinned at *A* to a vertical axle which rotates with constant angular velocity ω = 15 rad/s. The rod position is maintained by a horizontal wire *BC*.

Determine the tension in the wire and the reaction at *A*.

STRATEGY:

- Evaluate the system of effective forces by reducing them to a vector $m\vec{\bar{a}}$ attached at *G* and couple $\vec{H}_G$.
- Expressing that the system of external forces is equivalent to the system of effective forces, write vector expressions for the sum of moments about *A* and the summation of forces.
- Solve for the wire tension and the reactions at *A*.



MODELING and ANALYSIS:

• **Evaluate the system of effective forces by reducing them to a vector** $m\vec{\bar{a}}$ **attached at** ***G*** **and couple** $\vec{H}_G$.

$$\vec{\bar{a}} = \vec{a}_n = -r\omega^2\vec{I} = -\left(\tfrac{1}{2}L\cos\beta\right)\omega^2\vec{I}$$

$$= -\left(112.5\,\mathrm{m/s^2}\right)\vec{I}$$

$$m\vec{\bar{a}} = 20(-112.5) = -(2250\,\mathrm{N})\vec{I}$$

$$\vec{H}_G = \bar{I}_x\omega_x\vec{i} + \bar{I}_y\omega_y\vec{j} + \bar{I}_z\omega_z\vec{k}$$

$$\bar{I}_x = \tfrac{1}{12} mL^2 \qquad \bar{I}_y = 0 \qquad \bar{I}_z = \tfrac{1}{12} mL^2$$

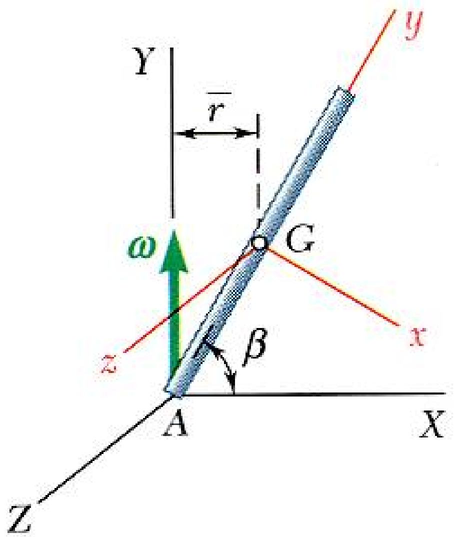
$$\omega_x = -\omega\cos\beta \quad \omega_y = \omega\sin\beta \quad \omega_z = 0$$

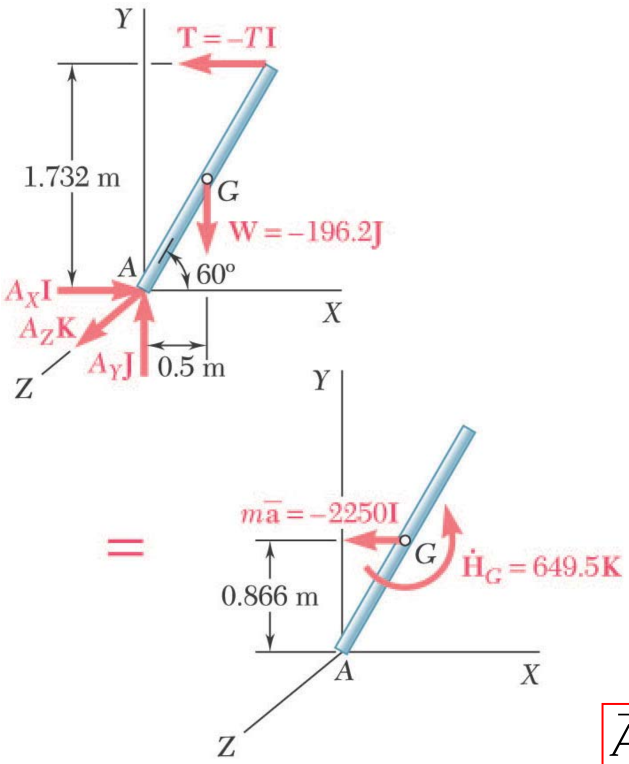
$$\vec{H}_G = -\tfrac{1}{12} mL^2 \omega \cos\beta\, \vec{i}$$

$$\dot{\vec{H}}_G = \left(\dot{\vec{H}}_G\right)_{Gxyz} + \vec{\omega} \times \vec{H}_G$$

$$= 0 + \left(-\omega\cos\beta\,\vec{i} + \omega\sin\beta\,\vec{j}\right) \times \left(\tfrac{1}{12} mL^2 \omega\cos\beta\,\vec{i}\right)$$

$$= \tfrac{1}{12} mL^2 \omega^2 \sin\beta\cos\beta\,\vec{k} = (649.5\,\mathrm{N \cdot m})\vec{k}$$





Expressing that the system of external forces is equivalent to the system of effective forces, write vector expressions for the sum of moments about *A* and the summation of forces.

$$\sum \vec{M}_A = \sum\left(\vec{M}_A\right)_{eff}$$

$$1.732\vec{J} \times \left(-T\vec{I}\right) + 0.5\vec{I} \times \left(-196.2\vec{J}\right) = 0.866\vec{J} \times \left(-2250\vec{I}\right) + 649.5\vec{K}$$

$$\left(1.732T - 98.1\right)\vec{K} = \left(1948.5 + 649.5\right)\vec{K}$$

$$\boxed{T = 1557\ \mathrm{N}}$$

$$\sum \vec{F} = \sum\left(\vec{F}\right)_{eff}$$

$$A_X\vec{I} + A_Y\vec{J} + A_Z\vec{K} - 1613\vec{I} - 196.2\vec{J} = -2250\vec{I}$$

$$\boxed{\vec{A} = -\left(697\ \mathrm{N}\right)\vec{I} + \left(196.2\ \mathrm{N}\right)\vec{J}}$$

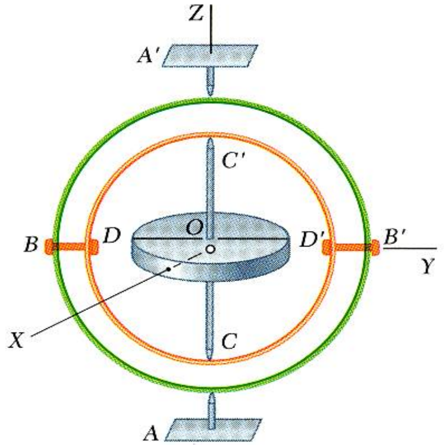
REFLECT and THINK:

- You could have obtained the value of T from $\mathbf{H}_A$ and Eq. (18.28). Even though the rod rotates with a constant angular velocity, the asymmetry of the rod causes a moment about the z axis. Note that we calculated the inertial term $\dot{H}_A$ by adding $r \times ma$ and the couple $\dot{H}_G$

Gyroscopes are used in the navigation system of the Hubble telescope, and can also be used as sensors . Modern gyroscopes can also be MEMS (Micro Electro-Mechanical System) devices, or based on fiber optic technology.

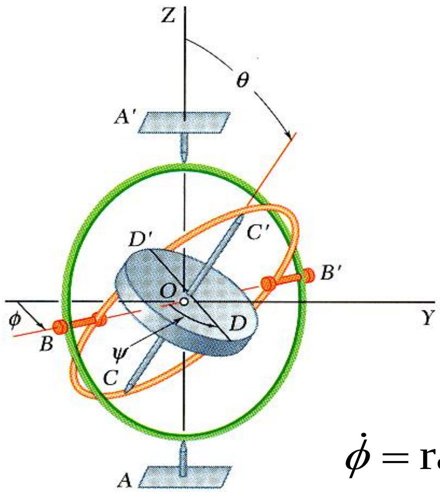
NASA/JSC

* Motion of a Gyroscope.   Eulerian Angles *



A gyroscope consists of a rotor with its mass center fixed in space but which can spin freely about its geometric axis and assume any orientation.

From a reference position with gimbals and a reference diameter of the rotor aligned, the gyroscope may be brought to any orientation through a succession of three steps:



a) rotation of outer gimbal through ϕ about *AA'*,

b) rotation of inner gimbal through θ about *BB'*,

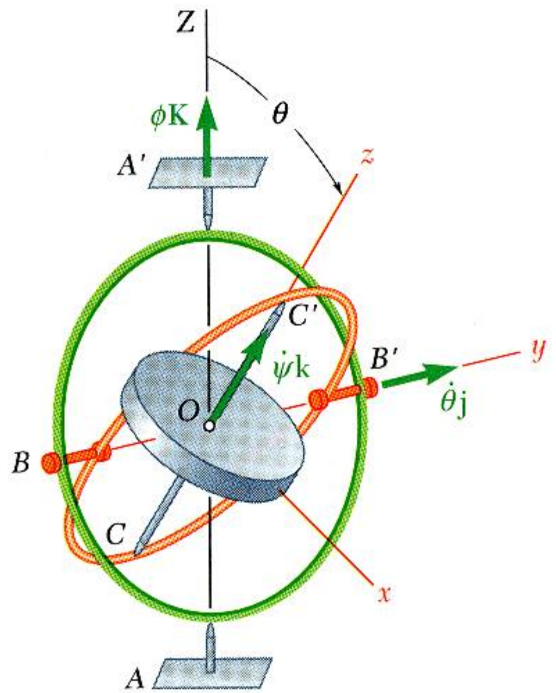
c) rotation of the rotor through ψ about *CC'*.

• $\phi\ \theta\ \psi$: *Eulerian Angles*

$$\dot{\phi} = \text{rate of precession}$$

$$\dot{\theta} = \text{rate of nutation}$$

$$\dot{\Psi} = \text{rate of spin}$$



The angular velocity of the gyroscope,

$$\vec{\omega} = \dot{\phi}\vec{K} + \dot{\theta}\vec{j} + \dot{\Psi}\vec{k}$$

with $\vec{K} = -\sin\theta\vec{i} + \cos\theta\vec{k}$

$$\vec{\omega} = -\dot{\phi}\sin\theta\vec{i} + \dot{\theta}\vec{j} + \left(\dot{\Psi} + \dot{\phi}\cos\theta\right)\vec{k}$$

Equation of motion,

$$\sum \vec{M}_O = \left(\dot{\vec{H}}_O\right)_{Oxyz} + \vec{\Omega} \times \vec{H}_O$$

$$\vec{H}_O = -I'\dot{\phi}\sin\theta\vec{i} + I'\dot{\theta}\vec{j} + I\left(\dot{\Psi} + \dot{\phi}\cos\theta\right)\vec{k}$$

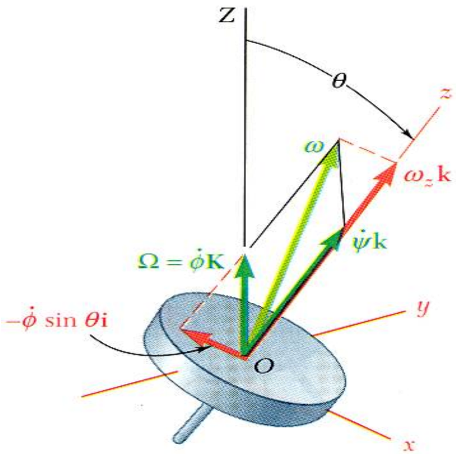
$$\vec{\Omega} = \dot{\phi}\vec{K} + \dot{\theta}\vec{j}$$

$$\sum M_x = -I'\left(\ddot{\phi}\sin\theta + 2\dot{\theta}\dot{\phi}\cos\theta\right) + I\dot{\theta}\left(\dot{\Psi} + \dot{\phi}\cos\theta\right)$$

$$\sum M_y = I'\left(\ddot{\theta} - \dot{\phi}^2\sin\theta\cos\theta\right) + I\dot{\phi}\sin\theta\left(\dot{\Psi} + \dot{\phi}\cos\theta\right)$$

$$\sum M_z = I\frac{d}{dt}\left(\dot{\Psi} + \dot{\phi}\cos\theta\right)$$

Steady Precession of a Gyroscope



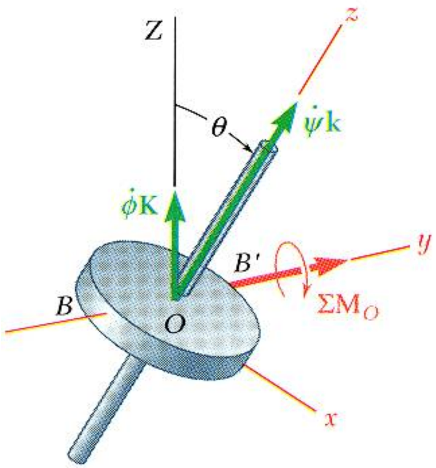
Steady precession,

$\theta, \dot\phi, \dot\psi$ are constant

$$\vec{\omega} = -\dot\phi \sin\theta \vec{i} + \omega_z \vec{k}$$

$$\vec{H}_O = -I'\dot\phi \sin\theta \vec{i} + I\omega_z \vec{k}$$

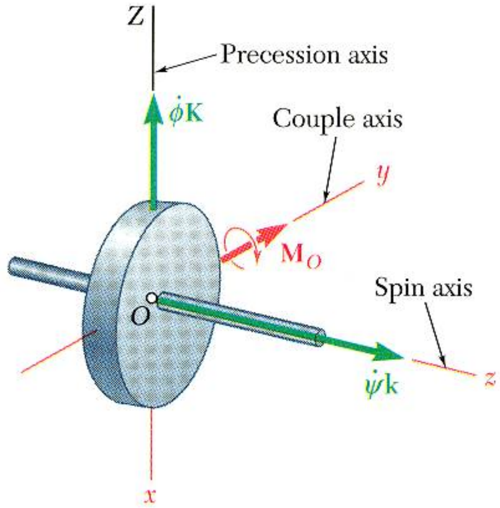
$$\vec{\Omega} = -\dot\phi \sin\theta \vec{i} + \dot\phi \cos\theta \vec{k}$$



$$\sum \vec{M}_O = \vec{\Omega} \times \vec{H}_O$$

$$= \left(I\omega_z - I'\dot\phi \cos\theta\right)\dot\phi \sin\theta \vec{j}$$

Couple is applied about an axis perpendicular to the precession and spin axes



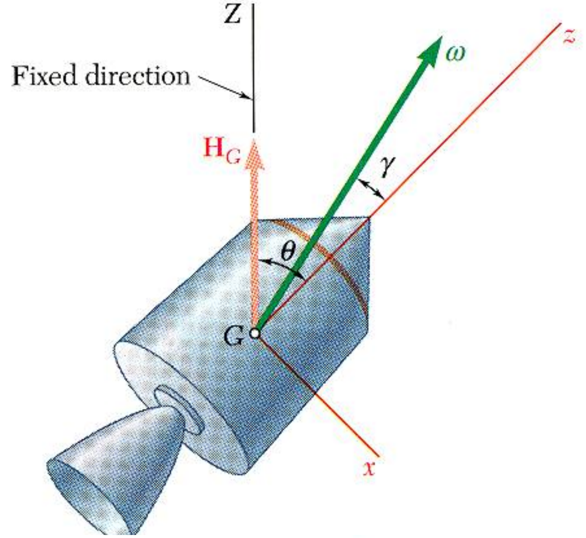
When the precession and spin axis are at a right angle,

$\theta = 90°$

$$\sum \vec{M}_O = I\dot{\Psi}\dot{\phi}\vec{j}$$

Gyroscope will precess about an axis perpendicular to both the spin axis and couple axis.

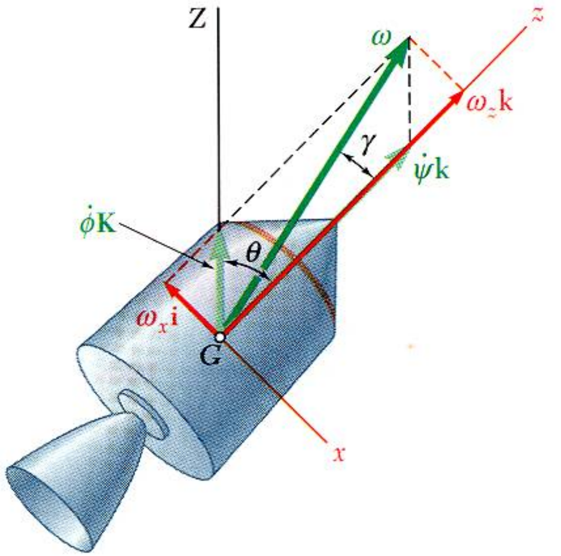
Motion of an Axisymmetrical Body Under No Force



Consider motion about its mass center of an axisymmetrical body under no force but its own weight, e.g., projectiles, satellites, and space craft.

$$\dot{\vec{H}}_G = 0 \qquad \vec{H}_G = \text{constant}$$

Define the Z axis to be aligned with $\vec{H}_G$ and z in a rotating axes system along the axis of symmetry. The x axis is chosen to lie in the Zz plane.

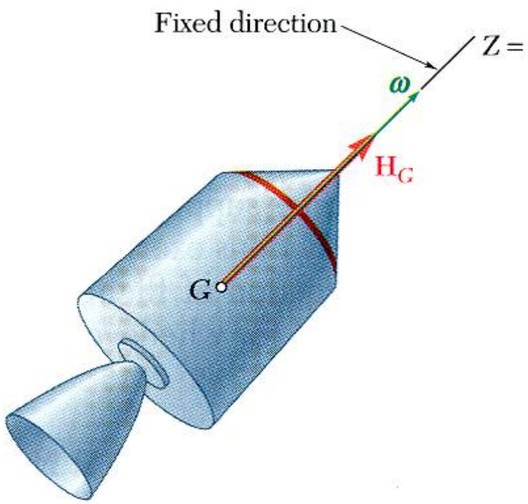


$$H_x = -H_G \sin\theta = I'\omega_x \qquad \omega_x = -\frac{H_G \sin\theta}{I'}$$

$$H_y = 0 = I'\omega_y \qquad \omega_y = 0$$

$$H_z = H_G \cos\theta = I\omega_z \qquad \omega_z = \frac{H_G \cos\theta}{I}$$

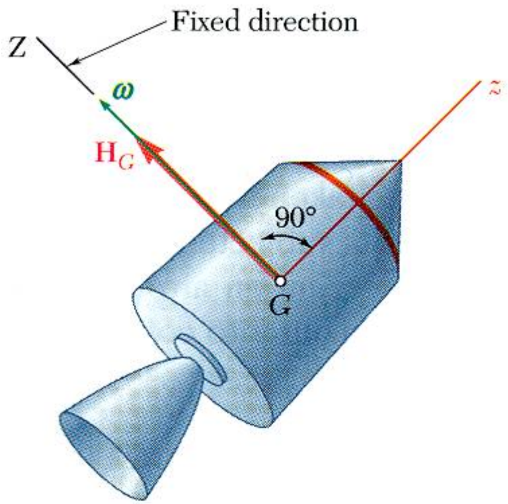
- θ = constant and body is in steady precession.
- Note: $$-\frac{\omega_x}{\omega_z} = \tan\gamma = \frac{I}{I'}\tan\theta$$



- Two cases of motion of an axisymmetrical body which under no force which involve no precession:
- Body set to spin about its axis of symmetry,

$$\omega_x = H_x = 0$$

$\vec{\omega}$ and $\vec{H}_G$ are aligned and body keeps spinning about its axis of symmetry.

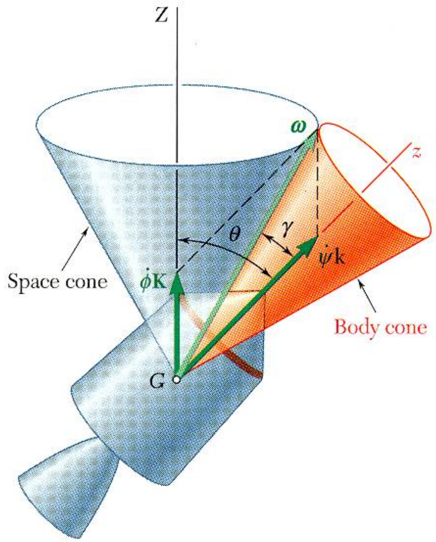


- Body is set to spin about its transverse axis,

$$\omega_z = H_z = 0$$

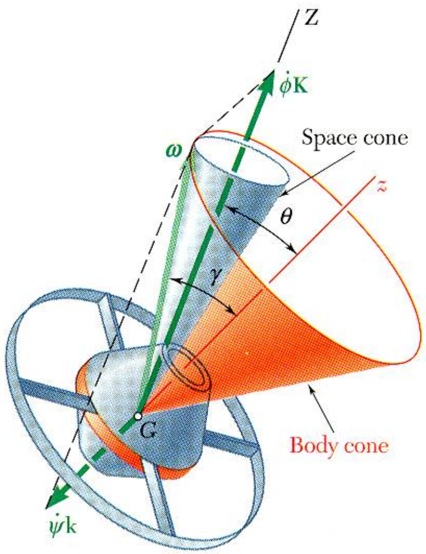
$\vec{\omega}$ and $\vec{H}_G$ are aligned

and body keeps spinning about the given transverse axis.



The motion of a body about a fixed point (or its mass center) can be represented by the motion of a body cone rolling on a space cone. In the case of steady precession the two cones are circular.

- $I < I'$. Case of an elongated body. $\gamma < \theta$ and the vector ω lies inside the angle *ZGz*. The space cone and body cone are tangent externally; the spin and precession are both counterclockwise from the positive *z* axis. The precession is said to be *direct*.



- $I > I'$. Case of a flattened body. $\gamma < \theta$ and the vector ω lies outside the angle *ZGz*. The space cone is inside the body cone; the spin and precession have opposite senses. The precession is said to be *retrograde*.