3.5 Differential Equilibrium Relationship

→ The conditions of equilibrium combined with a limiting process will lead us to differential equations connecting the load, the shear force, and the bending moment.

→ Integration of these relationships for particular cases furnishes us with an alternative method for evaluating shear forces and bending moments.

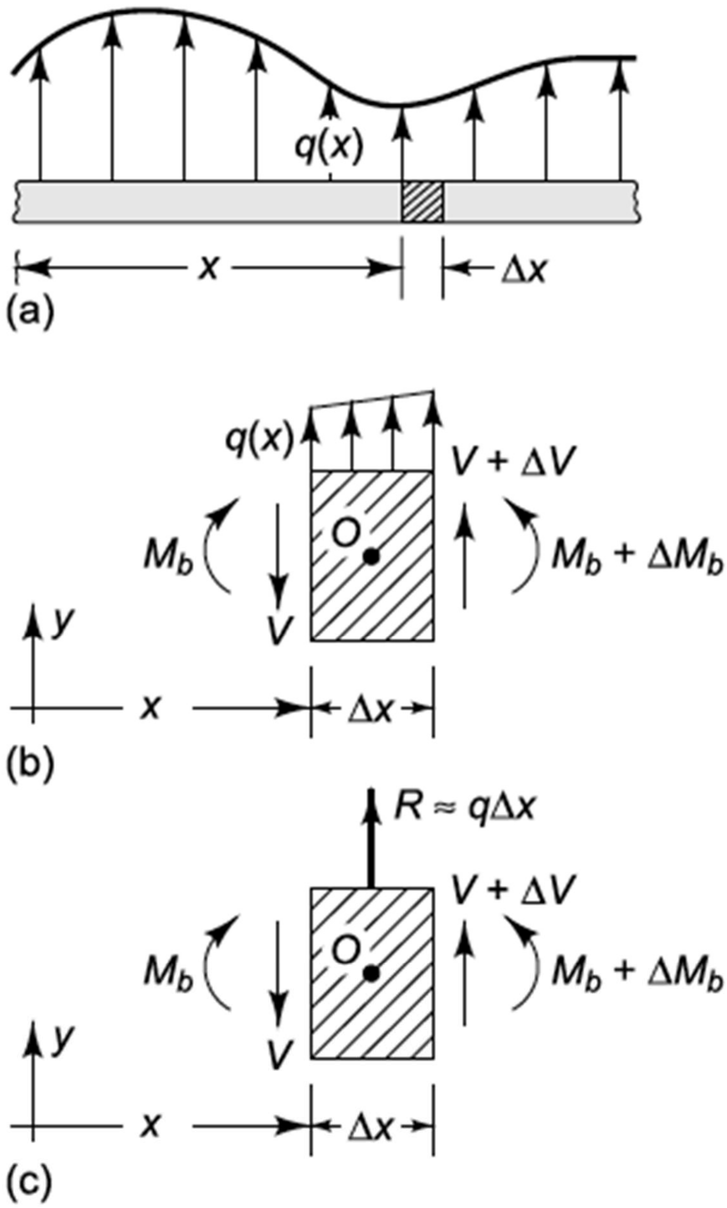

Fig. 3.14 *Free-body diagram of small element isolated from a beam under distributed loading*

▶ Assumption

→ Δx is already so small that we can safely take R to have the magnitude $q\Delta x$ and to pass through O.

▶ Equilibrium (see Fig. 3.14 (c))

$\sum F_y = (V + \Delta V) + q\Delta x - V = 0$

$\sum M_O = (M_b + \Delta M_b) + (V + \Delta V)\Delta x/2 + V\Delta x/2 - M_b = 0$

→ $\Delta M_b + V\Delta x + \Delta V \Delta x/2 = 0$ (3.9)

$\therefore\ \Delta V/\Delta x + q(x) = 0,\ \Delta M_b/\Delta x + V = -\Delta V/2$ (3.10)

For $\Delta x \to 0$;

$$dV/dx + q = 0 \quad (3.11)$$
$$dM_b/dx + V = 0 \quad (3.12)$$
$$\therefore\ d^2M_b/dx^2 - q = 0$$

▶ Example 3.6 Consider the beam shown in Fig. 3.16 (a) with simple transverse supports at A and B and loaded with a uniformly distributed load $q = -w_0$ over a portion of the length. It is desired to obtain the shear-force and bending-moment diagrams. In contrast with the previous example, it is not possible to write a single differential equation for V and M which will be valid over the complete length of the beam.

cf. At least without inventing a special notation, as will be done in the next section.

cf. Instead let subscripts 1 and 2 indicate values of variables in the loaded and unloaded segments of the beam.

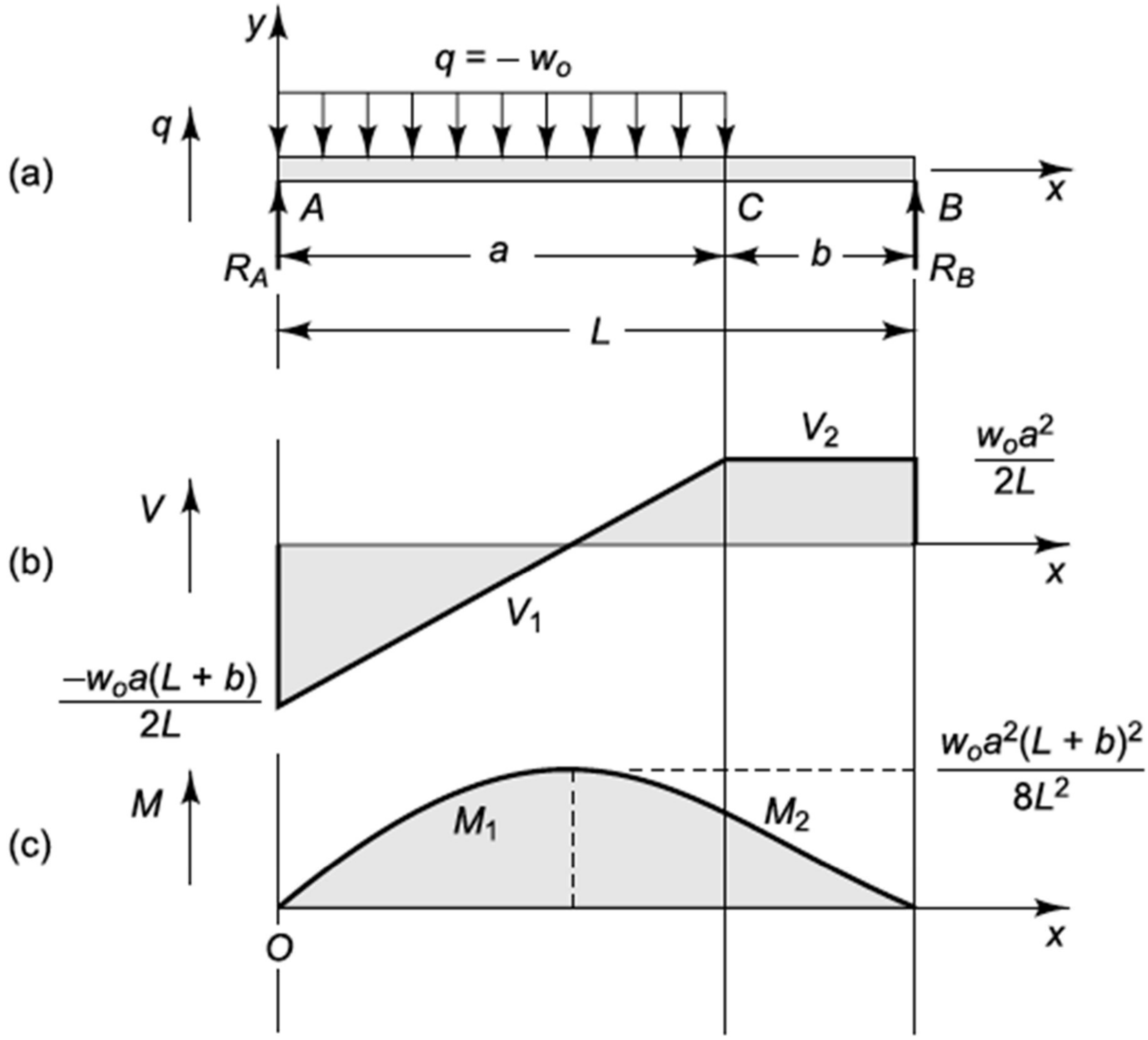

Fig. 3.16 *Example 3.6*

▷For $0 < x < a$

$dV_1/dx - w_0 = 0$

→ $V_1 - w_0 x = C_1$ (∵ Indefinite integration)

$dM_{b1}/dx + w_0 x + C_1 = 0$

→ $M_{b1} + (1/2) w_0 x^2 + C_1 x = C_3$ (∵ Indefinite integration)

▷For $a < x < L$

$dV_2/dx = 0$

→ $V_2 = C_2$ (∵ Indefinite integration)

$dM_{b2}/dx + C_2 = 0$

→ $M_{b2} + C_2 x = C_4$ (∵ Indefinite integration)

▷B.C.

i) $M_{b1}(0) = 0$

ii) $M_{b2}(L) = 0$

iii) $at\ \ x = a,\ V_1 = V_2$

iv) $at\ \ x = a,\ M_{b1} = M_{b2}$

$\therefore\ C_1 = -(1/2)w_0 a(L+b)/L \quad C_2 = (1/2)(w_0 a^2)/L$

$C_3 = 0, \qquad C_4 = (1/2)w_0 a^2$

$\therefore\ V_1(x) = w_0 x - (1/2)w_0 a(L+b)/L \quad (0 \le x \le a)$

$V_2(x) = (1/2)w_0 a^2/L \quad (a \le x \le L)$

$M_{b1}(x) = (1/2)w_0 a(L+b)x/L - (1/2)w_0 x^2 \quad (0 \le x \le a)$

$M_{b2}(x) = (1/2)w_0 a^2 - (1/2)w_0 a^2 x/L \quad (a \le x \le L)$

→ Clearly if the loading requires separate representations for a number of segments each with its own differential equation form, it becomes very awkward to carry along the additional arbitrary constants which are later eliminated by matching the *V*'s and *M*'s at the junctions of the segments.

3.6 Singularity Function

→ The unique characteristics of a special mathematical apparatus is to write a single differential equation with discontinuous functions.

▶ $F_n(x) = \langle x-a \rangle^n = \begin{cases} 0 & x < a \\ (x-a)^n & x \geq a \end{cases}$

(where $n = 0, 1, 2, 3, \cdots$)

$$\int_{-\infty}^{x} \langle x-a \rangle^n \; dx = \langle x-a \rangle^{n+1}/(n+1) \tag{3.16}$$

▶ $F_n(x) = \langle x-a \rangle_n = \begin{cases} 0 & x \neq a \\ \infty & x = a \end{cases}$

(where $n = -1, -2, \cdots$)

cf. $F_n(x) = \langle x-a \rangle^n = (x-a)^n \langle x-a \rangle^0$

$$\begin{aligned} &\int_{-\infty}^{x} \langle x-a \rangle_{-2} \, dx = \langle x-a \rangle_{-1} \\ &\int_{-\infty}^{x} \langle x-a \rangle_{-1} \, dx = \langle x-a \rangle^0 \end{aligned} \tag{3.17}$$

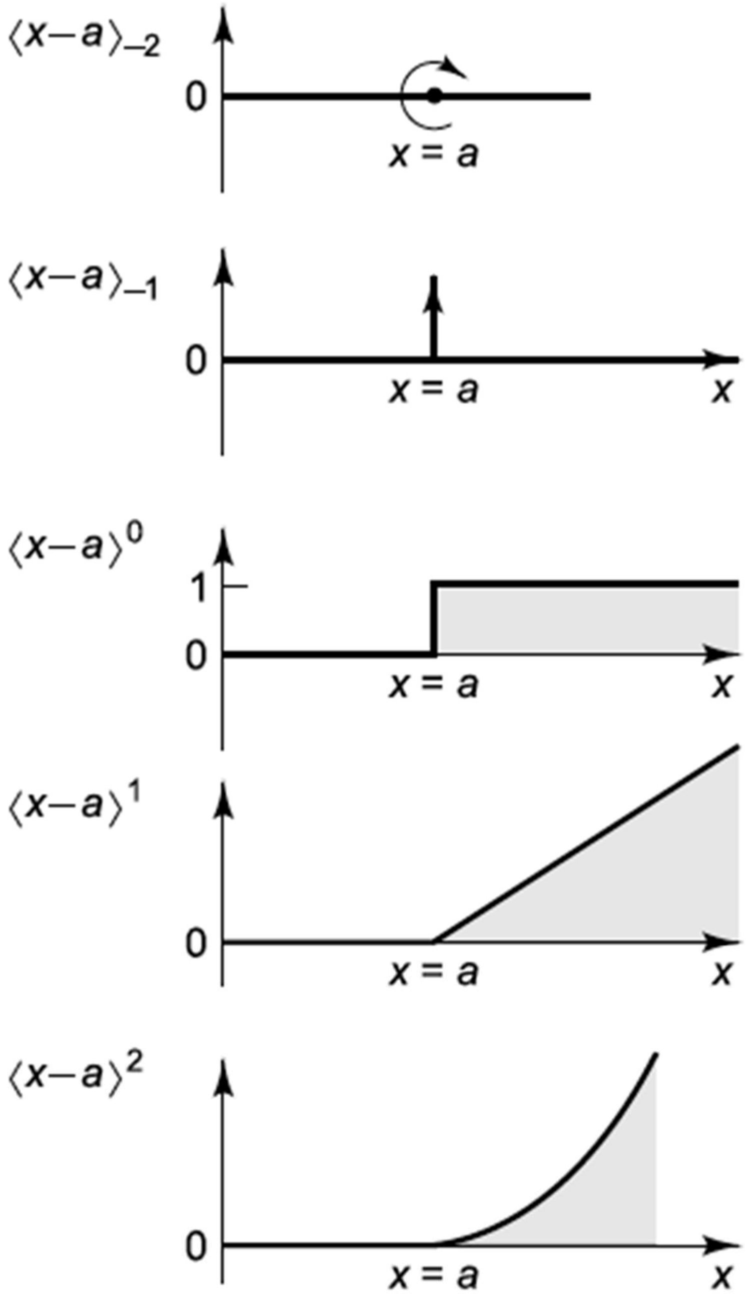

Fig. 3.17 *Family of singularity functions*

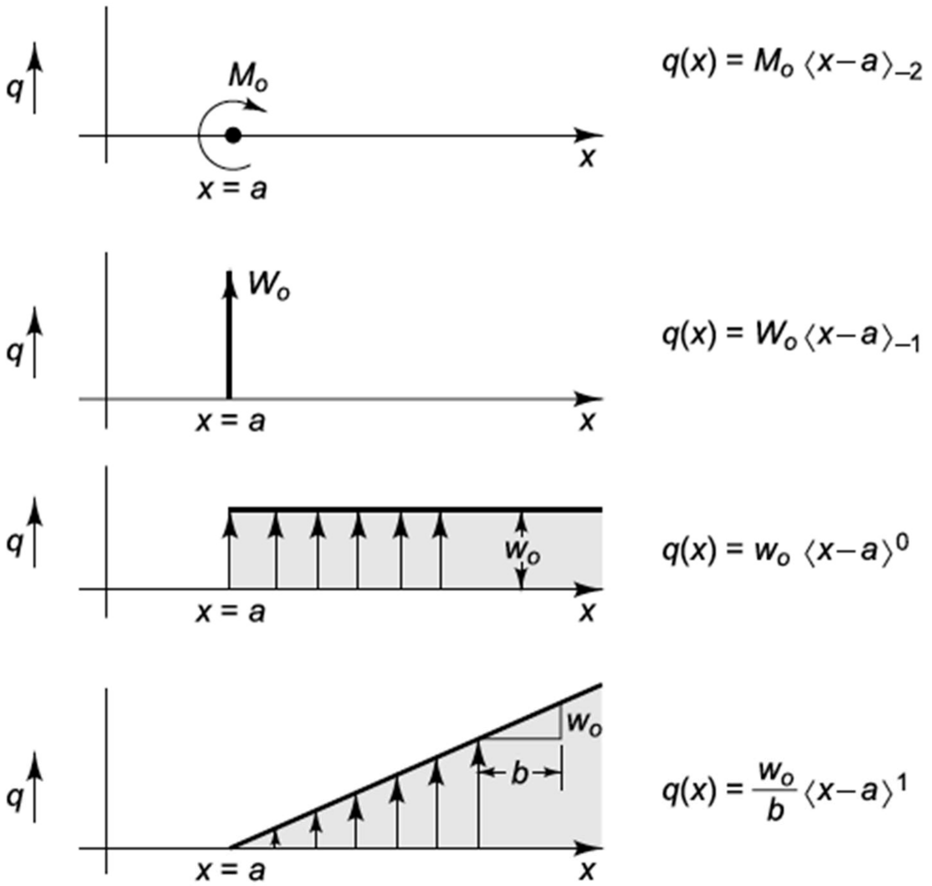

Fig. 3.18 *Examples of loading intensities represented by singularity functions*

▶ <u>Example 3.7</u> We consider the problem studied in Example 3.6 again, but we shall utilize the singularity functions.

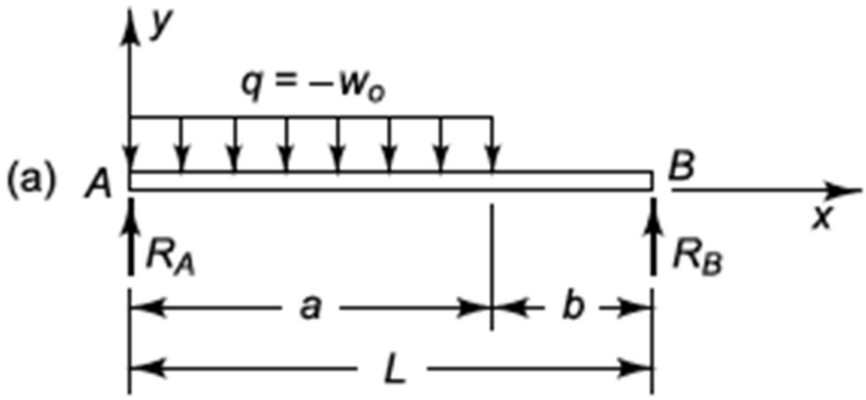

Fig. 3.19 *Example 3.7*

→ Fully aware of the cases ① and ②. Case ② is more powerful than case ①.

① Solve by calculating the support reactions separately (When the problem requires the maximum shear force and moment)

$$q(x) = -w_0 + w_0 < x - a >^0 \tag{a}$$

$$V(x) = w_0 x - w_0 < x - a >^1 + C_1 \tag{b}$$

B.C.) $V(0) = C_1 = -R_A$

From $\sum M_B = 0$;

$$-R_A = -\frac{w_0 a}{L}(b + a/2) \tag{c}$$

$$\therefore\ M_b(x) = -w_0 x^2/2 + \frac{w_0}{2} < x - a >^2 + \frac{w_0 a}{L}(b + a/2) + C_2 \tag{d}$$

B.C.) $M_b(0) = 0$;

$\therefore\ C_2 = 0$

② Solve by putting the support reactions into the unknown constants (When the problem requires the support reactions)

$$q(x) = R_A\langle x\rangle_{-1} - w_0\langle x\rangle^0 + w_0\langle x - a\rangle^0 + R_B\langle x - L\rangle_{-1} \tag{e}$$

Since $V(-\infty) = 0$,

$$-V(x) = \int_{-\infty}^{x} q(x)\ dx$$

$$= R_A\langle x\rangle^0 - w_0\langle x\rangle^1 + w_0\langle x - a\rangle^1 + R_B\langle x - L\rangle^0 \tag{f}$$

Since $M(-\infty) = 0$,

$$M_b(x) = -\int_{-\infty}^{x} V(x)\ dx$$

$$= R_A\langle x\rangle^1 - \frac{w_0}{2}\langle x\rangle^2 + \frac{w_0}{2}\langle x - a\rangle^2 + R_B\langle x - L\rangle^1 \tag{g}$$

→ If we make *x* just slightly larger than *x*=*L*, the shear force (*V*) and the bending moment (M_b) should vanish, that is,

$V(L) = R_A - w_0 L + w_0(L - a) + R_B = 0$ → Shear force balance

$M_b(L) = R_A L - \frac{w_0}{2}L^2 + \frac{w_0}{2}(L - a)^2 = 0$ → Bending moment balance

→ $R_A = \frac{w_0}{2}\frac{L^2-b^2}{L}$

cf. The satisfaction of the equilibrium requirements for every differential element of the beam implies satisfaction of the equilibrium requirements of the entire beam.

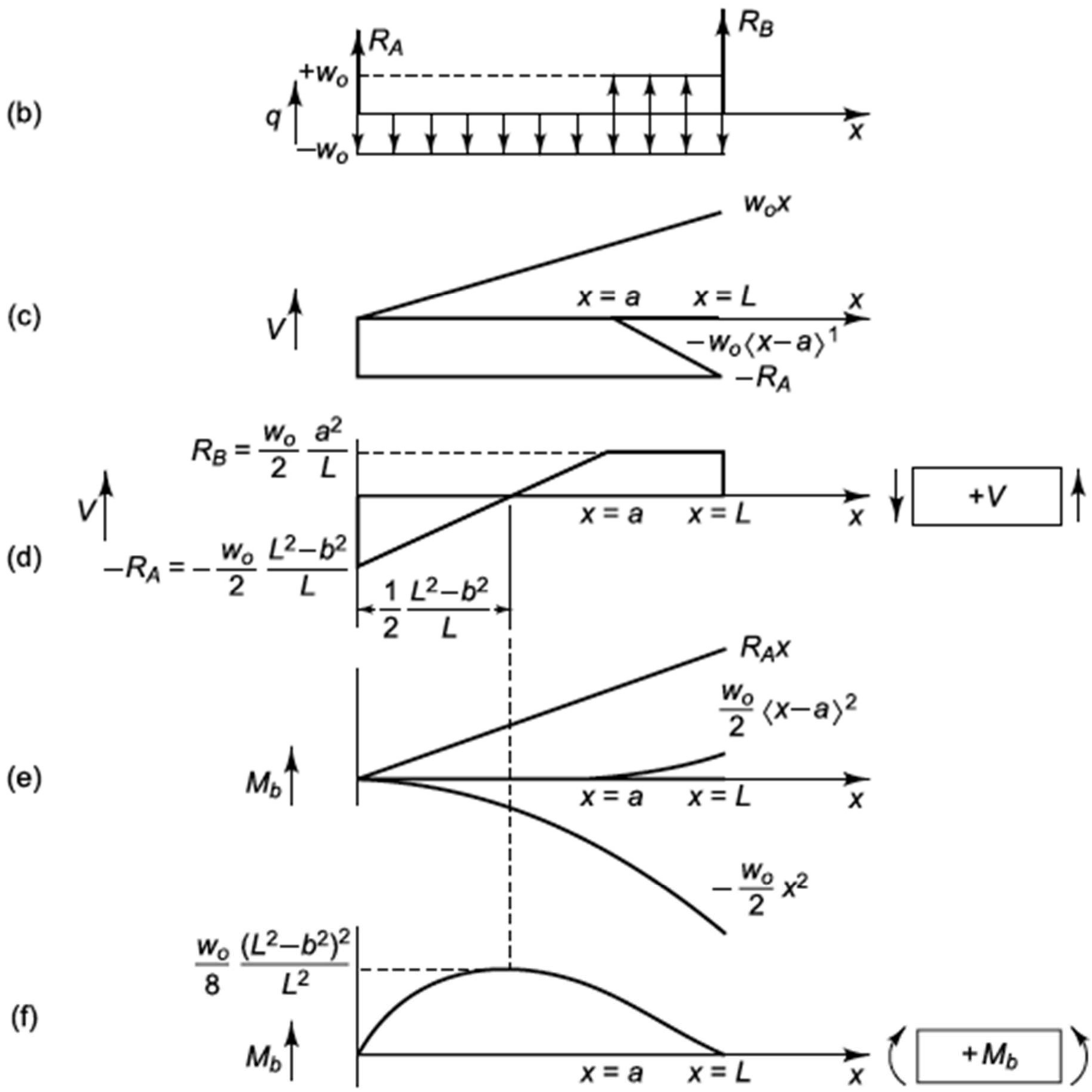

Fig. 3.19 *Example 3.7*

▶ Example 3.8 In Fig. 3.20 (a) the frame BAC is built-in at B and subjected to a load P at C. It is desired to obtain shear-force and bending-moment diagrams for the segment AB.

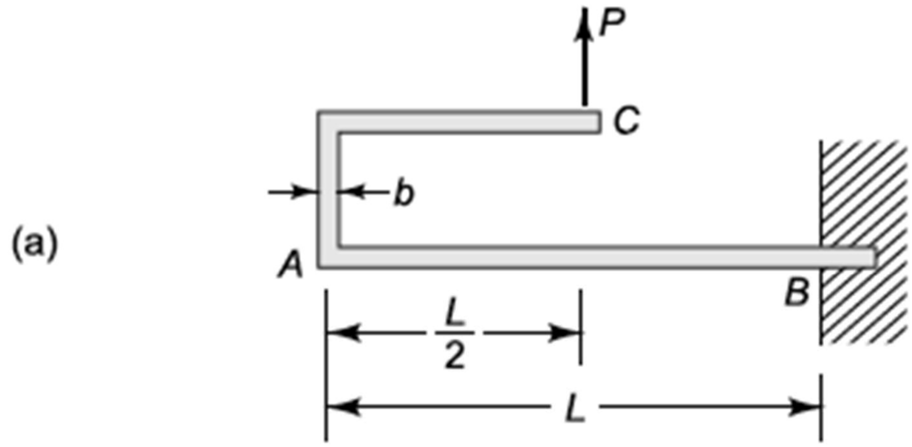

Fig. 3.20 *Example 3.8*

Sol) Since there is no loading between A and B,

$q(x) = 0$

$\therefore$ from $dV/dx + q = 0$;

$\therefore V(x) = C_1$ (a)

where $C_1 = -P$ because of the assumed concentrated force at A.

Integrating again using $dM_b/dx + V = 0$ we find

$M_b(x) = Px + C_2$ (b)

Now, $M_b(0) = -PL/2$

$\therefore C_2 = -PL/2$

$\therefore V(x) = -P$

$M_b(x) = Px - PL/2$

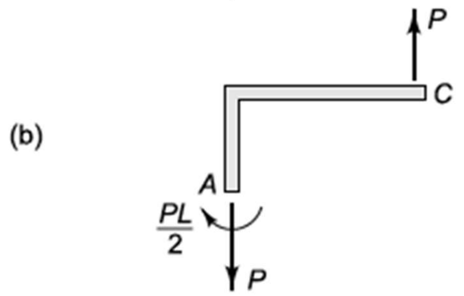

Fig. 3.20 *Example 3.8*

Fig. 3.20 *Example 3.8*

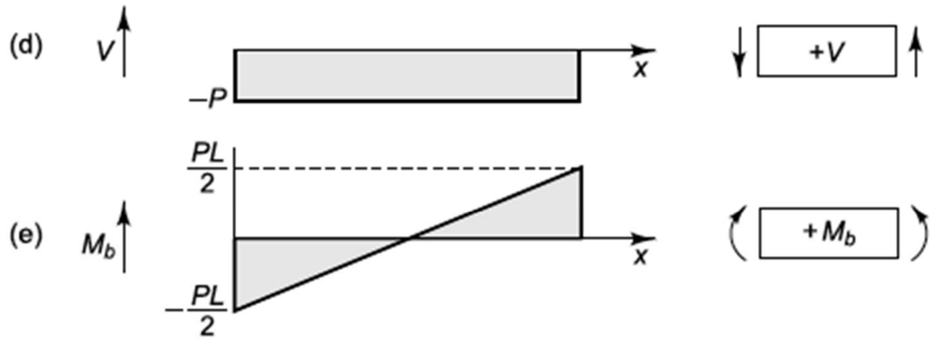

Fig. 3.20 *Example 3.8*

▶In general, the algebraic work is simplified if all the reactions are determined first from overall equilibrium (assuming that this can be done). However, whatever route is followed, all constants of integration must be evaluated carefully from the support conditions. Let us consider another example in which it is necessary to include the reactive forces into the loading term.

▶<u>Example 3.9</u> The loading on a beam is assumed to have the shape shown in Fig 3.2l (a). It is required to find the location of the supports A and B such that the bending moment at the midpoint is zero.

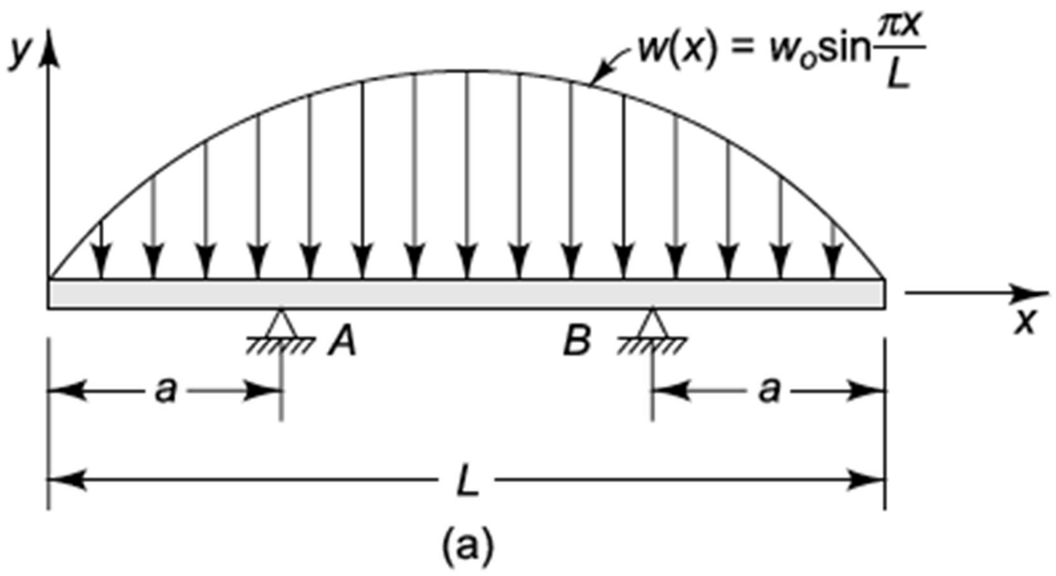

Fig. 3.21 *Example 3.9*

Sol)

$R_A = R_B = R/2$ (a)

Here, $R = \int_0^L w(x)dx = w_0 \int_0^L \sin\frac{\pi x}{L} \ dx = 2w_0 L/\pi$ (b)

$\therefore \ q(x) = -w_0 \sin\frac{\pi x}{L} + \frac{w_0 L}{\pi}\langle x-a\rangle_{-1} + \frac{w_0 L}{\pi}\langle x-(L-a)\rangle_{-1}$ (c)

→ $V(x) = -\frac{w_0 L}{\pi}\cos\frac{\pi x}{L} - \frac{w_0 L}{\pi}\langle x-a\rangle^0 - \frac{w_0 L}{\pi}\langle x-(L-a)\rangle^0 + C_1$ (d)

B.C.) $V(0) = 0$ → $C_1 = w_0 L/\pi$ (e)

$M_b(x) = -\frac{w_0 L}{\pi}\left(x - \frac{L}{\pi}\sin\frac{\pi x}{L}\right) + \frac{w_0 L}{\pi}\langle x-a\rangle^1$

$+\frac{w_0 L}{\pi}\langle x-(L-a)\rangle^1 + C_2$ (f)

B.C.) $M_b(0) = 0$ → $C_2 = 0$

cf. M_b will vanish at $x = L/2$ if $a = L/\pi$.

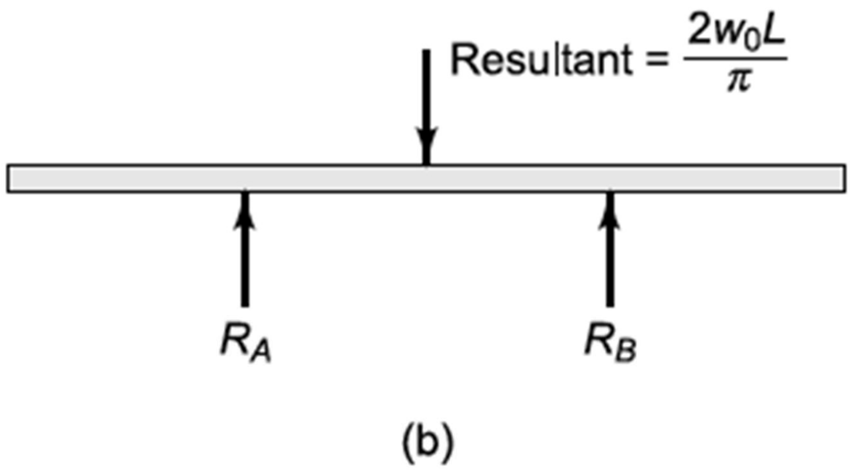

Fig. 3.21 *Example 3.9*

3.7 Fluid Force

→ In a liquid at rest the pressure at a point is the same in all directions.

▶A simple equilibrium consideration for a fluid under the action of gravity as shown in Fig 3.22.

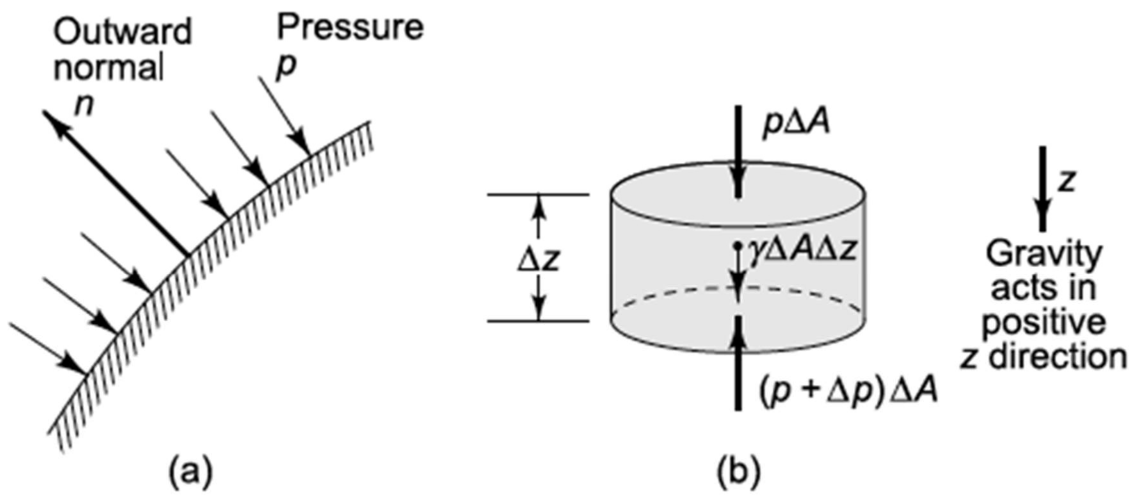

Fig. 3.22 *(a) Fluid pressure acts normal to the surface; (b) element of fluid in a gravity field*

$$p\Delta A + \gamma \Delta A \Delta z - (p + \Delta p)\Delta A = 0 \qquad (\gamma\text{: weight density})$$

In the limit,

$$dp/dz = \gamma$$

B.C.) if $p = p_0$ at $z = 0$

$$p = \gamma z + p_0$$

∴ Fluid pressure acts normal to a surface and is a linear function of depth.

▶<u>Example 3.10</u> Fig. 3.23 shows a 1.5-m-square gate which is retaining the water at half the length of the gate as shown. If it is assumed that the total pressure load on the gate is transmitted to the supports at A, B, D, and E by means of symmetrically located simply supported beams *AB* and *DE*, find the maximum bending moment in the beams. The bottom edge *DA* of the gate is 0.6 m below the water line, and $\gamma = 9.8\ \text{kN/m}^3$.

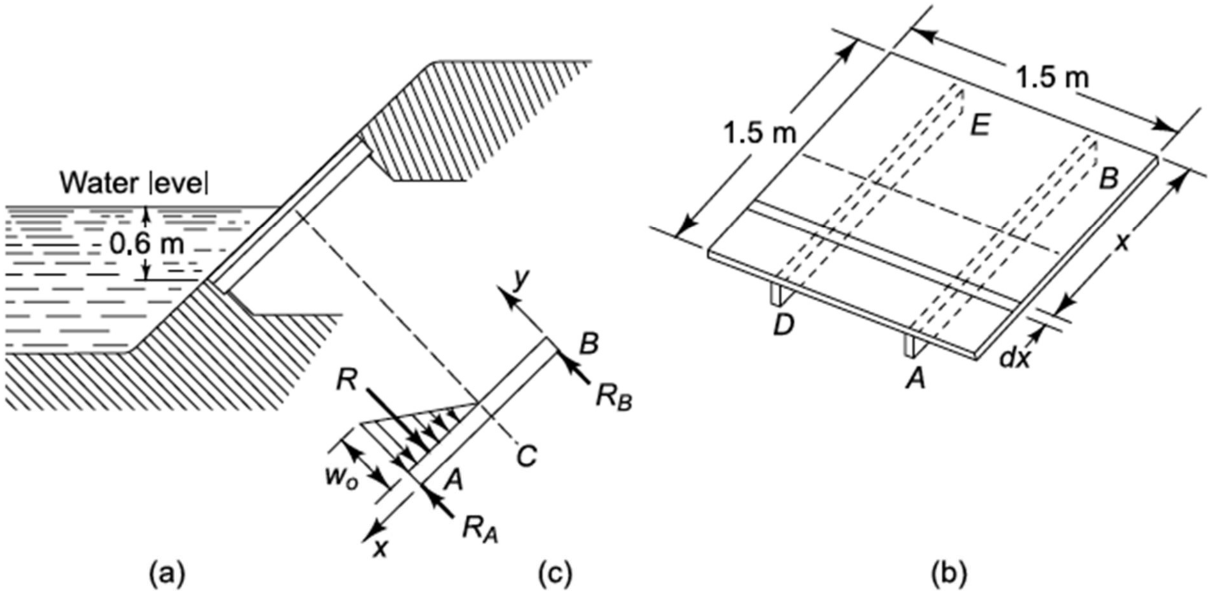

Fig. 3.23 *Example 3.10*

$p_A = \gamma z_A = (9.8)(0.6) = 5.88 \ \text{kN/m}^2$ (a)

$\therefore \ w(x) = (1.5/2) \times p(x)$ and $w_0 = 0.75 p_A$ (b)

▷For beam *AB*

$\therefore \ q(x) = -(w_0/0.75) < x - 0.75 >^1$ (c)

$-V(x) = -(2/3) \times w_0 < x - 0.75 >^2 + C_1$ (e)

B.C.) $V(0) = -R_B = -C_1$

Since, $\sum M_A = 0 \ \rightarrow \ 1.5R_B = (1/4)R \ \rightarrow \ R_B = (1/16)w_0$

$\rightarrow \ M_b(x) = -(2/9) \ w_0 < x - 0.75 >^3 + (1/16) \ w_0 x + C_2$ (h)

B.C.) $M_b(0) = 0 \ \rightarrow \ C_2 = 0$

$(M_b)_{max}$ is located in this case between A and C at the point where $V = 0$.

From Eq. (e)

$V(x) = 0 = \frac{2}{3}(4.41)\langle x - 0.75\rangle^2 - \frac{1}{16}(4.41)$

$\therefore \ x_0 = 1.056 \ \text{m}$

$\rightarrow \ M_b(x_0) = 319 \ \text{N} \cdot \text{m}$