Chap 3. Tools used in Plastic Analysis and Design

1. Introduction
2. Assumption of Ductility of Steel
3. Small deflection
4. Virtual work equations
5. Fundamental theorem
6. Upper and Lower Bound Solutions

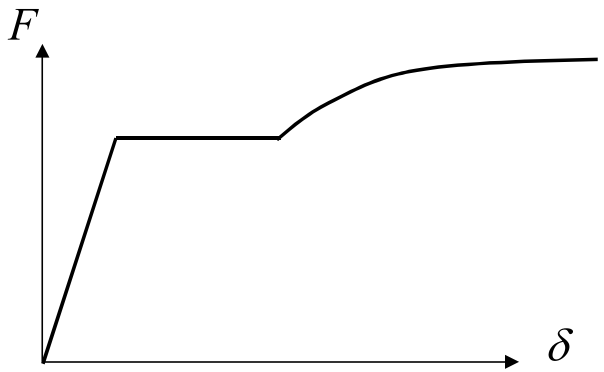
3.1 introduction

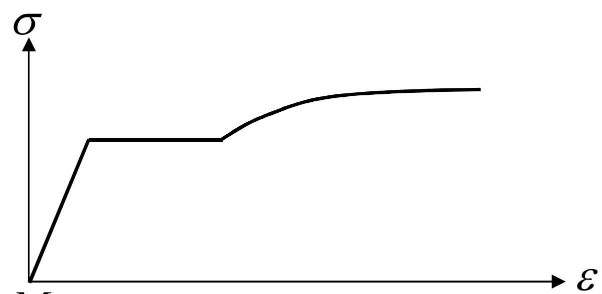
Mechanics	Limit Analysis
- Equilibrium	- Equilibrium
- Compatibility	- Failure mechanism
- Constitutive Relationship	- Yield condition
	- Virtual works

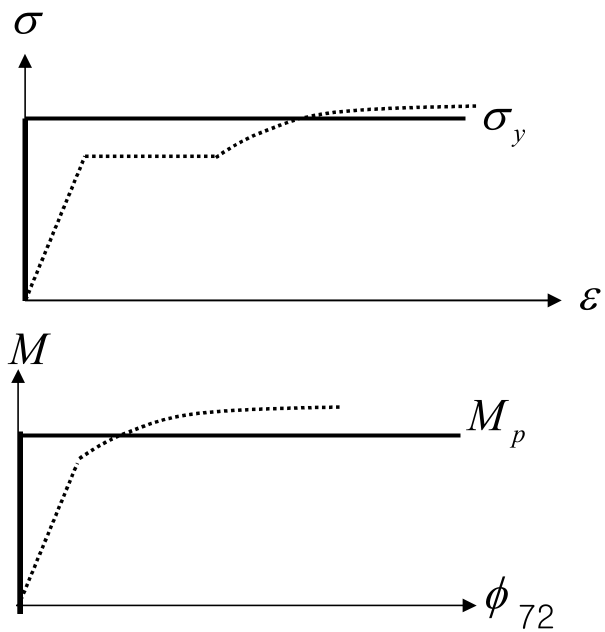
Lower bound

Upper bound

3.2 Ductility



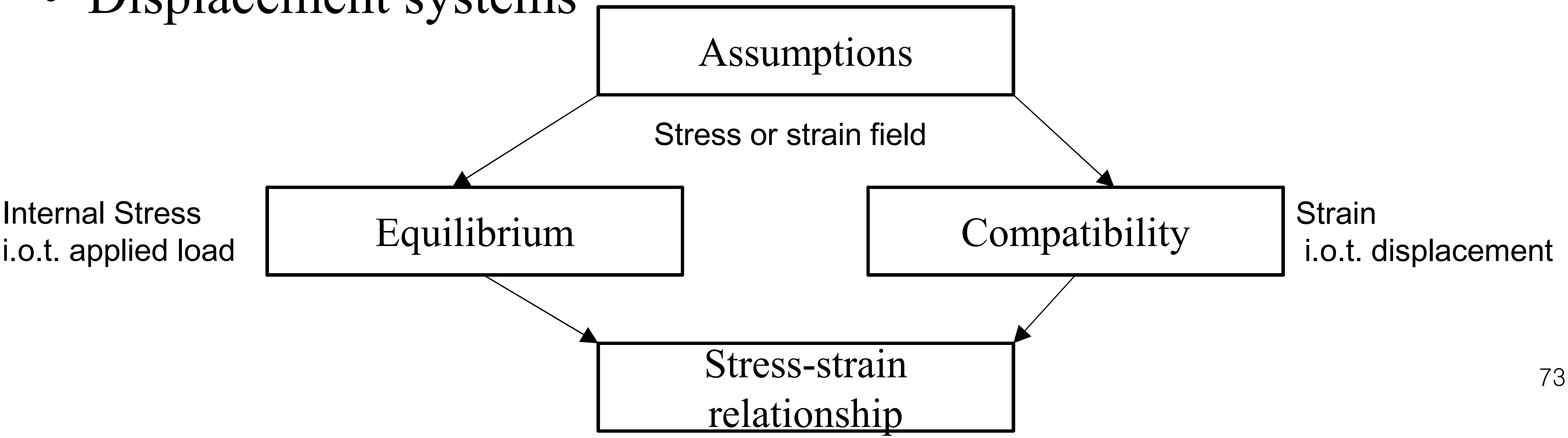




Plastic Idealization

Virtual Work Method

- The virtual work equation relates a system of forces in equilibrium to a system of compatible displacement.
- Equilibrium systems
- Displacement systems



Dual

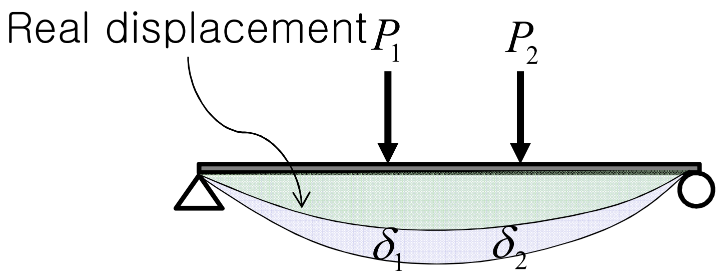
Statics ←-----------→ Kinematics

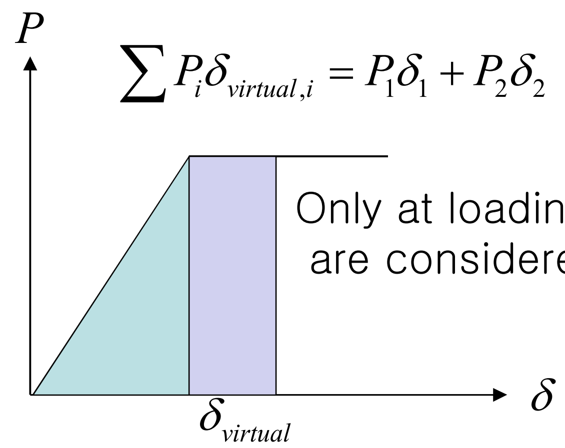
Equilibrium←-----------→ Compatibility

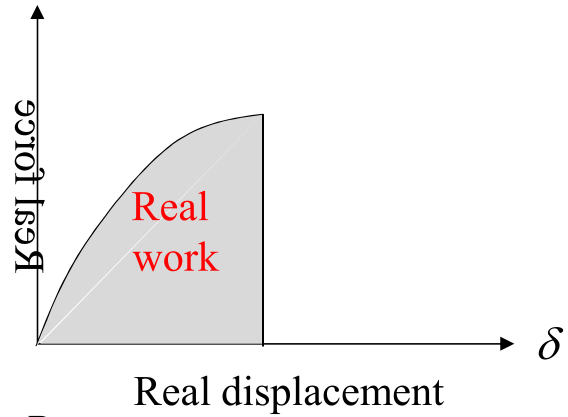
Stress ←-----------→ Strain

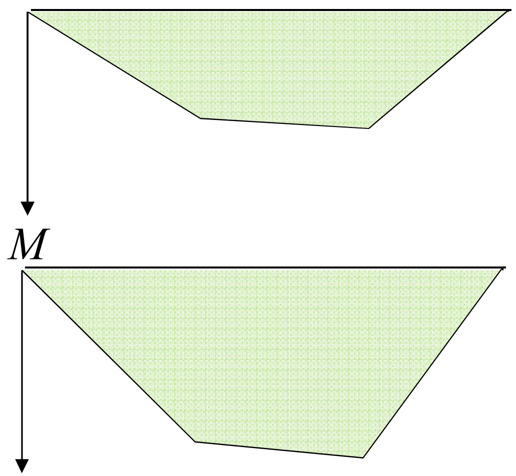
Force ←-----------→ Displacement

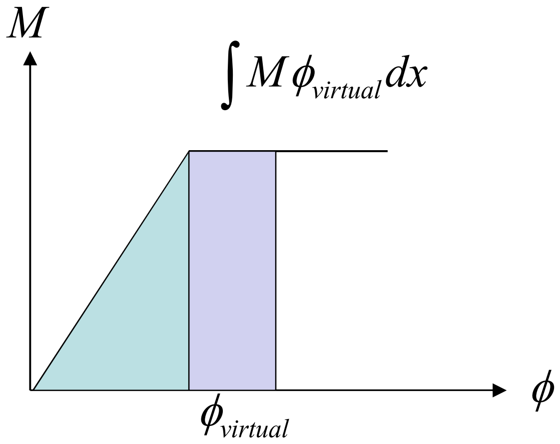
3.4 Virtual work theorem

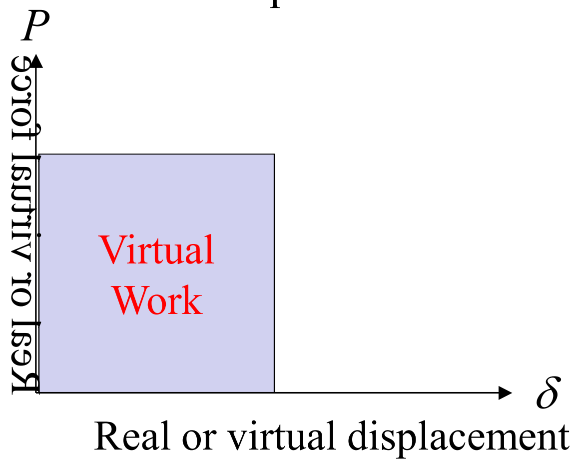






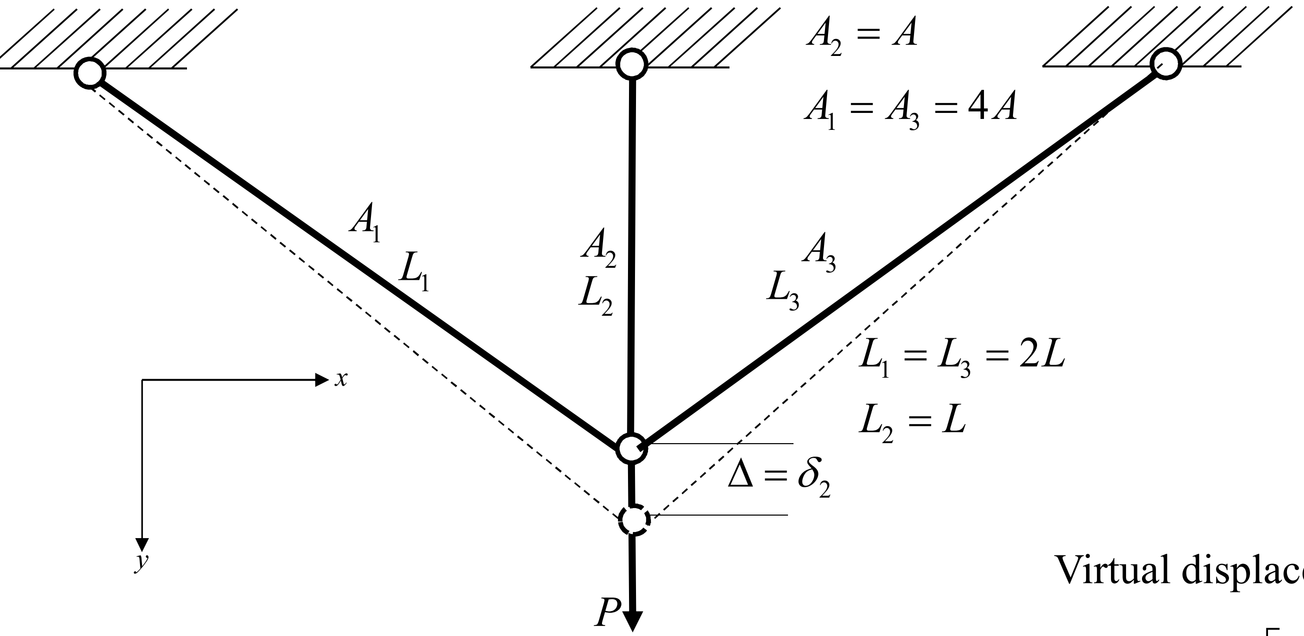


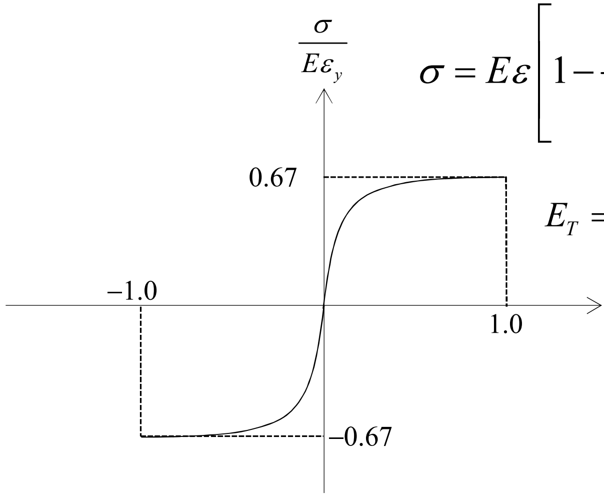




$$\phi_{real} = \frac{M}{EI} = \phi_{virtual}$$

$$\sum P_i \delta_{virtual,i} = \int M \phi_{virtual} dx$$





$$\sigma = E\varepsilon\left[1 - \frac{1}{3}\left(\frac{\varepsilon}{\varepsilon_y}\right)^2\right]$$

$$E_T = \frac{\partial\sigma}{\partial\varepsilon} = E\left[1 - \left(\frac{\varepsilon}{\varepsilon_y}\right)^2\right]$$

Virtual displacement

$$2 * 4AE\frac{\Delta}{2}\frac{1}{2L}\left[1 - \frac{1}{3}\left(\frac{\Delta}{2}\frac{1}{2L}\frac{1}{\varepsilon_y}\right)^2\right]\left(\frac{1}{2}\right) + AE\frac{\Delta}{L}\left[1 - \frac{1}{3}\left(\frac{\Delta}{L}\frac{1}{\varepsilon_y}\right)^2\right](1) = 1 * P$$

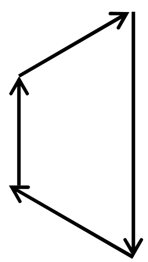
Unit displacement

Deformation

$$\delta_i = \frac{L_{ix}}{L_i}\delta_x + \frac{L_{iy}}{L_i}\delta_y + \frac{L_{iz}}{L_i}\delta_z$$

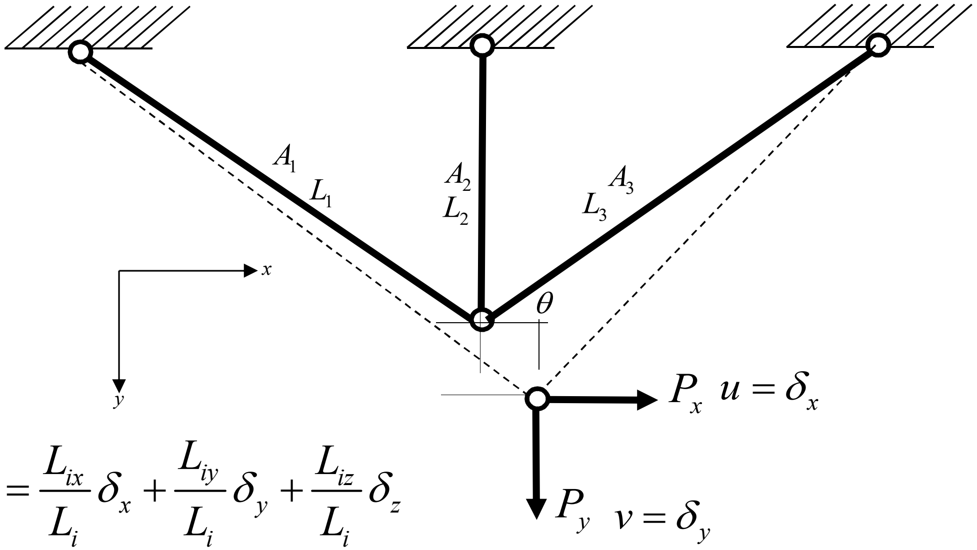
$$\begin{cases} \delta_1 = \dfrac{\Delta}{2} \\ \delta_2 = \Delta \\ \delta_3 = \dfrac{\Delta}{2} \end{cases}$$

Equilibrium

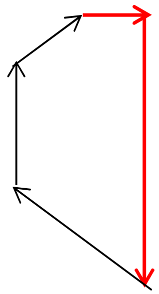


Load-displacement relationship

$$\frac{\Delta}{L}\left[2 - \frac{17}{48}\left(\frac{\Delta}{L}\frac{1}{\varepsilon_y}\right)^2\right] = \frac{P}{AE}$$



$$\delta_i = \frac{L_{ix}}{L_i}\delta_x + \frac{L_{iy}}{L_i}\delta_y + \frac{L_{iz}}{L_i}\delta_z$$

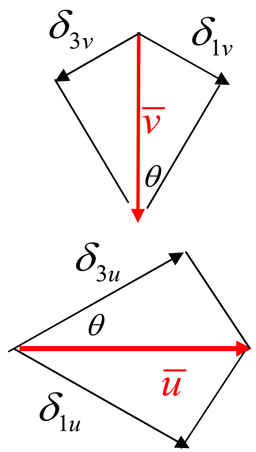


$$\begin{cases} N_1\cos\theta - N_3\cos\theta - P_x = 0 \\ N_1\sin\theta + N_3\sin\theta + N_2 - P_y = 0 \end{cases}$$

Equilibrium

The principle of virtual displacements

$$N_1\bar{\delta}_1 + N_2\bar{\delta}_2 + N_3\bar{\delta}_3 = P_x\bar{u} + P_y\bar{v}$$



Virtual displacement

$$\begin{cases} \bar{\delta}_1 = \bar{u}\cos\theta + \bar{v}\sin\theta \\ \bar{\delta}_2 = \bar{v} \\ \bar{\delta}_3 = -\bar{u}\cos\theta + \bar{v}\sin\theta \end{cases}$$

Then

$$\bar{u}\left(N_1\cos\theta - N_3\cos\theta - P_x\right)$$
$$+\bar{v}\left(N_1\sin\theta + N_3\sin\theta + N_2 - P_y\right) = 0$$

The principle of virtual forces

$$\bar{N}_1\delta_1 + \bar{N}_2\delta_2 + \bar{N}_3\delta_3 = \bar{P}_x u + \bar{P}_y v$$

Virtual forces

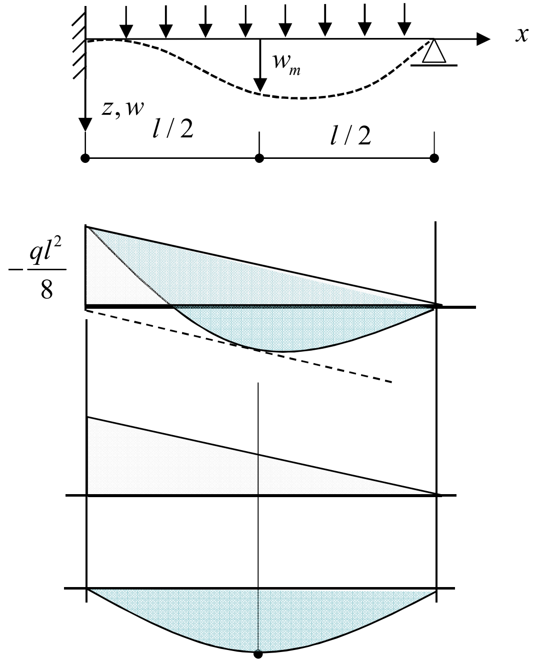
$$\begin{cases} \bar{N}_1\cos\theta - \bar{N}_3\cos\theta - \bar{P}_x = 0 \\ \bar{N}_1\sin\theta + \bar{N}_3\sin\theta + \bar{N}_2 - \bar{P}_y = 0 \end{cases}$$

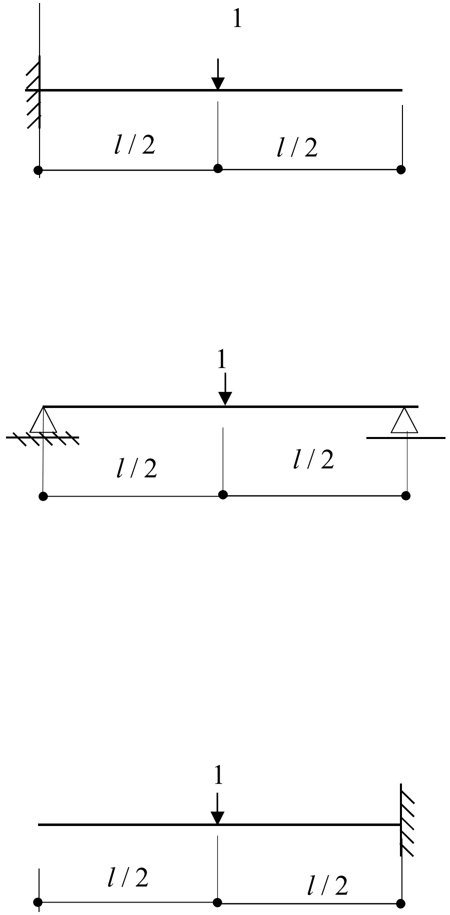
$$\bar{N}_1(\delta_1 - u\cos\theta - v\sin\theta) + \bar{N}_3\left(\delta_3 + u\cos\theta - v\sin\theta\right) + \bar{N}_1\left(\delta_2 - v\right) = 0$$

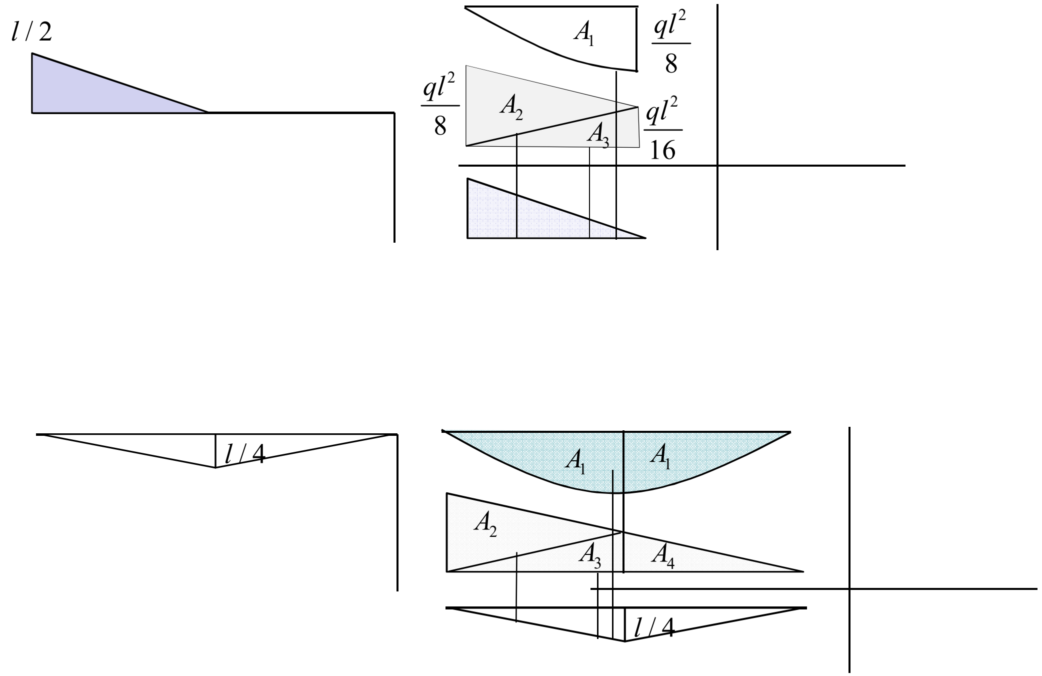
Compatibility equation

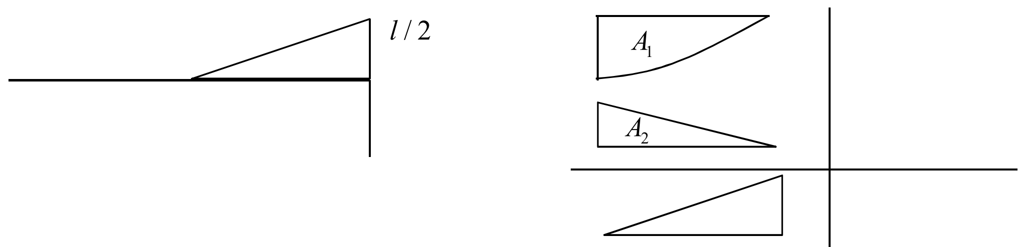
$$\begin{cases} \delta_1 - u\cos\theta - v\sin\theta = 0 \\ \delta_2 - v = 0 \\ \delta_3 + u\cos\theta - v\sin\theta = 0 \end{cases}$$

The principle of virtual forces

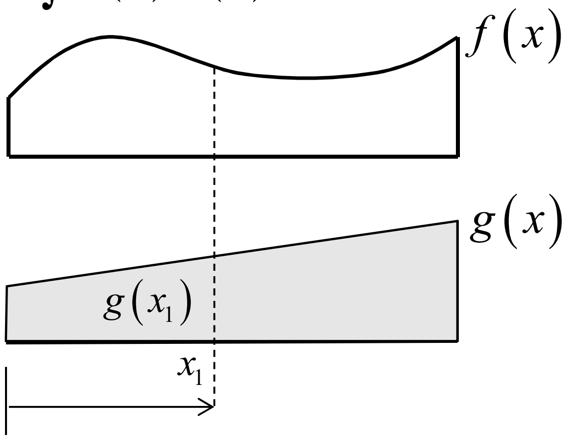


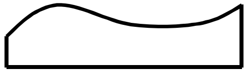


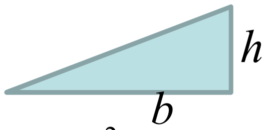
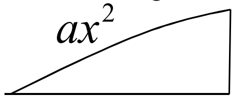
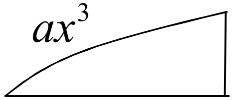
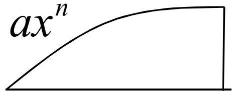


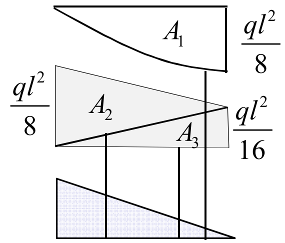


$$\int f(x)\,g(x)\,dx = ?$$



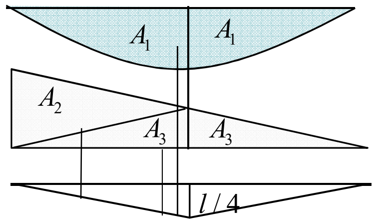
$$\int fg\,dx =$$  $$\times g(x_1)$$

	Area	Centroid
	$\frac{1}{2}bh$	$\frac{2}{3}b$
	$\frac{2}{3}bh$	$\frac{5}{8}b$
	$\frac{3}{4}bh$	$\frac{3}{5}b$
	$\frac{n}{n+1}bh$	$\frac{n+3}{2(n+2)}b$



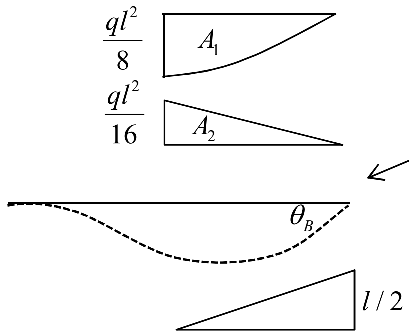
$$w_m = \frac{1}{EI}\left[\overbrace{-\frac{2}{3}\frac{l}{2}\frac{ql^2}{8}}^{A_1}\frac{3}{8}\frac{l}{2}+\overbrace{\frac{1}{2}\frac{l}{2}\frac{ql^2}{8}}^{A_2}\frac{2}{3}\frac{l}{2}+\overbrace{\frac{1}{2}\frac{l}{2}\frac{ql^2}{16}}^{A_3}\frac{1}{3}\frac{l}{2}\right]$$

$$=\frac{1}{EI}\left[\frac{l^2}{2}\frac{ql^2}{8}\left(-\frac{1}{8}+\frac{1}{6}+\frac{1}{24}\right)\right]=\frac{1}{EI}\frac{l^2}{2}\frac{ql^2}{8}\frac{1}{12}=\frac{ql^4}{192EI}$$



$$w_m = \frac{1}{EI}\left[-2*\frac{2}{3}\frac{l}{2}\frac{ql^2}{8}\frac{5}{8}\frac{l}{4}+\frac{1}{2}\frac{l}{2}\frac{ql^2}{8}\frac{1}{3}\frac{l}{4}+2*\frac{1}{2}\frac{l}{2}\frac{ql^2}{16}\frac{2}{3}\frac{l}{4}\right]$$

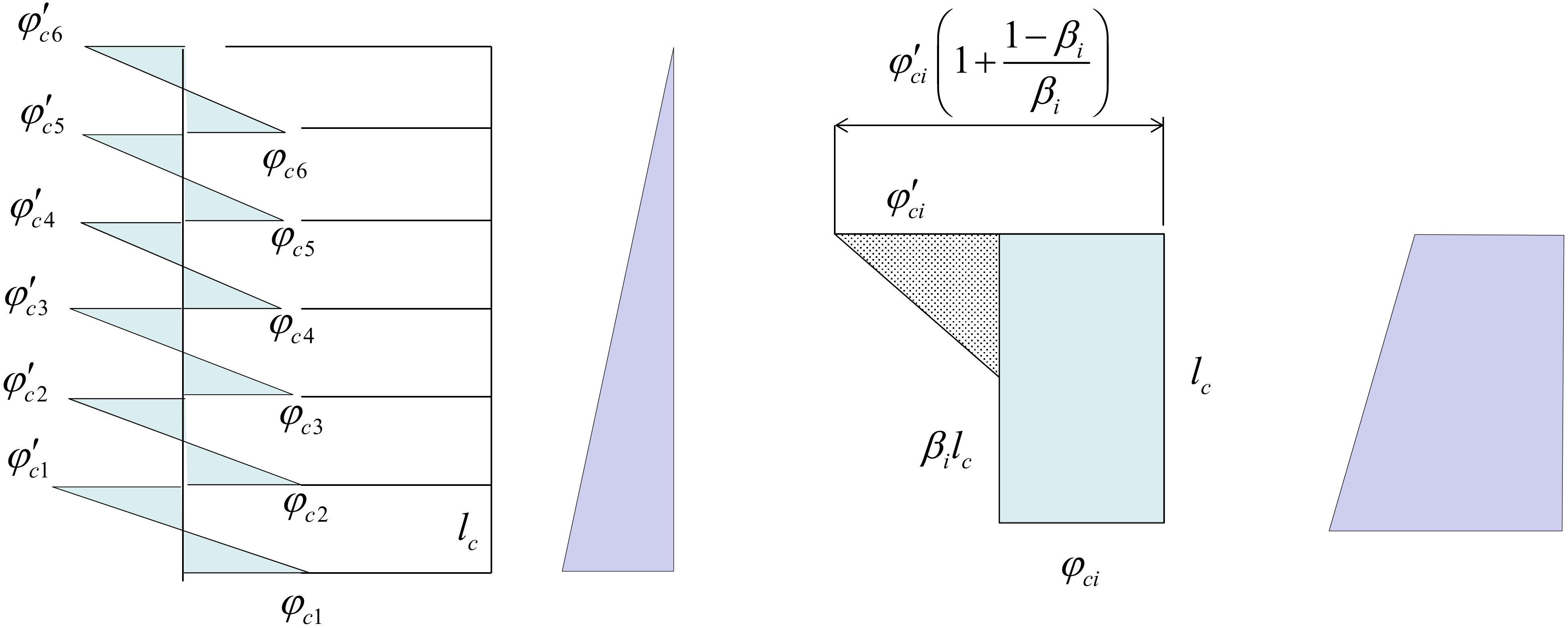
$$=\frac{1}{EI}\left[\frac{l^2}{2}\frac{ql^2}{8}\left(-\frac{5}{24}+\frac{1}{24}+\frac{1}{12}\right)\right]=\frac{1}{EI}\frac{l^2}{2}\frac{ql^2}{8}\frac{1}{12}=\frac{ql^4}{192EI}$$



$$w_m-\theta_B\frac{l}{2}=\frac{1}{EI}\left[-\frac{2}{3}\frac{l}{2}\frac{ql^2}{8}\frac{3}{8}\frac{l}{2}+\frac{1}{2}\frac{l}{2}\frac{ql^2}{16}\frac{1}{3}\frac{l}{2}\right]$$

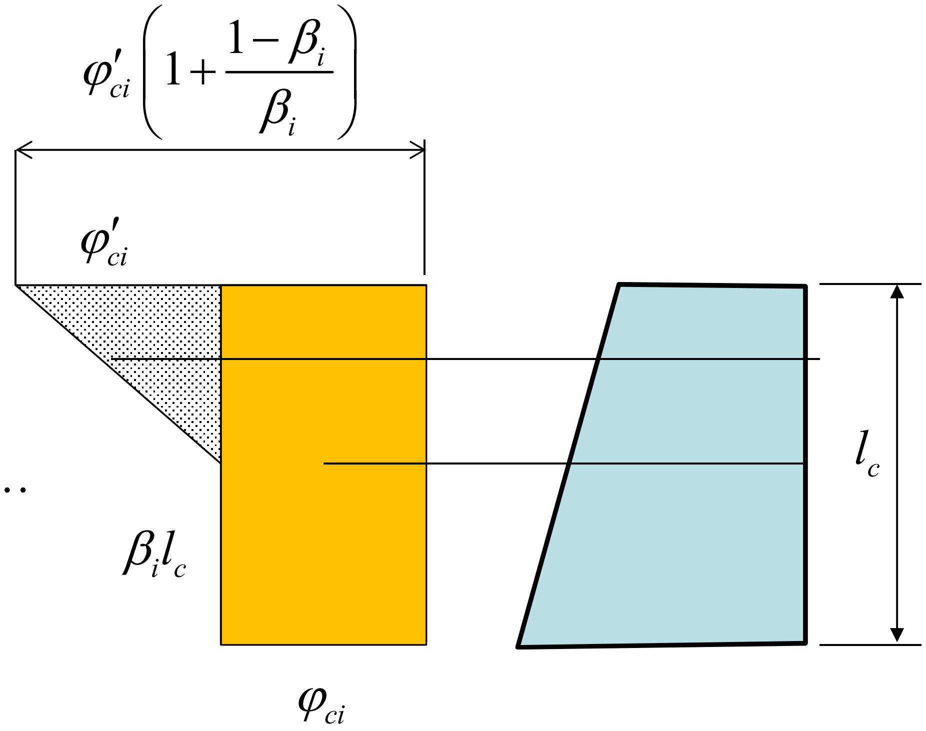
$$w_m=\frac{1}{EI}\left[\frac{l^2}{2}\frac{ql^2}{8}\left(-\frac{1}{8}+\frac{1}{24}\right)\right]+\frac{l}{2}\frac{ql^4}{48EI}$$

$$=\frac{1}{EI}\frac{ql^4}{16}\left(-\frac{1}{12}+\frac{1}{6}\right)=\frac{ql^4}{192EI}$$



$$
\begin{aligned}
\Delta_y &= \varphi_{c1} l_c\left(r l_c-\frac{l_c}{2}\right)-\varphi_{c1}\left(1+\frac{1-\beta_1}{\beta_1}\right)\frac{l_c}{2}\left(r l_c-\frac{2l_c}{3}\right)+\varphi_{c2} l_c\left(r l_c-\frac{3l_c}{2}\right)-\varphi_{c2}\left(1+\frac{1-\beta_2}{\beta_2}\right)\frac{l_c}{2}\left(r l_c-\frac{5l_c}{3}\right)+\cdots \\
&+\varphi_{ci} l_c\left[r l_c-\left(i-\frac{1}{2}\right)l_c\right]-\varphi_{ci}\left(1+\frac{1-\beta_i}{\beta_i}\right)\frac{l_c}{2}\left[r l_c-\left(i-\frac{1}{3}\right)l_c\right]+\cdots+\varphi_{cr}\frac{l_c^2}{2}-\varphi_{c2}\left(1+\frac{1-\beta_r}{\beta_r}\right)\frac{l_c^2}{6} \\
&=\frac{l_c^2}{6}\sum_{i=r}\frac{\varphi_{ci}}{\beta_i}\left[6\beta_i\left(r-i+0.5\right)-3\left(r-i\right)-1\right]
\end{aligned}
$$

$$
\begin{aligned}
\Delta_y &= \varphi_{c1} l_c \left(r l_c - \frac{l_c}{2} \right) - \varphi_{c1} \left(1 + \frac{1-\beta_1}{\beta_1} \right) \frac{l_c}{2} \left(r l_c - \frac{2 l_c}{3} \right) \\
&\quad + \varphi_{c2} l_c \left(r l_c - \frac{3 l_c}{2} \right) - \varphi_{c2} \left(1 + \frac{1-\beta_2}{\beta_2} \right) \frac{l_c}{2} \left(r l_c - \frac{5 l_c}{3} \right) + \cdots \\
&\quad + \varphi_{ci} l_c \left[r l_c - \left(i - \frac{1}{2} \right) l_c \right] - \varphi_{ci} \left(1 + \frac{1-\beta_i}{\beta_i} \right) \frac{l_c}{2} \left[r l_c - \left(i - \frac{1}{3} \right) l_c \right] + \cdots \\
&\quad + \varphi_{cr} \frac{l_c^2}{2} - \varphi_{c2} \left(1 + \frac{1-\beta_r}{\beta_r} \right) \frac{l_c^2}{6} \\
&= \frac{l_c^2}{6} \sum_{i=r} \frac{\varphi_{ci}}{\beta_i} \left[6\beta_i \left(r - i + 0.5 \right) - 3\left(r - i \right) - 1 \right]
\end{aligned}
$$



Application of virtual work method for plastic analysis

1.Obtain the geometric relationships of mechanism motion by assuming appropriate equilibrium.

2.Make a moment check for given mechanism by assuming appropriate displacement sets

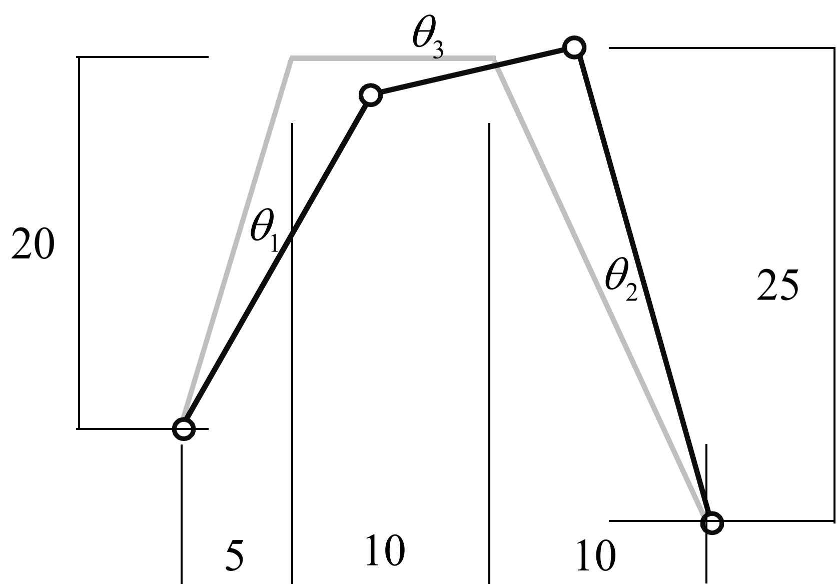
3. Prove the uniqueness, unsafe, and safe theorem

4. Obtain bounding solutions

5. Calculate deflections at collapse load

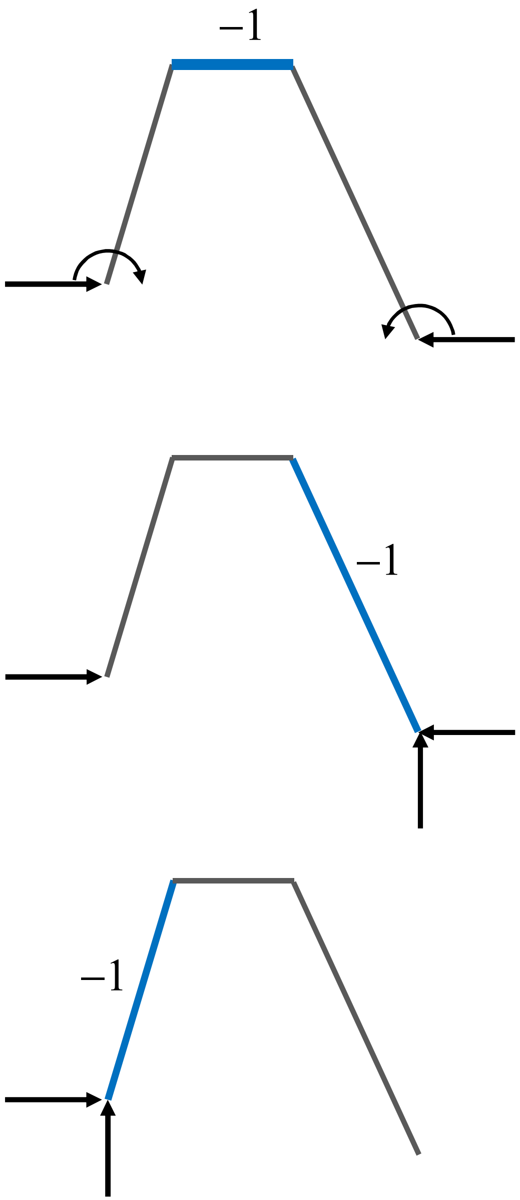
1.Obtain the geometric relationships of mechanism motion by assuming appropriate equilibrium.

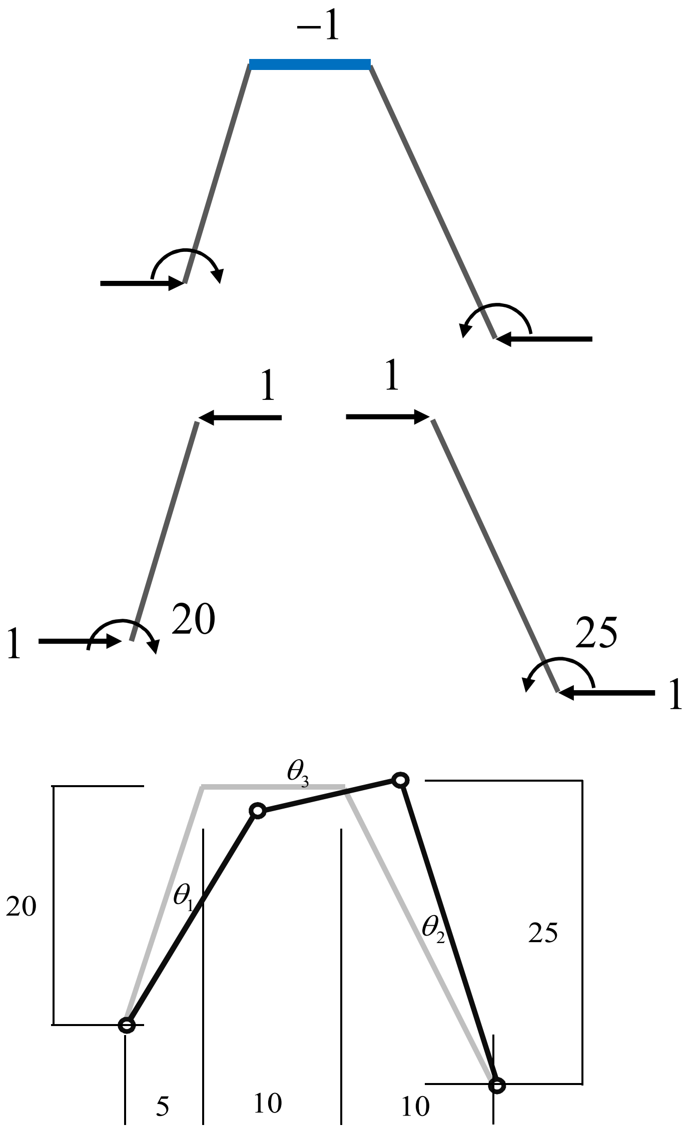
Approach
Make three set of equilibrium system



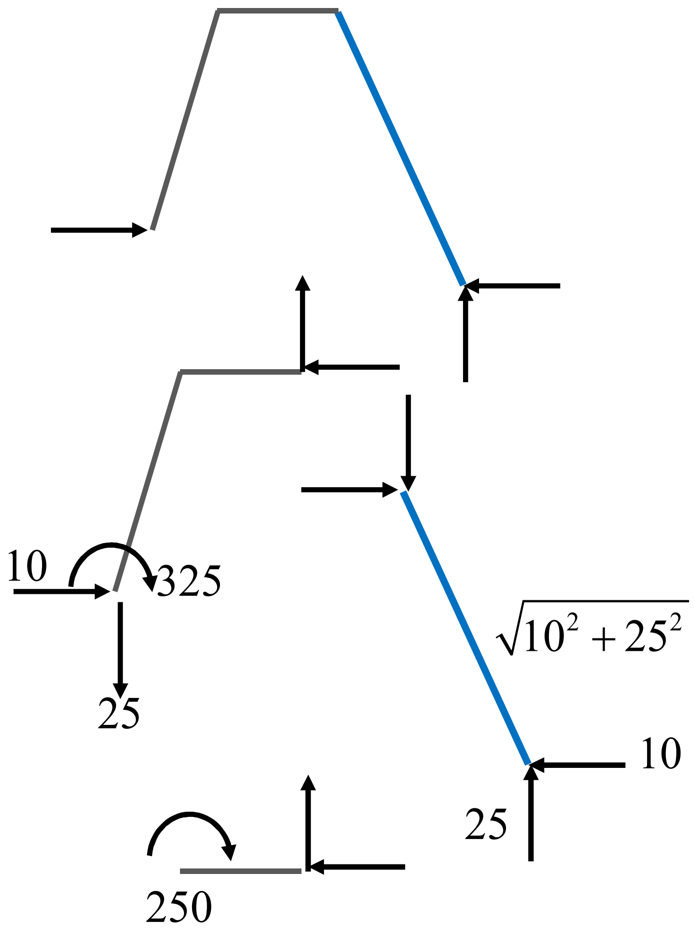
Displacement system

Equilibrium system

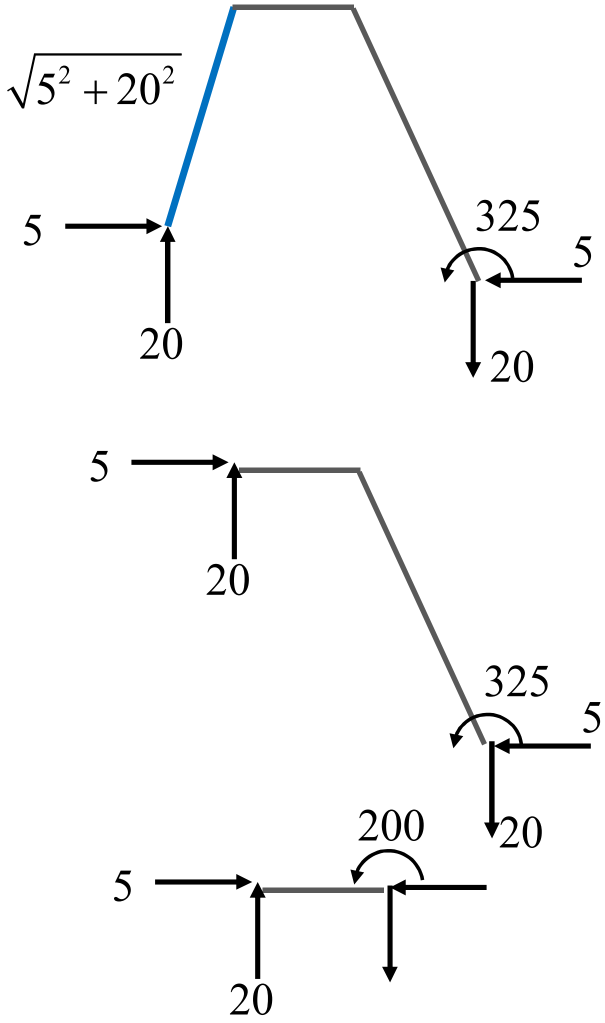




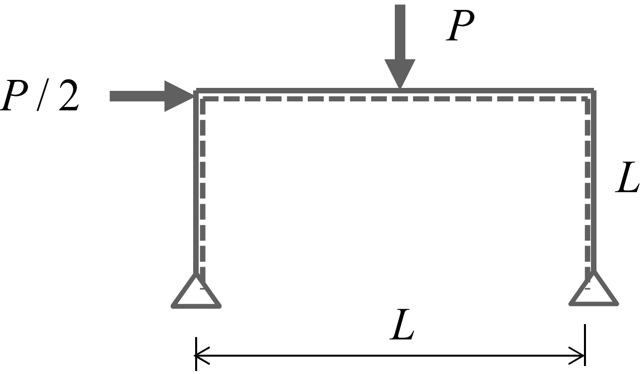
$$20\theta_1 - 25\theta_2 = 0$$

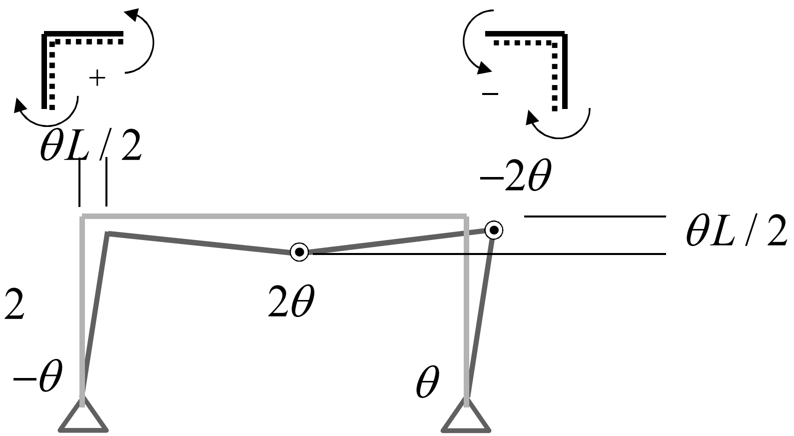


$$-250\theta_3 + 325\theta_1 = 0$$



2.Make a moment check for given mechanism by assuming appropriate displacement sets



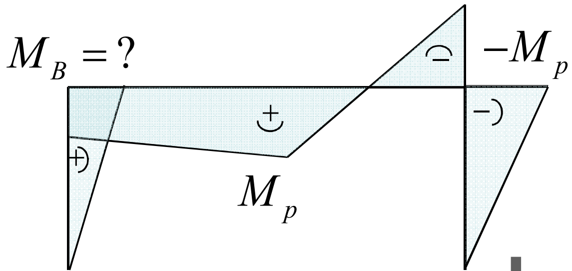


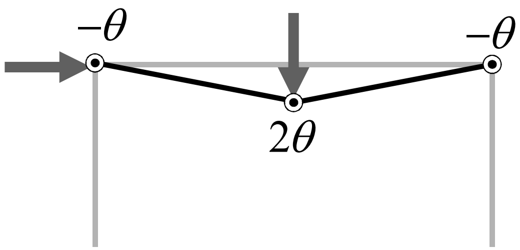
$$W_e = \theta L / 2 \times \frac{P}{2} + \theta L / 2 \times P$$

$$-W_d = \left(+M_p\right) \times \left(+2\theta\right) + \left(-M_p\right)\left(-2\theta\right)$$

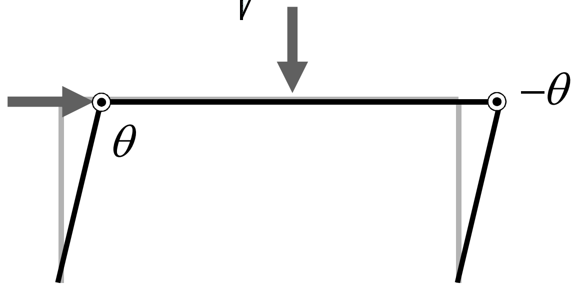
$$W_e - W_d = 0 \qquad \frac{3}{4} P \theta L = 4 M_p \theta$$

$$P_u = \frac{16}{3} \frac{M_p}{L}$$



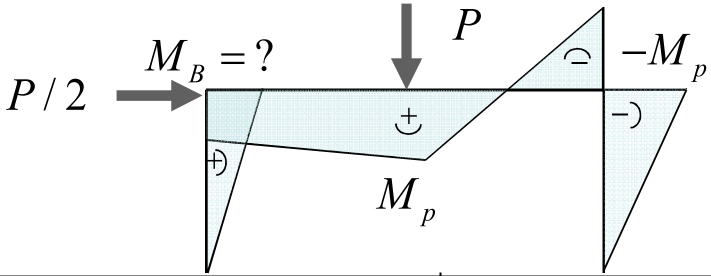
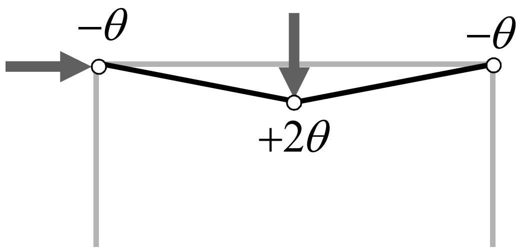
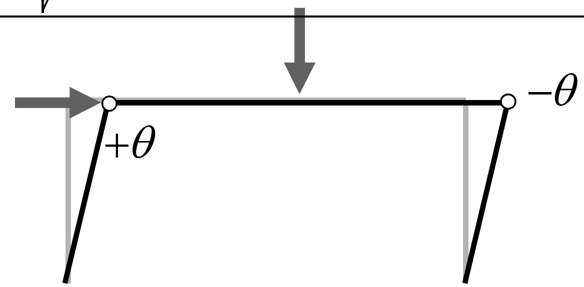


Beam mechanism

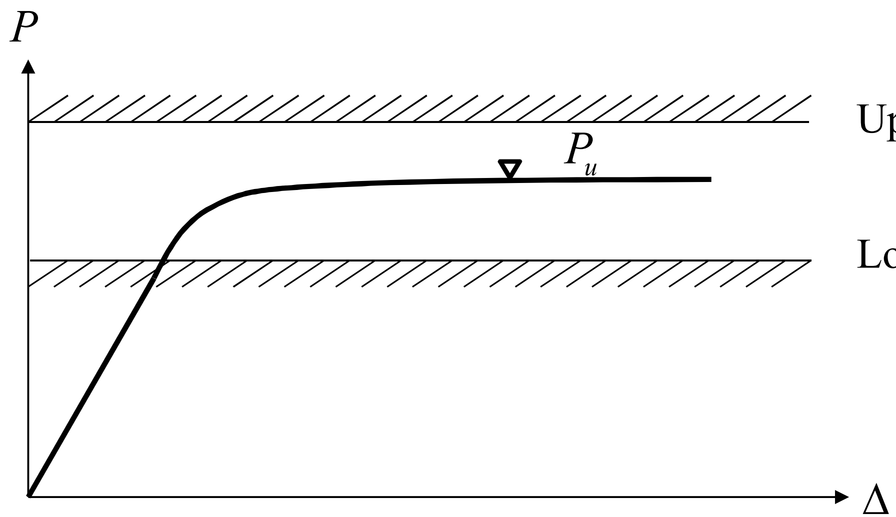


Side-sway mechanism

Virtual displacement

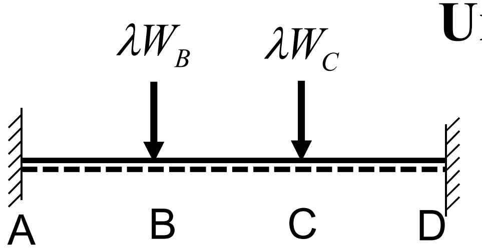
Equilibrium set		
Displacement set	 Displacement set 1	 Displacement set 2
	$P_u = \frac{16}{3}\frac{M_p}{L}$	$\sum P_i \times \delta_i = \sum M_{p_i} \times \theta_i$
	$P\frac{L}{2}\theta = M_B(-\theta) + (+M_p)(+2\theta) + (-M_p)(-\theta)$ $M_B = \frac{M_p}{3}$	$P\frac{L}{2}\theta = M_B(+\theta) + (-M_p)(-\theta)$ $M_B = \frac{M_p}{3}$

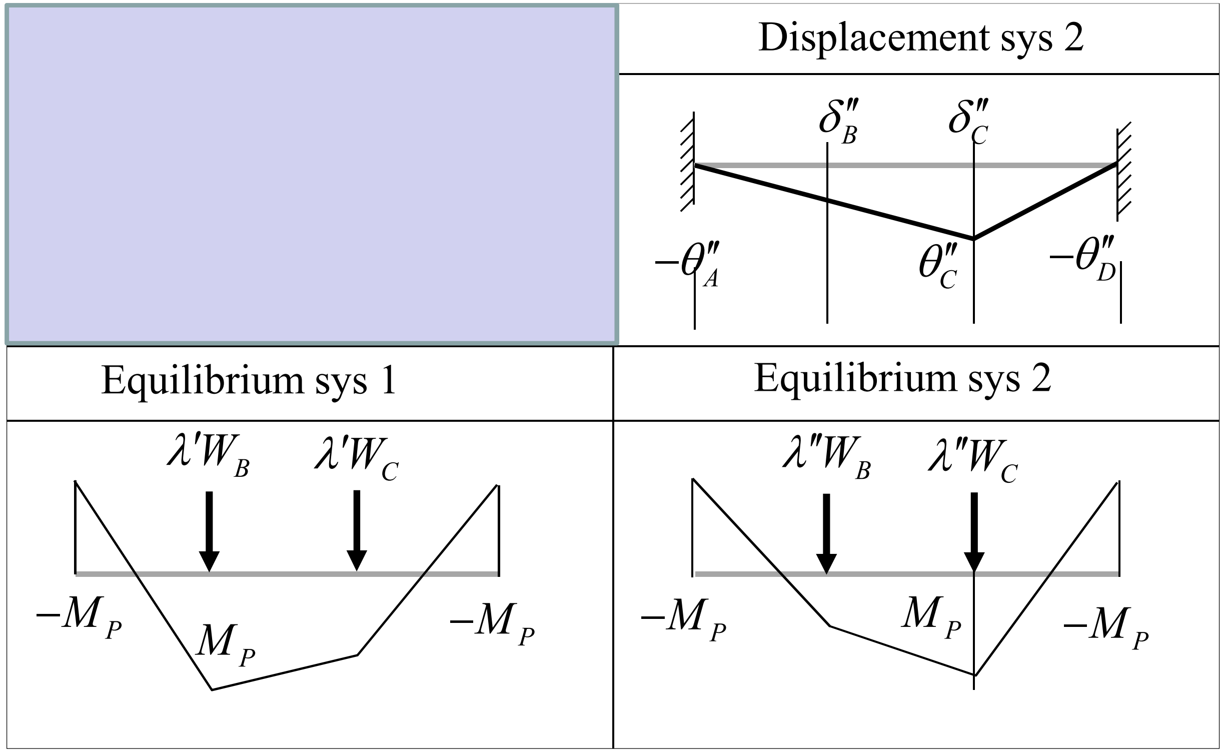
3. Prove the uniqueness, unsafe, and safe theorem



The ultimate load must exist between the lower and upper bounds

Uniqueness theorem





Displacement sys 1 **X** Equilibrium sys 1

$$\lambda' W_B \delta'_B + \lambda' W_C \delta'_C = M_P \theta'_A + M_P \theta'_B + M_P \theta'_D \quad \text{----} \quad (A)$$

Displacement sys 1 **X** Equilibrium sys 2

$$\lambda'' W_B \delta'_B + \lambda'' W_C \delta'_C = M_P \theta'_A + M_B \theta'_B + M_P \theta'_D \quad \text{----} \quad (B)$$

$$(A) - (B)$$

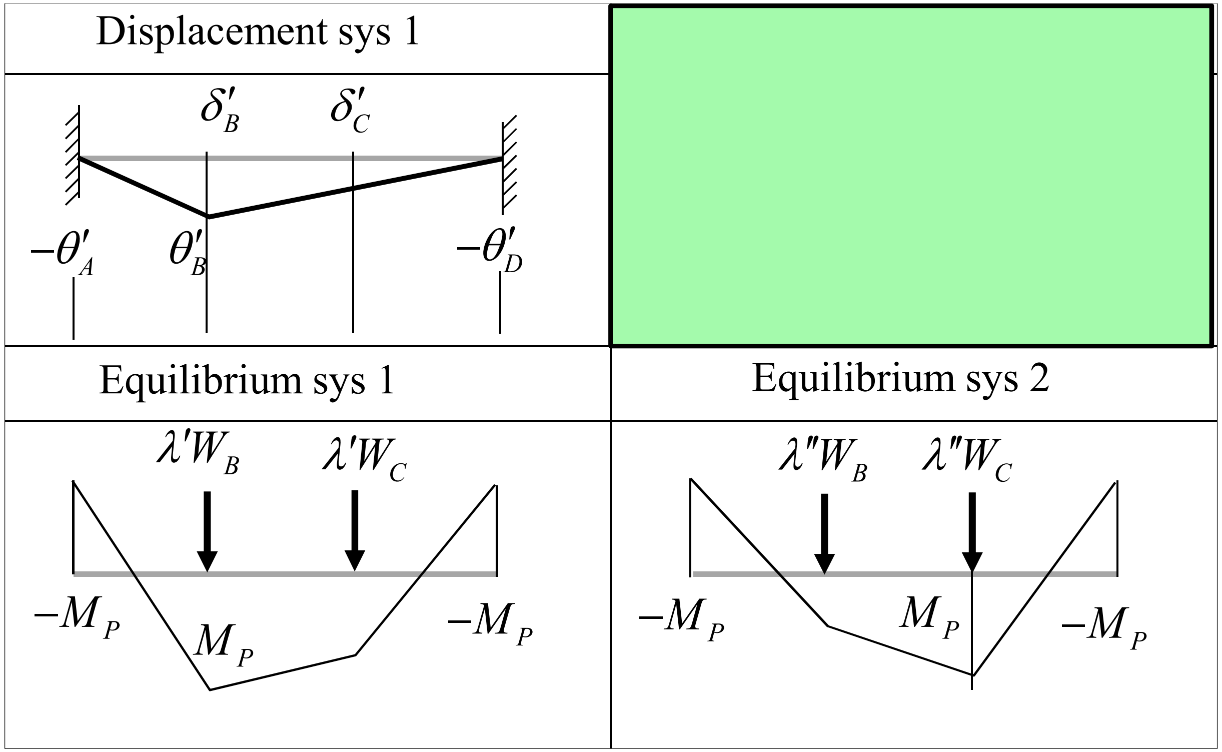
$$(\lambda' - \lambda'')(W_B \delta'_B + W_C \delta'_C) = \theta'_B (M_P - M_B)$$

Then

$$\theta'_B (M_P - M_B) \geq 0$$

so

$$\lambda' - \lambda'' \geq 0$$



Displacement sys 2 **X** Equilibrium sys 1

$$\lambda' W_B \delta''_B + \lambda' W_C \delta''_C = M_P \theta''_A + M_C \theta''_C + M_P \theta''_D \quad \text{---- (A)}$$

Displacement sys 2 **X** Equilibrium sys 2

$$\lambda'' W_B \delta''_B + \lambda'' W_C \delta''_C = M_P \theta''_A + M_P \theta''_C + M_P \theta''_D \quad \text{---- (B)}$$

(B)- (A)

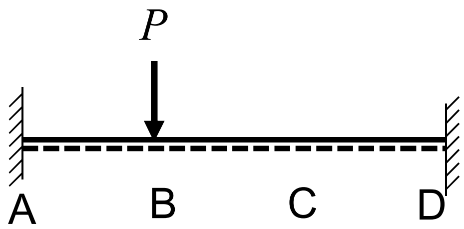
$$\left(\lambda'' - \lambda'\right)\left(W_B \delta''_B + W_C \delta''_C\right) = \theta''_C \left(M_P - M_C\right)$$

$$\theta''_C \left(M_P - M_C\right) \geq 0$$

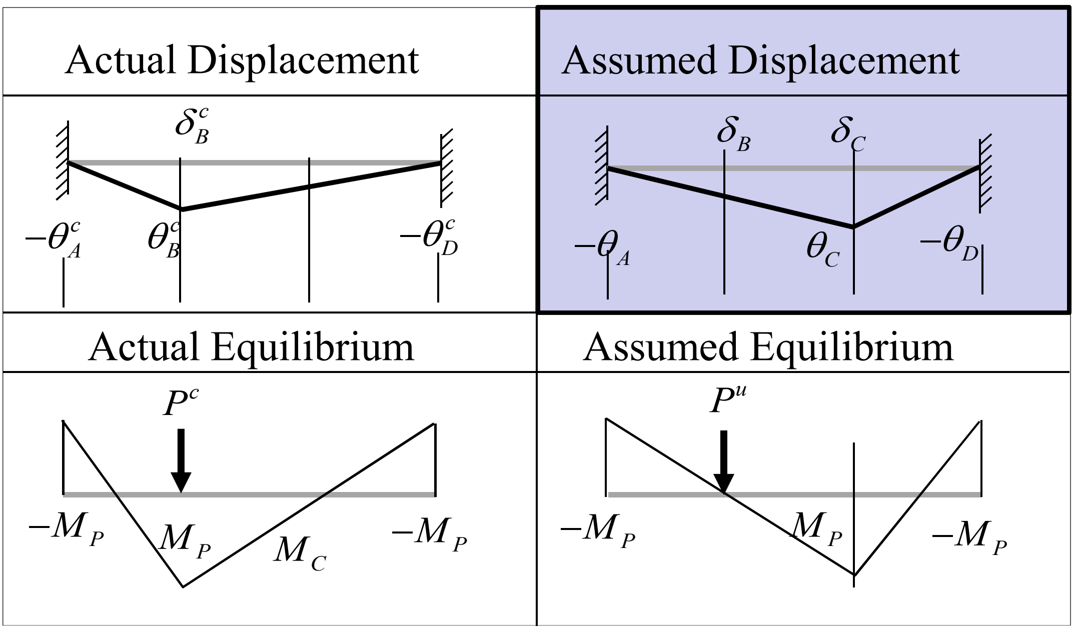
$$\lambda'' - \lambda' \geq 0$$

$$\lambda' - \lambda'' \geq 0$$

$$\Rightarrow \quad \lambda' = \lambda''$$



Unsafe theorem



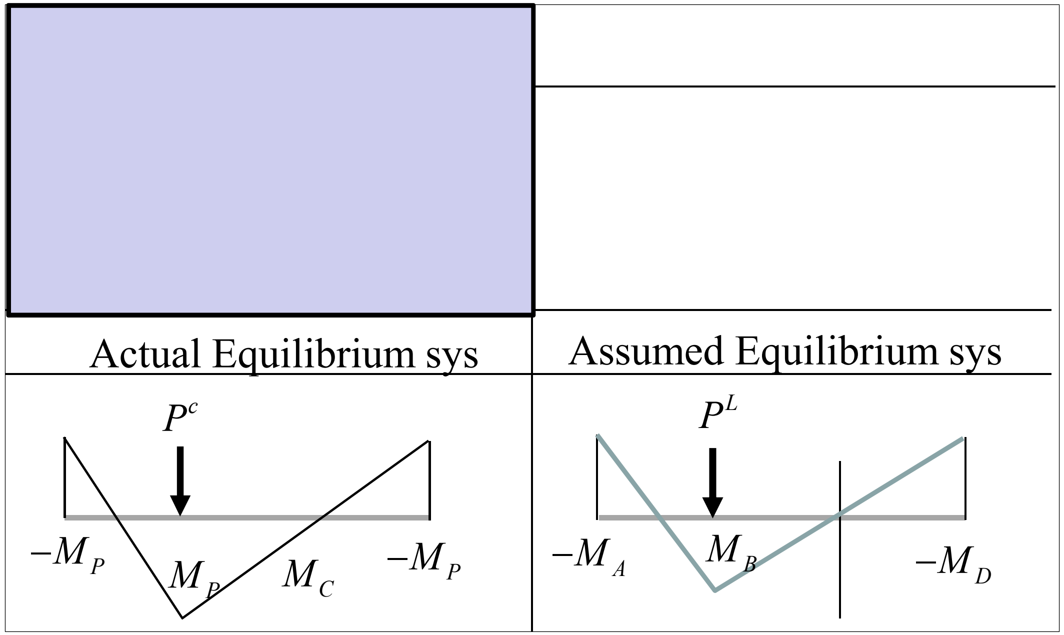
Assum Dis **X** Assum Equil $P^u\delta_B = M_P\theta_A + M_P\theta_C + M_P\theta_D$

Assum Dis **X** Act Equil $P^c\delta_B = M_P\theta_A + M_C\theta_C + M_P\theta_D$

$$\left(P^u - P^c\right)\delta_B = \left(M_P - M_C\right)\theta_C$$

$$P^u \geq P^c$$

Safe theorem



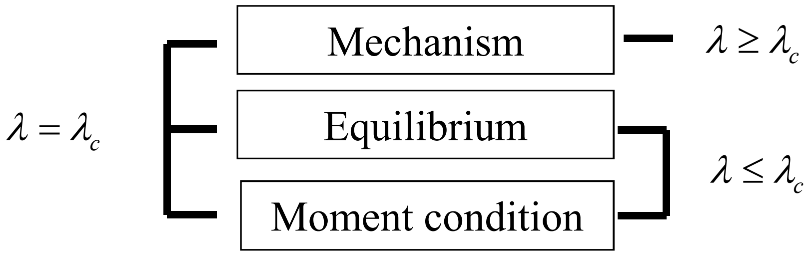
Act Dis sys **X** Ass Equil sys $P^L\delta_B^c = M_A\theta_A^c + M_B\theta_B^c + M_D\theta_D^c$

Act Dis sys **X** Act Equil sys $P^c\delta_B^c = M_P\theta_A^c + M_P\theta_B^c + M_P\theta_D^c$

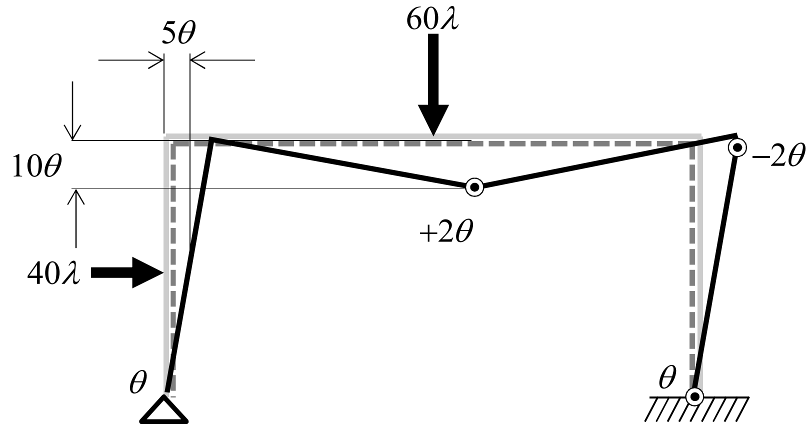
$$\left(P^L - P^c\right)\delta_B^c = \left(M_A - M_P\right)\theta_A^c + \left(M_B - M_P\right)\theta_B^c + \left(M_D - M_P\right)\theta_D^c$$

$$P^L \leq P^c$$

3.5.4 Corollaries

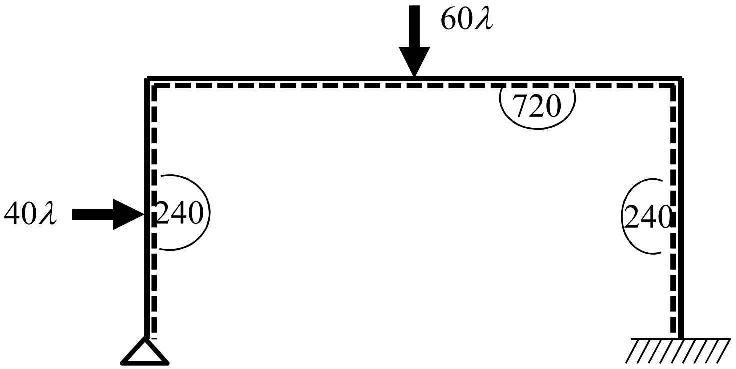


Obtain bounding solutions



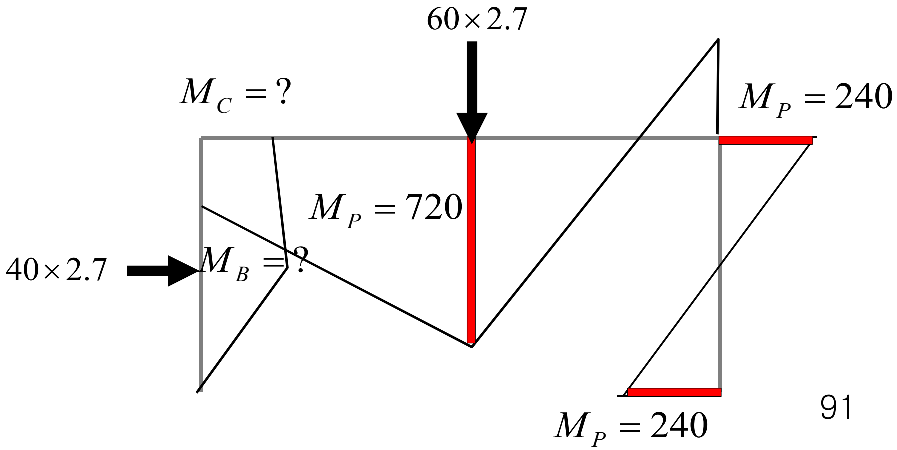
$$\sum P_i \times \delta_i = \sum M_{p_i} \times \theta_i$$

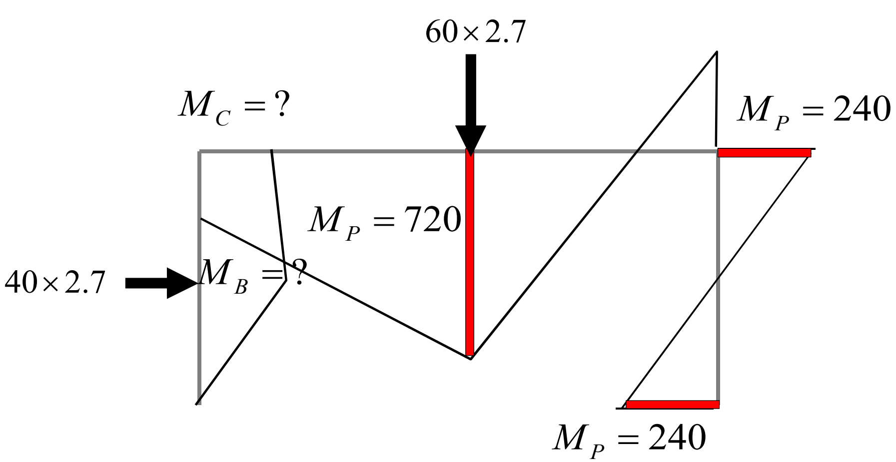
3.6 Upper-and-Lower Bound Solutions



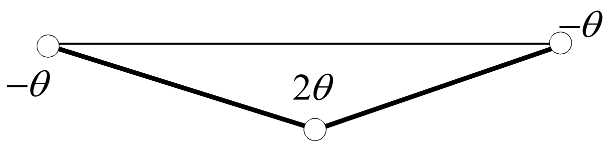
$$40\lambda(5\theta) + 60\lambda(10\theta) = (+720)(+2\theta) + (-240)(-2\theta) + (+240)(+\theta)$$

$$\lambda = 2.7$$



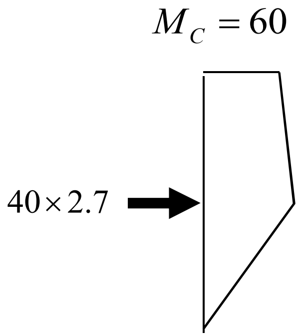


Equilibrium System

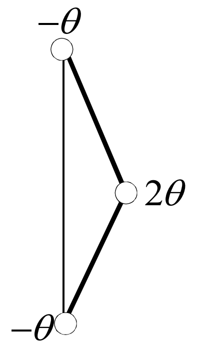


Displacement (virtual) system 1

$$60\times2.7\times10\theta=(+M_c)(-\theta)+(+720)(+2\theta)+(-240)(-\theta)$$
$$\Rightarrow M_c=60$$



Equilibrium system

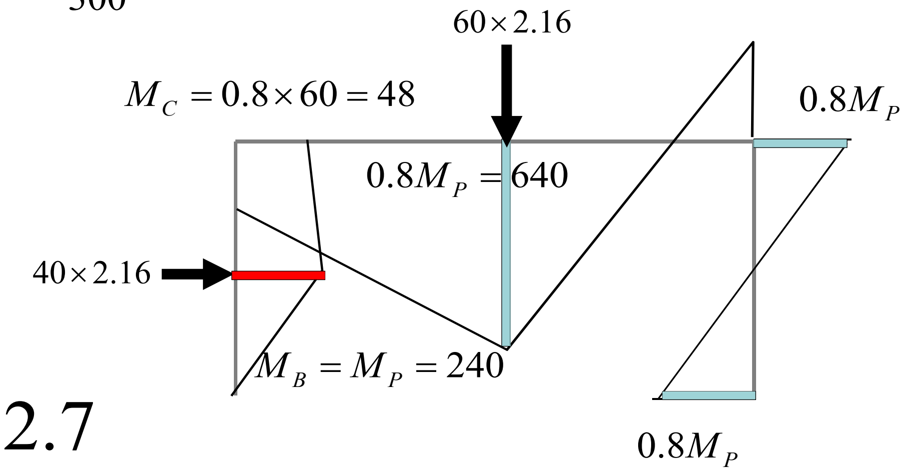


Displacement (virtual) system 2

$$40\times2.7\times5\theta=(+60)(-\theta)+(M_B)(+2\theta)$$
$$\Rightarrow M_B=300$$

$$M_P=240$$
$$M_B>M_P$$

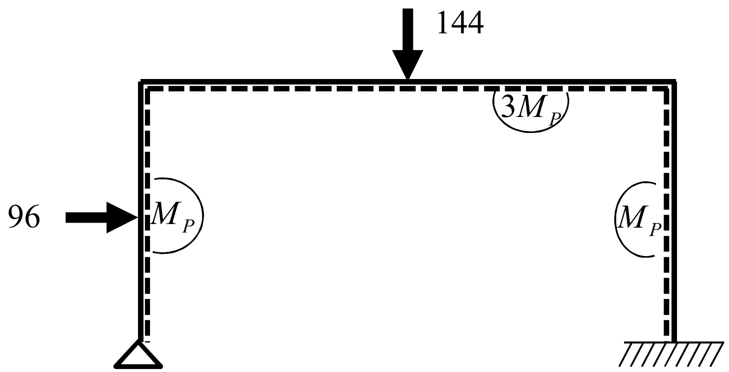
$$\frac{240}{300}=0.8\Rightarrow2.7\times0.8=2.16$$



$$2.16\le\lambda_c\le2.7$$

3.6.2 design

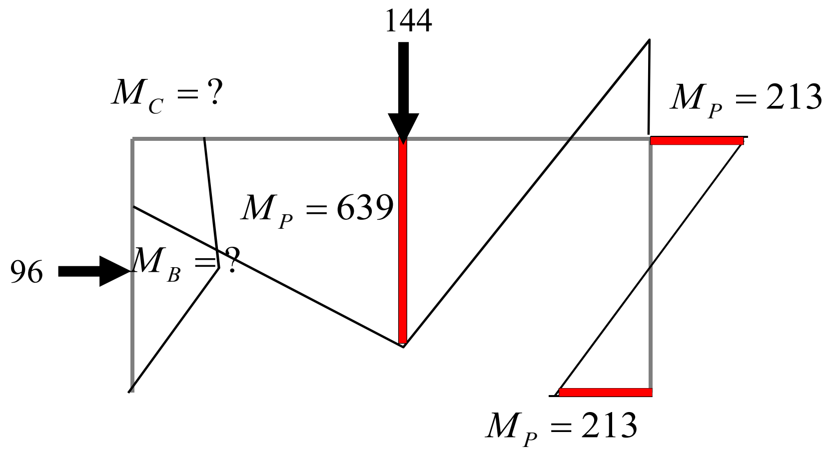
$$\sum P_i \times \delta_i = \sum M_{p_i} \times \theta_i$$

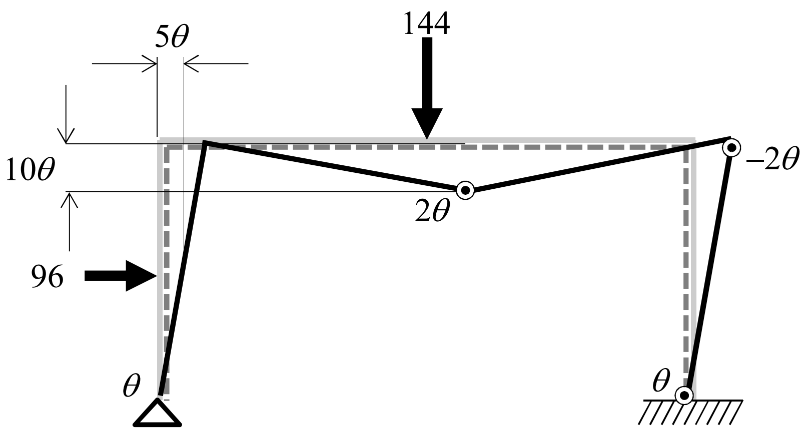


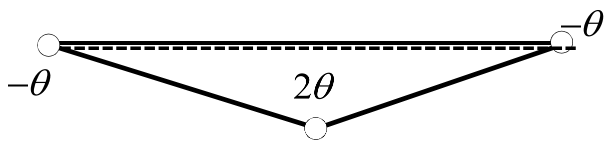
Find Mp for given load

$$96(5\theta)+144(10\theta)=(3M_P)(2\theta)+(M_P)(2\theta)+(M_P)(\theta)$$

$$M_P = 213$$

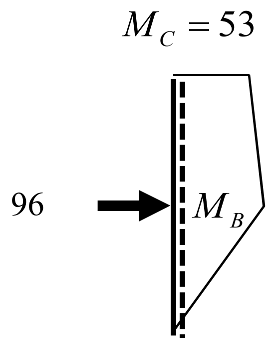




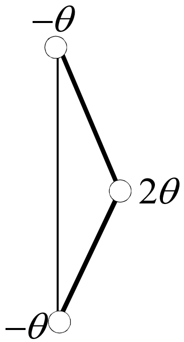


Displacement system 1

$$(+M_c)(-\theta)+(639)(2\theta)+(-213)(-\theta)$$
$$=144\times 10\theta$$
$$\Rightarrow M_c = 53$$



Equilibrium system

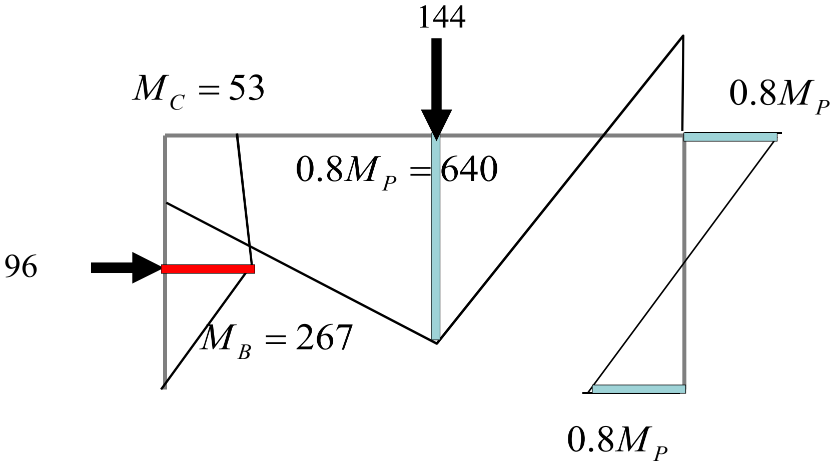


Displacement system 2

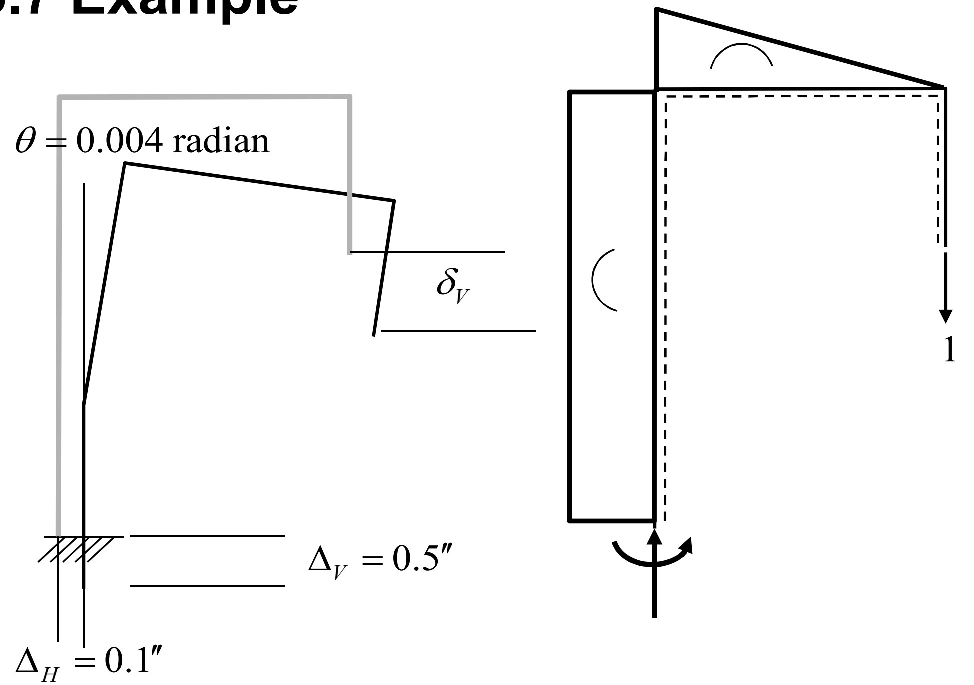
$$(+53)(-\theta)+(M_B)(2\theta)=96\times 5\theta$$

$$\Rightarrow M_B = 267$$

$$213 \le M_P \le 267$$



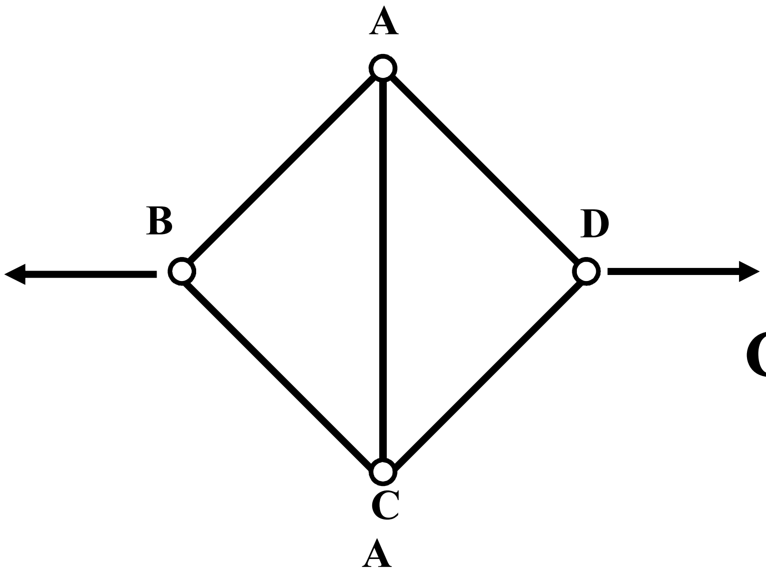
3.7 Example



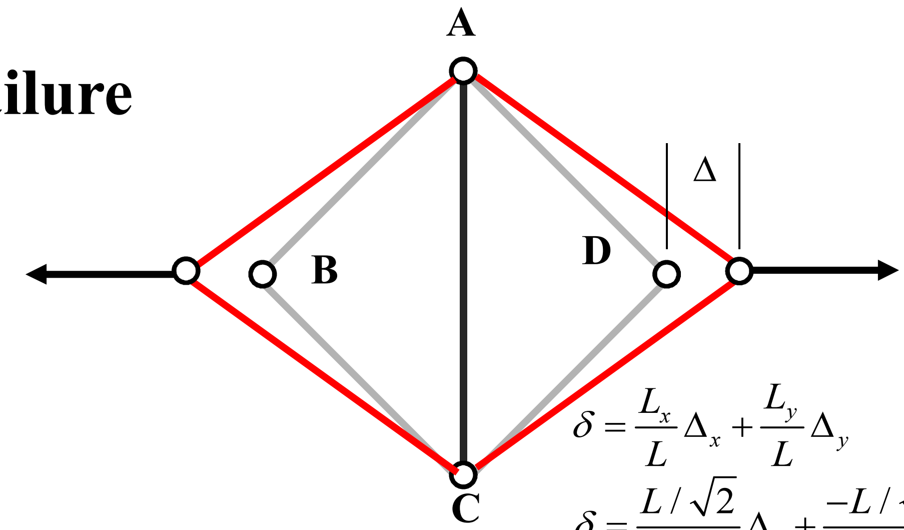
$$\sum P_i \times \delta_i = \sum M_{p_i} \times \theta_i$$

$$(1)(\delta_v)+(-1)(0.5)=(-5\times 12)(-0.004)$$

$$\Rightarrow \delta_v = -0.74''$$



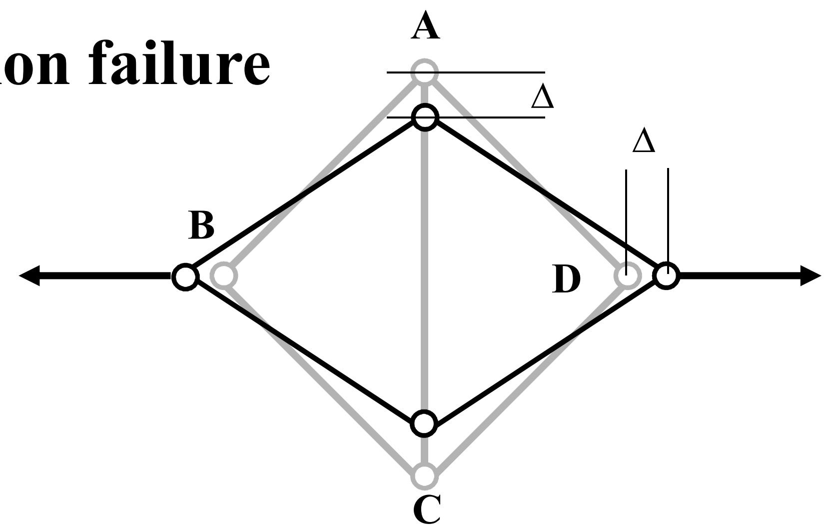
Tension failure

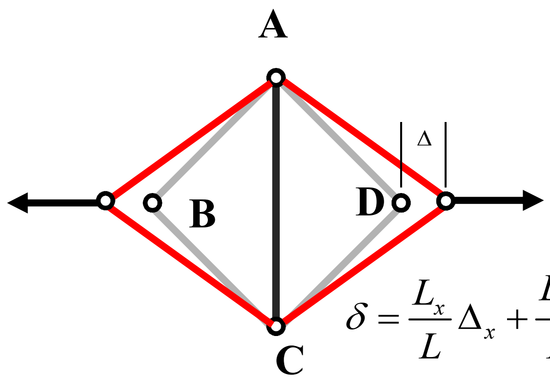


$$\delta = \frac{L_x}{L}\Delta_x + \frac{L_y}{L}\Delta_y$$

$$\delta = \frac{L/\sqrt{2}}{L}\Delta_x + \frac{-L/\sqrt{2}}{L}0 = \frac{1}{\sqrt{2}}\Delta$$

Compression failure





$$\delta = \frac{L_x}{L}\Delta_x + \frac{L_y}{L}\Delta_y$$

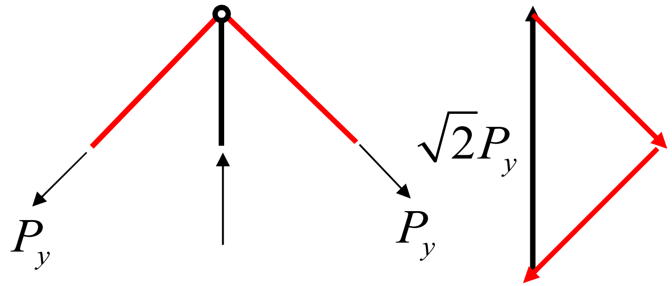
$$\delta = \frac{L/\sqrt{2}}{L}\Delta_x + \frac{-L/\sqrt{2}}{L}0 = \frac{1}{\sqrt{2}}\Delta$$

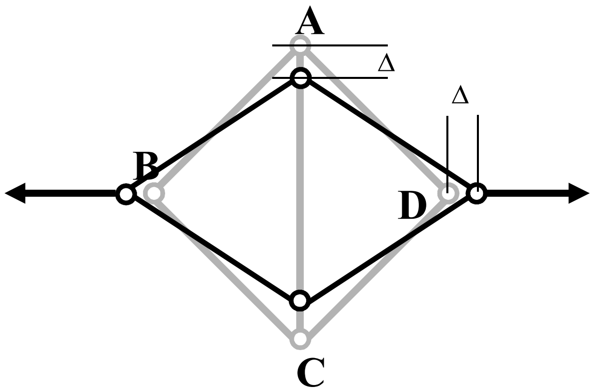
Mechanism 1

$$2P^u\Delta = 4P_y\delta$$

$$\delta = \frac{1}{\sqrt{2}}\Delta$$

$$P^u = \sqrt{2}P_y$$



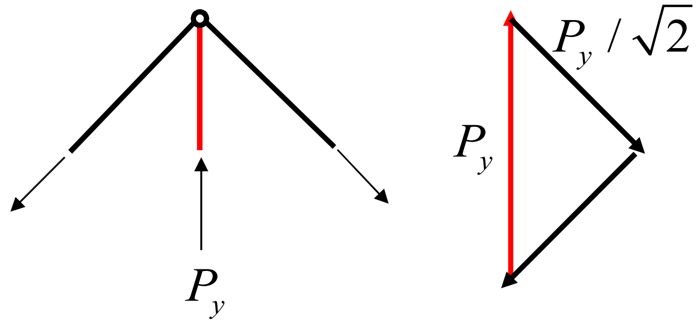


Mechanism 2

$$2P^u\Delta = P_y\left(2\delta\right)$$

$$\delta = \Delta$$

$$P^u = P_y$$



Example of highly redundant system