4.8 Analysis of Deformation

→ By a geometrically compatible deformation of a continuous body we mean one in which no voids are created in the body. This is purely a problem in the geometry of a continuum and is independent of the equilibrium requirements established in the foregoing sections of this chapter.

▶ The displacement of a continuous body may be considered as the sum of two parts:

i) A translation and/or rotation of the body as a whole

ii) A motion of the points of the body relative to each other

cf. The translation and rotation of the body as a whole is called *rigid-body motion* because it can take place even if the body is perfectly rigid. And the motion of the points of a body relative to each other is called a *deformation*.

→ The remaining sections of this chapter will be devoted to a study of the deformation at a point in a continuous body.

4.9 Definition of Strain Component

▶ Plane strain

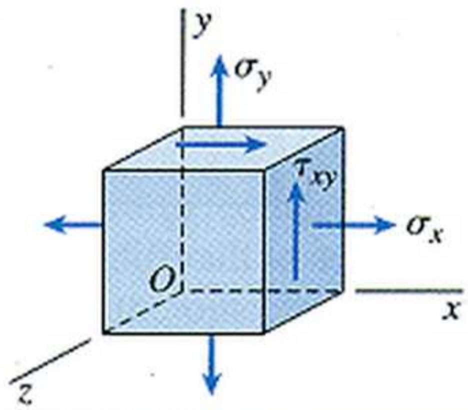
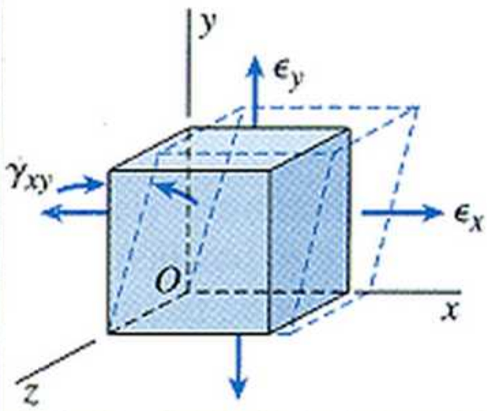
→ A body whose particles all lie in the same plane and which deforms only in this plane

▷ Condition of plane strain

i) $\epsilon_z = 0, \gamma_{xz} = 0, \gamma_{yz} = 0$

ii) $\sigma_z = 0, \tau_{xz} = 0, \tau_{yz} = 0$ (plane stress)

iii) Plane stress and plane strain does not occur simultaneously.
[exception: in case of $\sigma_x = -\sigma_y (\therefore \epsilon_z = 0)$ and $\nu = 0$]

	Plane stress	Plane strain
		
Stresses	$\sigma_z = 0 \quad \tau_{xz} = 0 \quad \tau_{yz} = 0$ σ_x, σ_y, and τ_{xy} may have nonzero values	$\tau_{xz} = 0 \quad \tau_{yz} = 0$ σ_x, σ_y, σ_z, and τ_{xy} may have nonzero values
Strains	$\gamma_{xz} = 0 \quad \gamma_{yz} = 0$ ϵ_x, ϵ_y, ϵ_z, and γ_{xy} may have nonzero values	$\epsilon_z = 0 \quad \gamma_{xz} = 0 \quad \gamma_{yz} = 0$ ϵ_x, ϵ_y, and γ_{xy} may have nonzero values

<Plane stress and Plane strain>

▶ Normal strain

→ A measure of the elongation or contraction of a line

▶ Shear strain

→ A measure of the relative rotation of two lines

▶ State of uniform strain (see Fig. 4.27 (b))

i) All elements in the block have been deformed the same amount.

ii) Originally straight lines are straight in the deformed state, but they may have changed their length or rotated.

ex) The lines *AE* and *CG* do not rotate and line *AE* remains unchanged in length while *CG* shortens. By contrast, the lines *BF* and *DH* rotate equal and opposite amounts and both change in length by the same increment.

iii) Any other type of transformation of an originally straight line does not occur.

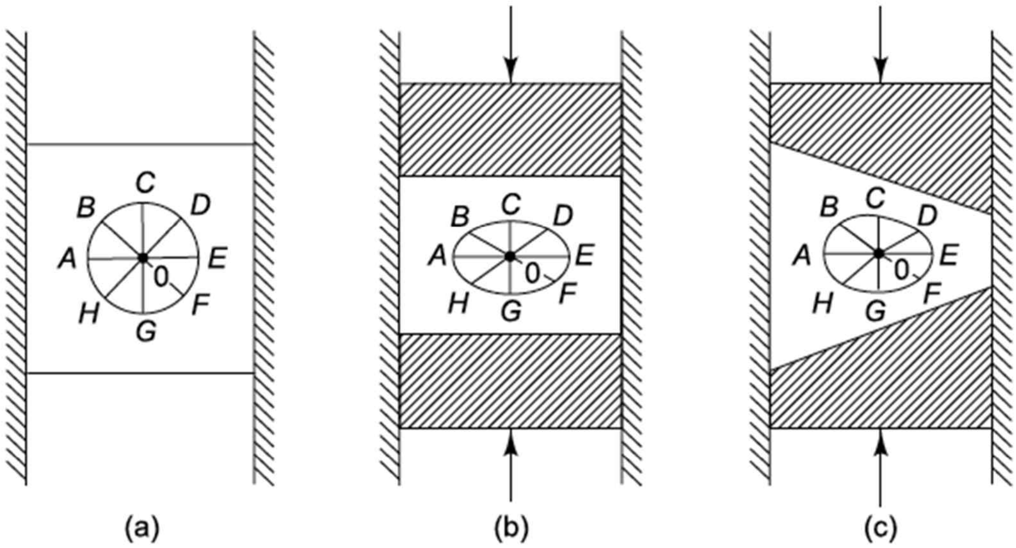


Fig. 4.27 *(a) Underformed block of rubber with superimposed diagram. (b) Rubber block of (a) deformed in unifrom strain. (c) Rubber block of (a) deformed in nonuniform strain*

▶ State of non-uniform strain (see Fig. 4.27 (c))

i) Originally straight lines are not necessarily straight in the deformed state.

ii) Within the small area the deformation is approximately uniform.

▶ Confer

i) Shear strain γ may be defined as the tangent of the change in angle between two originally perpendicular axes. When the axes rotate so that the first and third quadrants become smaller, the shear strain is positive.

ii) For small shear strains (those of engineering interest are mostly less than 0.01) it is adequate to define shear strain in terms of the change in angle itself (in radians) instead of the tangent of this angle change.

4.10 Relation Between Strain and Displacement in Plane Strain

▶ The displacement vector $\mathbf{u_0}$ of point *O*:

$$\mathbf{U}_0 = u\boldsymbol{i} + v\boldsymbol{j}$$

u and *v* are a continuous function of *x* and *y* to ensure that no voids or holes are created by the displacement. → Geometrically compatible.

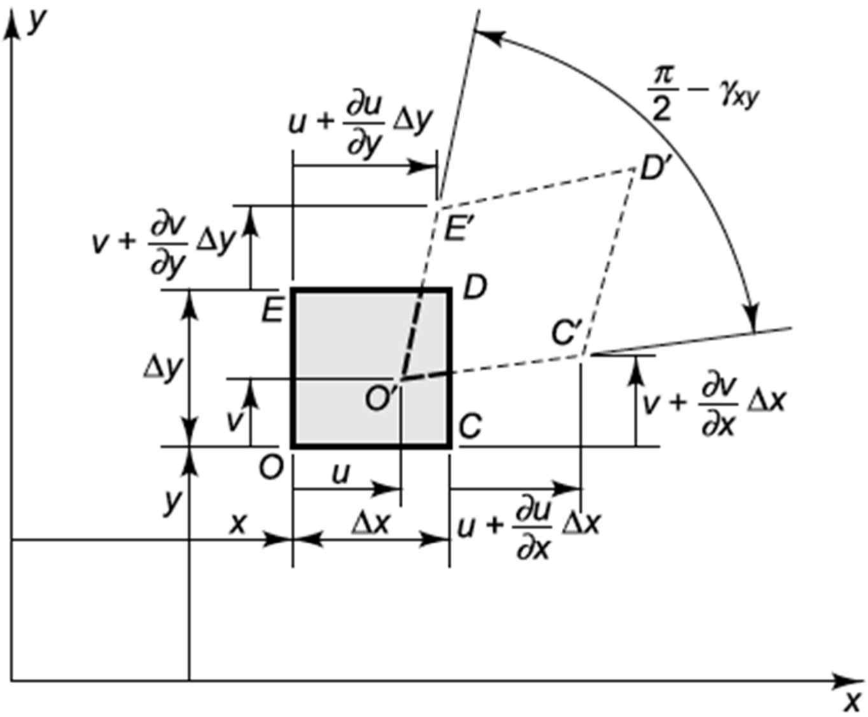


Fig. 4.29 *Plane strain deformation expressed in terms of the components u and v and their partial derivatives*

▶ The strain components (ϵ_x, ϵ_y and γ_{xy})

→ Under the assumption that the strains are small compared with unity;

$$\epsilon_x = \lim_{\Delta x \to 0} \frac{O'C' - OC}{OC} = \lim_{\Delta x \to 0} \frac{[\Delta x + (\partial u/\partial x)\Delta x] - \Delta x}{\Delta x} = \frac{\partial u}{\partial x}$$

$$\epsilon_y = \lim_{\Delta y \to 0} \frac{O'E' - OE}{OE} = \lim_{\Delta y \to 0} \frac{[\Delta y + (\partial v/\partial y)\Delta y] - \Delta y}{\Delta x} = \frac{\partial v}{\partial y} \qquad (4.31)$$

$$\gamma_{xy} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left[\frac{\pi}{2} - \angle \mathrm{C'O'E'}\right] = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left\{\frac{\pi}{2} - \left[\frac{\pi}{2} - \frac{(\partial v/\partial x)\Delta x}{\Delta x} - \frac{(\partial u/\partial y)\Delta y}{\Delta y}\right]\right\} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

▶ The rotation component ω_z (average (small) rotation of the element)

for line *OC*

$$(\omega_z)_{OC} = ([v + (\partial v/\partial x)/\Delta x] - v)/\Delta x = \partial v/\partial x$$

for line *OE*

$$(\omega_z)_{OE} = (-[u + (\partial u/\partial y)/\Delta y] + u)/\Delta y = -\partial u/\partial y$$

$$\therefore \omega_z = 1/2\,[(\omega_z)_{OC} + (\omega_z)_{OE}] = \frac{1}{2}(\partial v/\partial x - \partial u/\partial y) \qquad (4.32)$$

cf.

i) Derivation for the normal and shear strains is valid under the assumption of small displacement derivatives compared to unity.

ii) We speak of the state of plane strain at a given point in a two-dimensional body as given by the strain components $\begin{bmatrix} \epsilon_x & \gamma_{xy} \\ \gamma_{yx} & \epsilon_y \end{bmatrix}$ where we define $\gamma_{yx} = \gamma_{xy}$.

iii) Eq. (4.33) indicates that the three components of strain cannot vary arbitrarily in a field of non-uniform strain.

▶ Abstract.

$$\epsilon_x = \frac{\partial u}{\partial x} \qquad \epsilon_y = \frac{\partial v}{\partial y} \qquad \epsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \qquad (4.33)$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \qquad \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

▶ The stress and strain components in different coordinates

▷ Three dimensional rectangular coordinate system

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0$$

$$\epsilon_x = \frac{\partial u}{\partial x} \qquad \epsilon_y = \frac{\partial v}{\partial y} \qquad \epsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \qquad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

▷ Cylindrical coordinate system

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + 2\frac{\tau_{r\theta}}{r} = 0$$

$$\frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{zr}}{r} = 0$$

$$\epsilon_r = \frac{\partial u}{\partial r} \qquad \epsilon_\theta = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} \qquad \epsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} \qquad \gamma_{\theta z} = \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z} \qquad \gamma_{zr} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

4.11 Strain Component Associated with Arbitrary Set of Axes

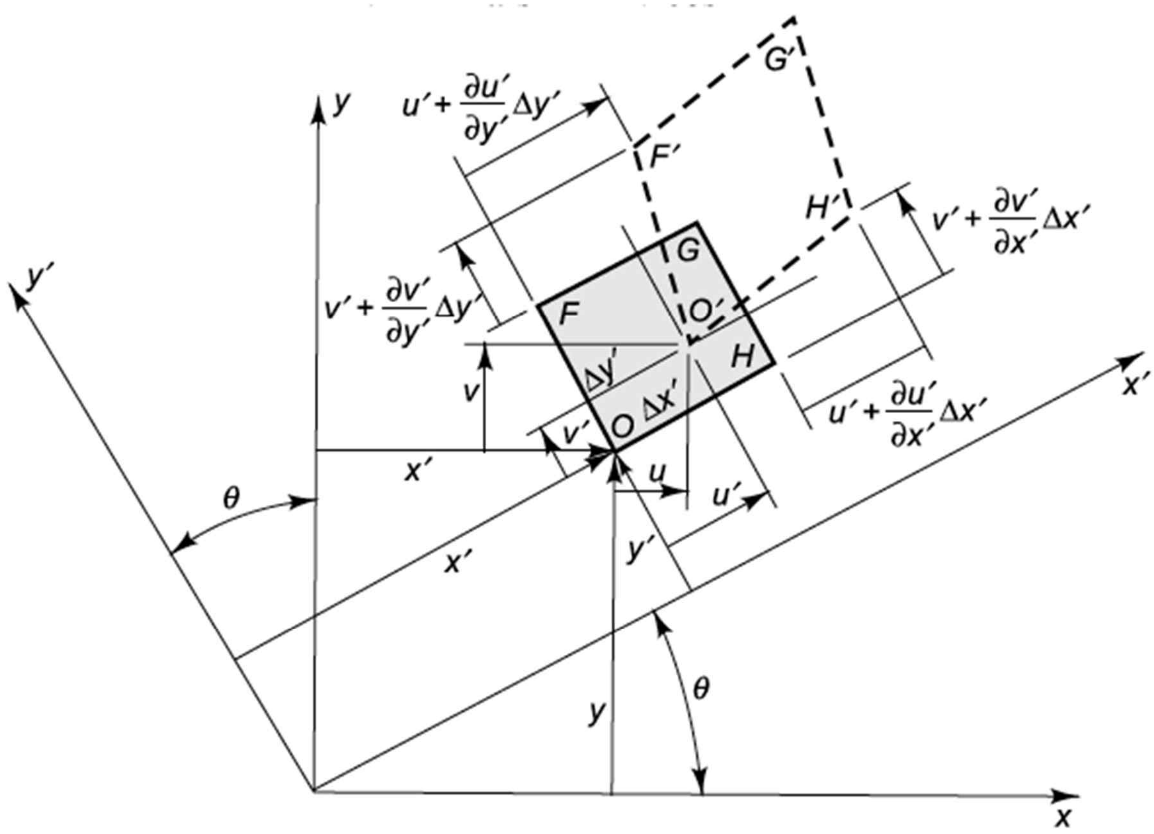


Fig. 4.30 *Plane strain. Deformation of a small element with sides originally parallel to the x′ and y′ set of axes*

▶ From chain Rule;

$$\epsilon_{x\prime} = \frac{\partial u\prime}{\partial x'} = \frac{\partial u'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial x'}$$

$$\epsilon_{y\prime} = \frac{\partial v'}{\partial y'} = \frac{\partial v'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial y'} \tag{4.38}$$

$$\gamma_{x\prime y\prime} = \frac{\partial v'}{\partial x\prime} + \frac{\partial u'}{\partial y'} = \left(\frac{\partial v'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial x'}\right) + \left(\frac{\partial u'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial y'}\right)$$

The following relationship is substituted into the preceding equation and summarized,

$$x = x'\cos\theta - y'\sin\theta \ , \qquad u' = u\cos\theta + v\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta \qquad v' = -u\sin\theta + v\cos\theta \qquad (4.39)$$

$$\epsilon_{x'} = \frac{\epsilon_x+\epsilon_y}{2} + \frac{\epsilon_x-\epsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$
$$\epsilon_{y'} = \frac{\epsilon_x+\epsilon_y}{2} - \frac{\epsilon_x-\epsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta \qquad (4.41)$$
$$\frac{\gamma_{x'y'}}{2} = -\frac{\epsilon_x-\epsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$$\epsilon_{x'} + \epsilon_{y'} = \epsilon_x + \epsilon_y$$
$$\tan 2\theta_p = \frac{\gamma_{xy}}{\epsilon_x-\epsilon_y}$$
$$\epsilon_{1,2} = \frac{\epsilon_x+\epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x-\epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$
$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\epsilon_x-\epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = \frac{\epsilon_1-\epsilon_2}{2}$$

cf. In the case of the state of plane stress that rotates on the principal axes, $\tau = 0$, so that the $\gamma = 0$. That is, the principal plane is coincide with each other in the case of plane stress and plane strain.

4.12 Mohr's Circle Representation of Plane Strain

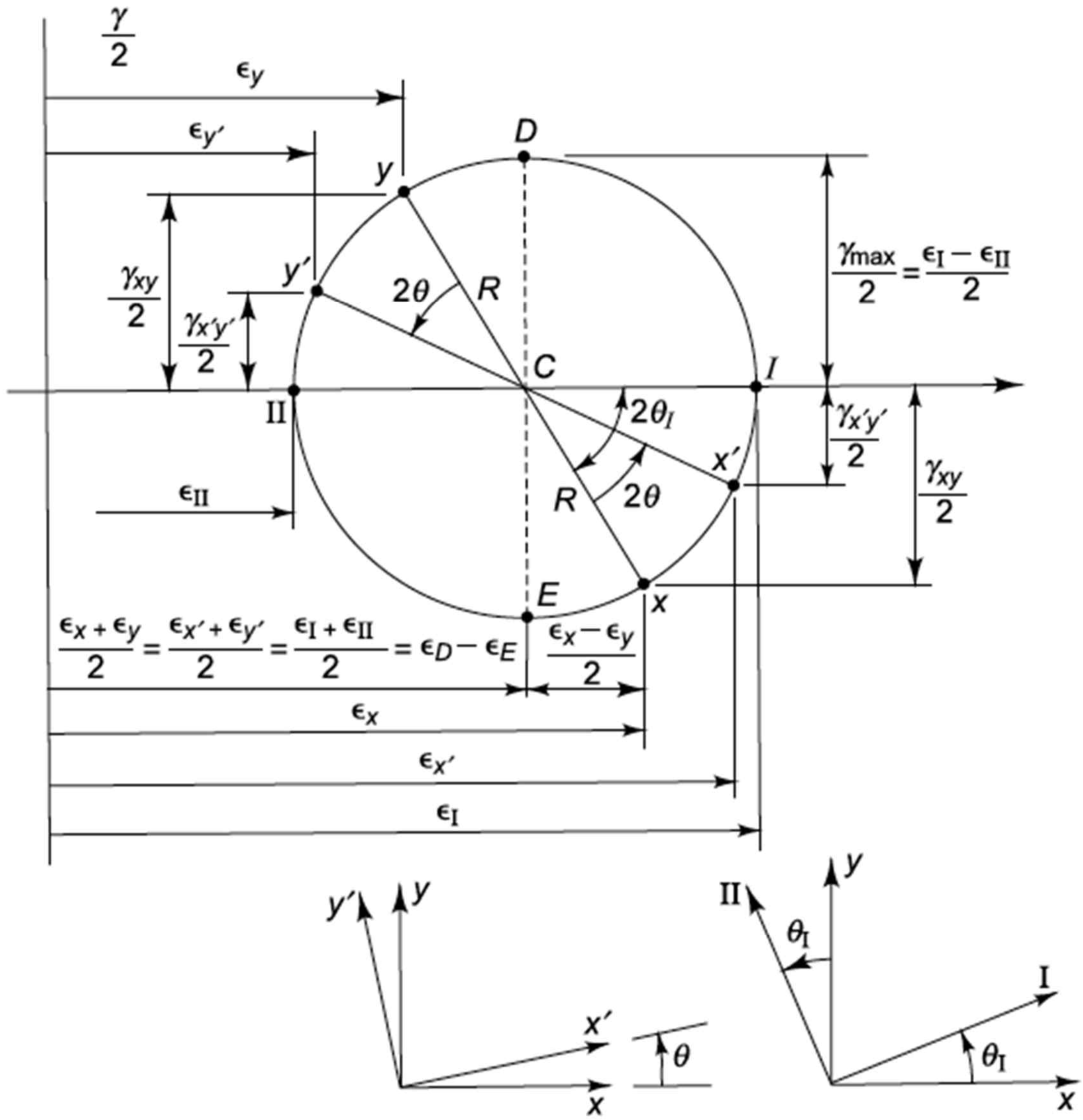


Fig. 4.31 *Mohr's circle for plain strain*

▶ <u>Plane stress and plane strain have similarities in the transformation equation, and use the following table</u>.

Stresses	Strains
σ_x	ϵ_x
σ_y	ϵ_y

$$\boxed{\boldsymbol{\tau_{xy}} \qquad \gamma_{xy}/2}$$