5.8 Strain Energy in an Elastic Body

→ In Sec. 2.6 the concept of elastic energy was introduced in terms of springs and uniaxial members. Here we extend the concept to arbitrary linearly elastic bodies subjected to small deformations.

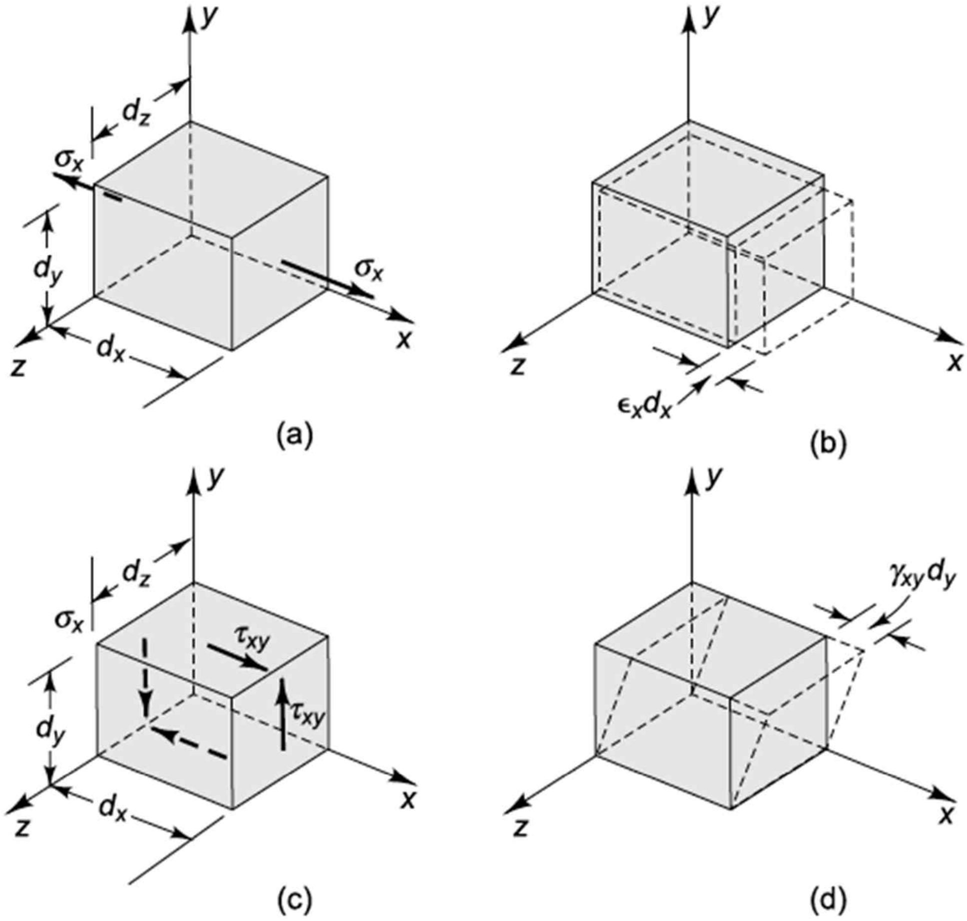


Fig. 5.20 *Infinitesimal element subjected to: uniaxial tension (a), with resulting deformation (b); pure shear (c), with resulting deformation (d)*

$$U = \frac{1}{2} P\delta \tag{5.11}$$

▶ <u>The strain energy stored in the element (in a linearly elastic material)</u>

<u>From Fig. 5.20 (a)</u>

$$dU = \frac{1}{2}(\sigma_x dydz)(\epsilon_x dx) = \frac{1}{2}\sigma_x \epsilon_x dV \tag{5.12}$$

$$\rightarrow U = \frac{1}{2}\int_v \sigma_x \epsilon_x \, dV \tag{5.13}$$

Since $\sigma_x = P/A$, $\epsilon_x = \delta/L$;

$$U = \frac{1}{2}\left(\frac{P}{A}\right)\left(\frac{\delta}{L}\right)\int_v dV$$

$$= \frac{1}{2}P\delta \qquad (5.14)$$

From Fig. 5.20 (c)

$$dU = \frac{1}{2}(\tau_{xy}\,dxdz)(\gamma_{xy}\,dy)$$

$$= \frac{1}{2}\tau_{xy}\gamma_{xy}\,dV \qquad (5.15)$$

cf. $U = T\phi/2$

cf. The individual strain components may depend on more than one stress component, but we assume that the dependence is linear. Thus, if we imagine a gradual loading process in which all stress components maintain the same relative magnitudes as in the final stress state, the strain components will also grow in proportion, maintaining the same relative magnitudes as in the final strain state. During this process in which all stresses and strains are growing, a single stress component such as σ_x will do work only on the deformation due to its corresponding strain ϵ_x.

▶ The total stain energy stored in the element

$$dU = \frac{1}{2}(\sigma_x\epsilon_x + \sigma_y\epsilon_y + \sigma_z\epsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})dV \qquad (5.16)$$

∴ In general, the final stresses and strains vary from point to point in the body. The strain energy stored in the entire body is obtained by integrating (5.16) over the volume of the body.

$$U = \frac{1}{2}\int_v(\sigma_x\epsilon_x + \sigma_y\epsilon_y + \sigma_z\epsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})dV \qquad (5.17)$$

cf. In the case of plane stress or plane strain

$$U = \frac{1}{2}\int_v(\sigma_x\epsilon_x + \sigma_y\epsilon_y + \tau_{xy}\gamma_{xy})dV \qquad (5.18)$$

In Chapter 6 and 7 we shall use these results to develop special formulas for strain energy in torsion and bending.

Overall Summary

▶ Hooke's law

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] + \alpha(T - T_0) \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\epsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_z + \sigma_x)\right] + \alpha(T - T_0) \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \qquad (5.8)$$

$$\epsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] + \alpha(T - T_0) \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

In case of statically determinate structure, the thermal strain does not generate the stress. But in the case of statically indeterminate structure, it generates the stress.

By strain-term

In 2D case

$$\sigma_x = \frac{E}{1-\nu^2}\left(\epsilon_x + \nu\epsilon_y\right)$$

$$\sigma_y = \frac{E}{1-\nu^2}\left(\epsilon_y + \nu\epsilon_x\right)$$

In 3D case

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left\{(1-\nu)\epsilon_x + \nu\left(\epsilon_y + \epsilon_z\right)\right\}$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left\{(1-\nu)\epsilon_y + \nu(\epsilon_z + \epsilon_x)\right\}$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left\{(1-\nu)\epsilon_z + \nu\left(\epsilon_x + \epsilon_y\right)\right\}$$

▶ Unit volume change

$$e = \frac{\Delta V}{V_0} = \epsilon_x + \epsilon_y + \epsilon_z = \frac{1-2\nu}{E}(\sigma_x + \sigma_y + \sigma_z)$$

Spherical stress : In the case of $\sigma_x = \sigma_y = \sigma_z = \sigma_o$ and shear stress components are absent. In addition, the Mohr's circle of stress and strain is indicated by a point.

$$\epsilon_x = \epsilon_y = \epsilon_z = \epsilon_0 = \frac{\sigma_0}{E}(1-2\nu)$$

$$e = \frac{\Delta V}{V_0} = \frac{3(1-2\nu)\sigma_0}{E} = 3\epsilon_0$$

∴ This stress distribution is called hydrostatic stress distribution.

▶ Relation between E and G

$$G = \frac{E}{2(1+\nu)} \tag{5.3}$$

▶ Strain energy density $(u = \mathrm{U/V})$

By stress-term

$$u = \frac{1}{2}(\sigma_x\epsilon_x + \sigma_y\epsilon_y + \sigma_{xz}\epsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})$$

$$= \frac{1}{2E}\left(\sigma_x^2 + \sigma_y^2 + \sigma_y^2\right) - \frac{\nu}{E}\left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x\right) + \frac{1}{2G}(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)$$

By strain-term

$$u = \frac{E(1-\nu)}{2(1+\nu)(1-2\nu)}\left(\epsilon_x + \epsilon_y + \epsilon_z\right)^2 - \frac{E}{1+\nu}\left\{\epsilon_x\epsilon_y + \epsilon_y\epsilon_z + \epsilon_z\epsilon_x - \frac{1}{4}\left(\gamma_{xy}^2 + \gamma_{yx}^2 + \gamma_{zx}^2\right)\right\}$$

$$= \frac{E}{2(1+\nu)(1-2\nu)}\left[(1-\nu)\left(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2\right) + 2\nu\left(\epsilon_x\epsilon_y + \epsilon_y\epsilon_z + \epsilon_z\epsilon_x\right)\right] - \frac{G}{2}(\gamma_{xy}^2 + \gamma_{yx}^2 + \gamma_{zx}^2)$$

5.9 Stress Concentration

$$\begin{cases} \sigma_r = -\dfrac{p_i[(r_0/r)^2-1]+p_0[(r_0/r_i)^2-(r_0/r)^2]}{(r_0/r_i)^2-1} \\ \sigma_\theta = \dfrac{p_i[(r_0/r)^2+1]-p_0[(r_0/r_i)^2+(r_0/r)^2]}{(r_0/r_i)^2-1} \end{cases} \tag{5.9}$$

▶ Stress concentration

The local increase in stress caused by the irregularity in geometry

▶ Stress concentration factor

$K_t = \sigma_{max}/\sigma_{nom}$

σ_{max}: The maximum stress in the presence of a geometric irregularity or discontinuity.

σ_{nom}: The nominal stress which would exist at the point if the irregularity were not there.

→ The magnitude of this factor depends upon the particular geometry and loading involved, but factors of 2 or more are common.

cf. In case of plastic flow or ductile fracture, strain concentration might be more important than stress concentration.

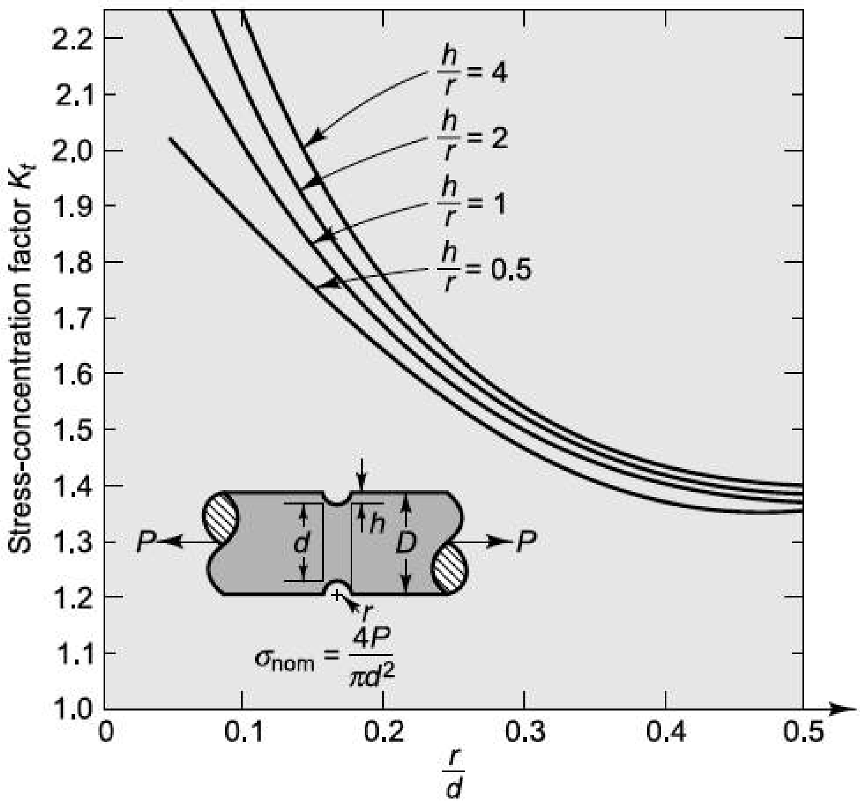


Fig. 5.21 *Stress concentration factor K_t for a circular groove in a solid circular shaft with tensile force P. (From C. Lipson and R. Juvinall, "Handbook of Stress and Strength," The Macmillan Company, New York, 1963)*

5.11 Criteria for Initial Yielding

We now turn to the problem of what happens when, in a general state of stress, the material is stressed to the point where it no longer behaves in a linearly elastic manner.

For most materials, including metals, the deviation from proportionality in a uniaxial tensile test is an indication of the beginning of plastic flow (yielding).

→ We shall restrict ourselves to polycrystalline materials which are at least statistically isotropic.

▶ Dislocation

i) During elastic deformation of a crystal, there is a uniform shifting of the whole planes of atoms relative to each other.

ii) Plastic deformation depends on the motion of individual imperfections in the crystal structure.

iii) Under the presence of a shear stress, one kind of imperfection called an edge dislocation will tend to migrate until there has been a displacement of the upper part of the crystal relative to the lower by approximately one atomic spacing.

iv) By a combination of such motions, plastic strain can be produced.

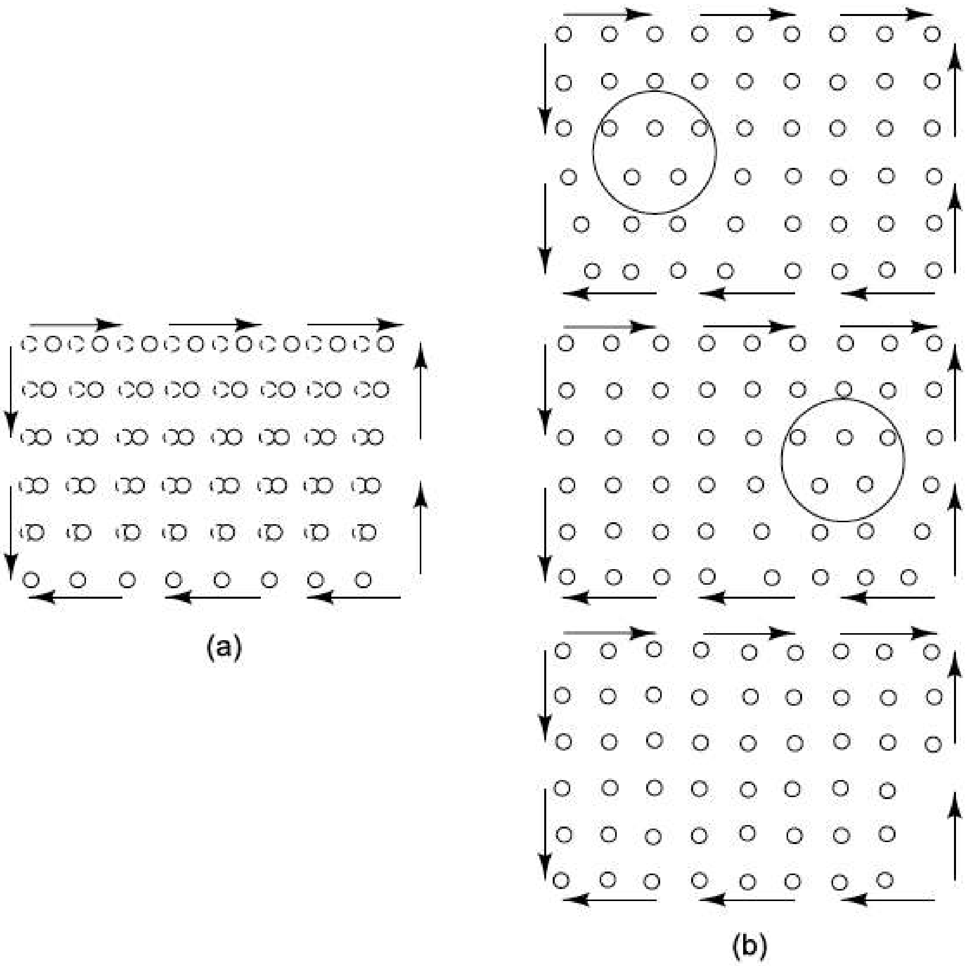


Fig. 5.27 *Deformation of a crystal lattice. (a) Elastic deformation; (b) plastic deformation*

→ It is important to note that a consequence of this simple model is that shear stress is the dominant agent in the migration of these dislocations.

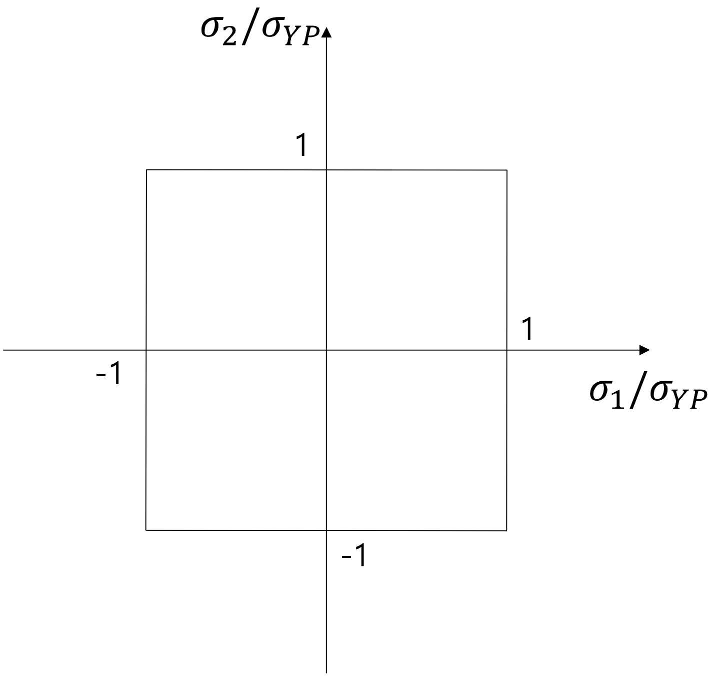
▶ Yielding Criteria

i) The state of stress can be described completely by giving the magnitude and orientation of the principal stresses.

ii) Since we are considering only isotropic materials, the orientation of the principal stresses is unimportant, thus the criteria for yielding are based only on the magnitude of the principal stresses.

iii) Since experimental work that a hydrostatic state of stress does not affect yielding, above two criteria are based not on the absolute magnitude of the principal stresses but rather on the magnitude of the differences between the principal stresses.

▶ Maximum Stress Theory



Yielding can occur when the any principal stress at arbitrary point reaches the same value which the stress has when yielding occurs in the tensile test

$\therefore \quad (\sigma_1)_{YP} = \sigma_{YP}$ or $|(\sigma_2)_{YP}| = |\sigma_{YP}|$

cf. Limitations: 1) $(\sigma_{YP})_{Tensile} \neq (\sigma_{YP})_{Compression}$,

2) $(\tau_{max})_{YP}$ differs for different materials

▶ Von Mises Criterion

→ It is also called the maximum distortion-energy theory and applied to the ductile materials.

Yielding condition

Yielding can occur in a three-dimensional state of stress when the root mean squares of the differences between the principal stresses reaches the same value which it has when yielding occurs in the tensile test.

Since $\sigma_1 = Y,\ \sigma_2 = \sigma_3 = 0,$ the yielding occurs when the stress condition is satisfied.

$$\sqrt{\frac{1}{3}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]}$$

$$= \sqrt{\frac{1}{3}[(Y-0)^2+(0-0)^2+(0-Y)^2} = \sqrt{2/3}\,Y$$

→ For general stress state, we can derive

$$\sqrt{\frac{1}{2}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]} = Y \qquad (5.23)$$

→ In case of non-principal stress axis, we can derive

$$\sqrt{\frac{1}{2}\left[\left(\sigma_x-\sigma_y\right)^2+\left(\sigma_y-\sigma_z\right)^2+(\sigma_z-\sigma_x)^2\right]+3\tau_{xy}^2+3\tau_{yz}^2+\tau_{zx}^2}$$

$$= Y$$

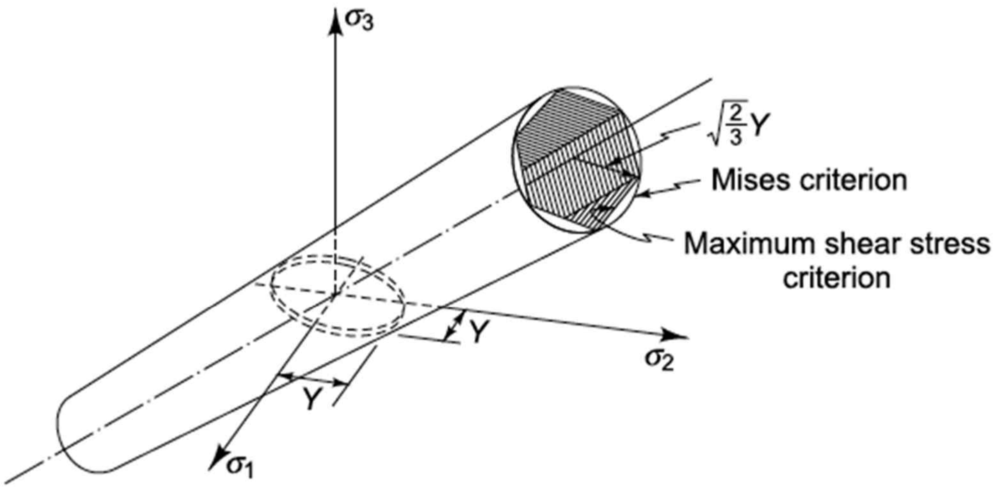


Fig. 5.30 *Geometrical representation in principal stress space of the Mises and maximum shear-stress yield criteria*

cf. The criterion (5.23) then is represented in this space by a right-circular cylinder of radius $\sqrt{\frac{2}{3}}\boldsymbol{Y}$ whose axis makes equal angles with the $\boldsymbol{\sigma_1, \sigma_2}$ ***and*** $\boldsymbol{\sigma_3}$ coordinate axes, as illustrated in Fig. 5.30. Yielding occurs for any state of stress which lies on the surface of this circular cylinder.

Yielding condition in plain stress

$$\left(\frac{\sigma_1}{Y}\right)^2 - \frac{\sigma_1\sigma_2}{Y^2} + \left(\frac{\sigma_2}{Y}\right)^2 = 1$$

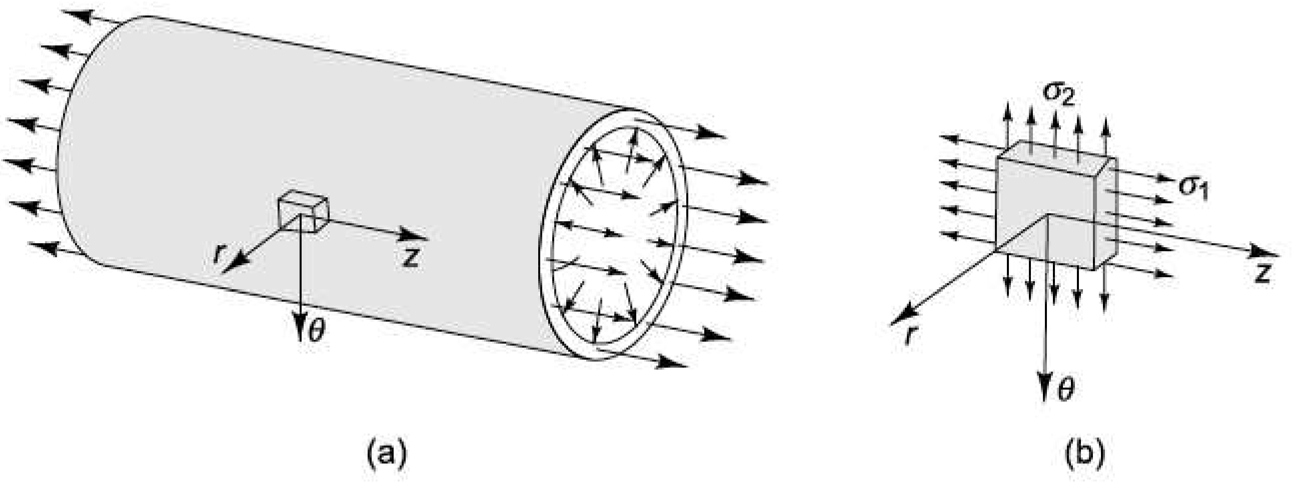


Fig. 5.28 *Example of biaxial stress in a thin-walled cylinder*

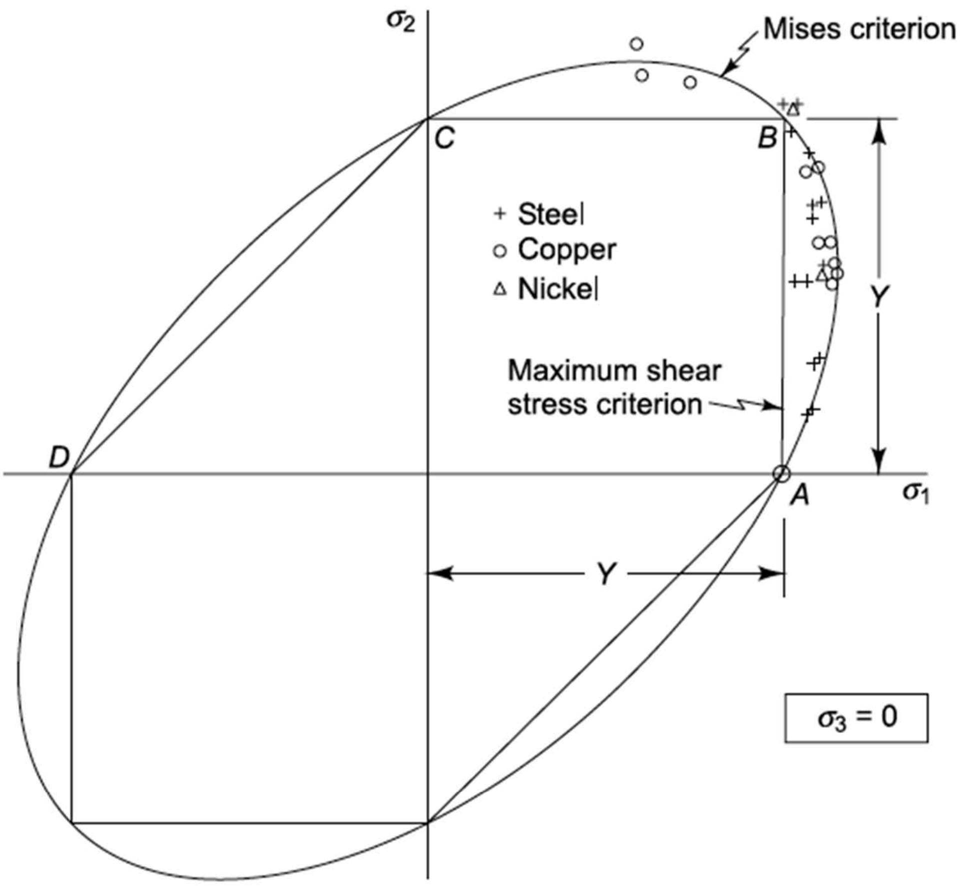


Fig. 5.29 *Yielding of thin-walled tubes under combined stress. (From W. Lode, Versuche uber den Einfluss der mittleren Hauptspannung auf das Fliessen der Metalle Eisen, Kupfer, und Nickel, Z. Physik, vol. 36, pp. 913–939, 1926)*

▶ Tresca Criterion

→ It is also called the maximum shear-stress criterion and applied to the elastic ductile materials.

Yielding condition

Yielding occurs whenever the maximum shear stress reaches the value it has when yielding occurs in the tensile test.

$$\tau_{max} = \frac{\sigma_{max}-\sigma_{min}}{2} = \frac{Y}{2} \qquad (5.25)$$

cf. The criterion (5.25) can be represented by a hexagonal cylinder inscribed within the right-circular cylinder of the Von Mises criterion.

Yielding condition in Plain stress

Refer to Fig. 5.29

Application of the Tresca Criterion (see Fig. 5.28, 5.29)

i) When only internal pressure (σ_2) increases, it corresponds to proceeding along the straight line from *A* toward *B*. (Fig. 5.29)

cf. Further increasing the inner pressure (σ_2), the axial load or σ_1 no more influence on yielding condition, and thus

$$\tau_{max} = 1/2(\sigma_z - \sigma_r) = 1/2(\sigma_1 - \sigma_3)$$

cf. In case of $\sigma_\theta = \sigma_z$ ($\therefore \sigma_1 = \sigma_2$), it corresponds to the point B.

ii) If the axial load (σ_1) decreases, it corresponds to proceeding along the straight line from B toward C.

cf. If axial load (σ_1) changes from tensile to compressive, it corresponds to proceeding along the straight line from *C* toward *D*.

→ This means that the internal pressure (σ_2) must be decreased in order to avoid the yielding.

In this case, as σ_{min}, σ_{max} are important, it's not possible to apply the Von Mises criterion to this situation directly.

→ Check the figures below

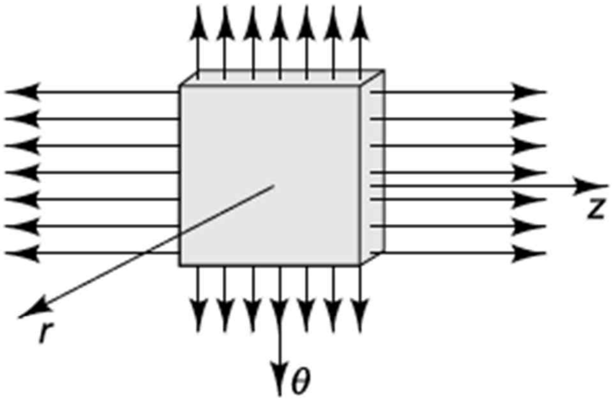


Fig. 5.31 *State of stress in cylinder wall of Fig. 5.28(a) when σ_z and σ_r determine the maximum shear stress*

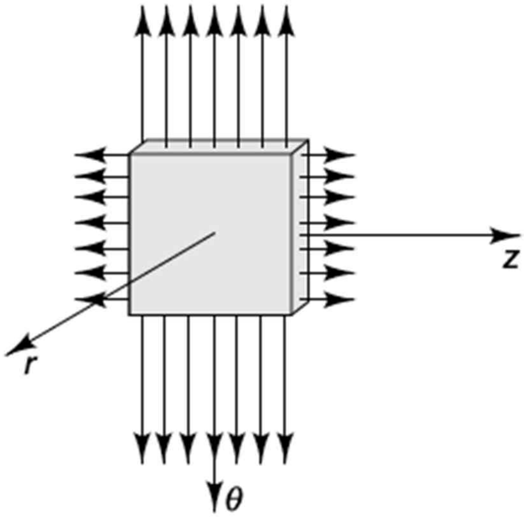


Fig. 5.32 *State of stress in cylinder wall of Fig. 5.28(a) when σ_θ and σ_r determine the maximum shear stress*

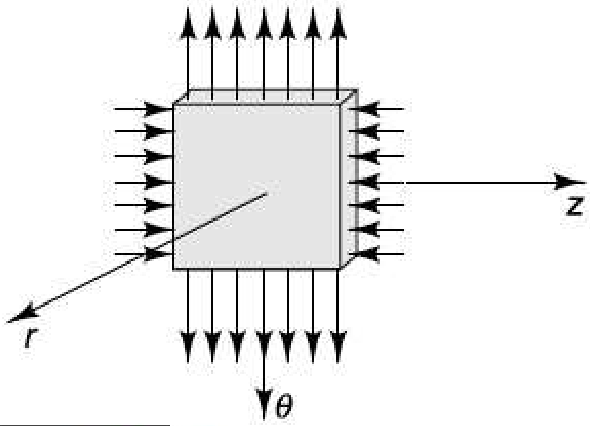


Fig. 5.33 *State of stress in cylinder wall of Fig. 5.28(a) when σ_θ and σ_z determine the maximum shear stress*

▶ <u>Comparison of the criteria</u>

These criteria are identical in case of uniaxial stress.

Thus, one of the principal stress at arbitrary point is greater than the others, these criteria have identical values in the majority of case.

On the other hands, in case that the absolute value of principal stress is same, these criteria have distinguished difference.

5.12 Behavior Beyond Initial Yielding in the Tensile Test

→ The following description is an idealized description of the behavior of a real material during loading and unloading beyond initial yielding.

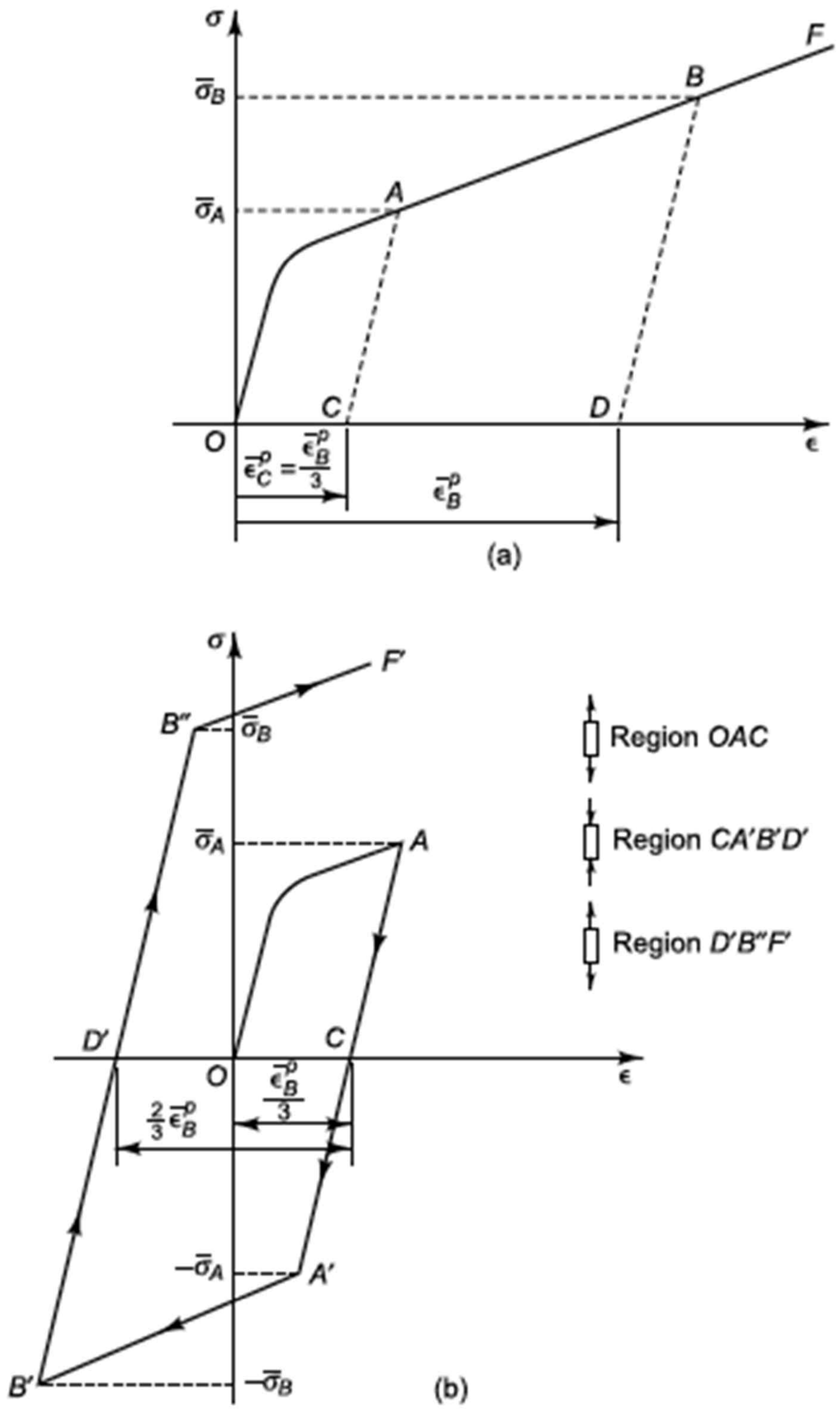


Fig. 5.34 *Example of simple loading path. (a) Stress-strain curve in uniaxial tensile test; (b) stress-strain behavior in alternate uniaxial tension and compression*

▶ For Fig. 5.34 (b)

i) A fresh specimen of the material is stretched in tension to point A, where the plastic extensional strain is $\frac{1}{3}\bar{\epsilon}_B^P$ and the stress is $\bar{\sigma}_A$.

ii) The load is released, bringing the specimen to point C, and then reapplied as compression.

iii) Further yielding begins when the stress $-\bar{\sigma}_A$ is reached at point A'.

iv) As the compressive load is increased, yielding continues along the curve $A'B'$, which has the same shape as the curve AB in Fig. 5.34 (a).

v) When the point B' is reached, a compressive plastic strain of $\frac{2}{3}\bar{\epsilon}_B^P$ has occurred between A' and B', and the stress required to cause further yielding has reached the value $-\bar{\sigma}_B$.

vi) If the load is now released, the material returns to D'.

vii) A reapplication of the tensile load will cause the material to move along the curve $D'B'F'$, which is identical with the curve DBF in Fig. 5.34 (a).

cf. All the plastic-strain increments along the loading path have contributed in a positive manner to the strain-hardening so that the material in state D' has been strain-hardened the same amount as the material in state D in Fig. 5.34 (a).

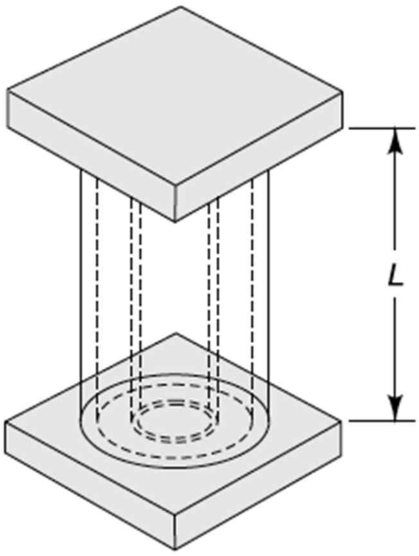


Fig. 5.8 Example 5.1

▶ Example 5.3 Returning to Example 5.1, we ask, what will happen if we remove the load P after we have strained the combined assembly so that both the steel and the aluminum are in the plastic range, that is, beyond a strain of 0.005?

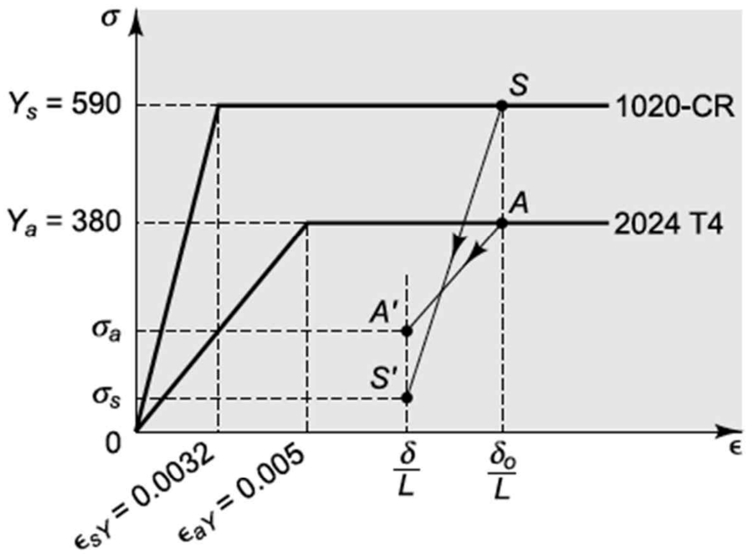


Fig. 5.35 *Example 5.3. Stress-strain behavior of the assembly of Example 5.1 when the load P is decreased after both the steel and aluminum alloy have been strained plastically*

We can again use the model of Fig. 5.9, and the equilibrium relation (e) and geometric compatibility relation (a) still remain valid. We need new stress-strain relations which will be valid during unloading.

▷ Stress-strain relation

δ_0: deflection when the assembly is loaded by P

δ: deflection after the load has been decreased somewhat

Then,

$$\begin{cases} \sigma_s = Y_s - E_s \dfrac{(\delta_0 - \delta)}{L} \\ \sigma_a = Y_a - E_a \dfrac{(\delta_0 - \delta)}{L} \end{cases} \tag{f}$$

Substituting (f) into Eq. (e) of Example 5.1 and setting P = 0, we obtain

$$\sum F_y = \sigma_s A_s + \sigma_a A_a - P = 0 \tag{e}$$

$$A_s\left(Y_s - E_s\frac{\delta_0-\delta}{L}\right) + A_a\left(Y_a - E_a\frac{\delta_0-\delta}{L}\right) = 0 \tag{g}$$

$$\therefore \frac{\delta_0-\delta}{L} = \frac{A_sY_s+A_aY_a}{(A_sE_s+A_aE_a)} \tag{h}$$

Substituting (h) into (f), we find the residual stresses which remain in the assembly after the load has been removed

$$\begin{aligned}(\sigma_s)_{RESIDUAL} &= Y_s\frac{1-\frac{Y_a/E_a}{Y_s/E_s}}{1+\frac{E_sA_s}{E_aA_a}} = Y_s\frac{1-\frac{\epsilon_{aY}}{\epsilon_{sY}}}{1+\frac{E_sA_s}{E_aA_a}} \\ (\sigma_a)_{RESIDUAL} &= Y_a\frac{1-\frac{Y_s/E_s}{Y_a/E_a}}{1+\frac{E_aA_a}{E_sA_s}} = Y_s\frac{1-\frac{\epsilon_{sY}}{\epsilon_{aY}}}{1+\frac{E_aA_a}{E_sA_s}}\end{aligned} \tag{i}$$

→ Since in the present case $\epsilon_{aY} > \epsilon_{sY}$, the Eq. (i) show that the steel will be in compression and the aluminum in tension.

cf. The residual stresses will be zero only when the initial yield strains $\epsilon_{sY} = Y_s/E_s$ and $\epsilon_{aY} = Y_a/E_a$ are equal.

▶ Engineering stress-strain

1 ▷ Engineering stress

$$\sigma_E = \frac{load}{original\ \ (before\ \ loading\ \ )\ area}$$

→ The maximum value of the engineering stress is termed the tensile strength.

2 ▷ Engineering strain

$$\varepsilon_E = \Delta L/L_0 = (L_f - L_0)/L_0 \tag{5.26}$$

where L_0: original length between two dots of specimen,

L_f: length between two dots of specimen after loading.

▶ True stress-strain

1 ▷ True stress

$$\sigma_T = \frac{bad}{actual\ (under\ bading\ )\ area}$$

→ Even when the axial strain has reached the relatively large (for engineering purposes) value of 0.05, the true stress is only about 5 percent greater than the engineering stress.

2 ▷ True strain

The strain, obtained by adding up the increments of strain which are based on the current dimensions, is called a true strain. Sometimes it is called logarithmic strain or natural strain.

$$\epsilon_T = \int_{L_0}^{L_f} (1/L)\, dL = \ln\left(L_f/L_0\right) \tag{5.27}$$

→ For very small strain, assume that $A_0 L_0 = A_f L_f$.

$$\epsilon_T = \ln \frac{A_o}{A_f} = 2 \ln \frac{D_0}{D_f} \tag{5.28}$$

▷ ▷ Confer

1) Most of the dislocation processes are more conveniently described by an incremental concept of strain.

2) When a ductile metal is tested both in tension and in compression, the true-stress and true strain curves practically coincide, whereas the two curves are quite different when engineering strain is used.

∴ When deciding which definition of strain to use in describing the behavior beyond initial yielding in the tensile test, the balance is in favor of using true strain.

▶ Necking

→ It is difficult to decide the time when necking starts.

cf. Details about necking will be discussed in Ch. 9-7.

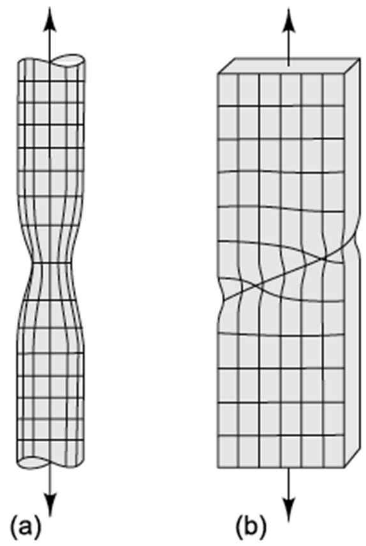


Fig. 5.37 *Examples of necking*

▶ Reduction of area (R.A.)

$$R.A. = (A_0 - A_f)/A_0 = 1 - A_f/A_0 = 1 - e^{-\epsilon_f}$$

→ The ductility of a material can be described by the reduction of area (R.A.).

▶ Elongation

$$Elongation = \Delta L/L_0 = (L_f - L_0)/L_0$$

→ Elongation is defined as the change in gage length to final fracture divided by the original gage length (i.e., the engineering strain at fracture).

→ As a measure of ductility of the material, the elongation has the disadvantage that it is an engineering, rather than a true, strain.

→ It is very dependent on the length as well as on the cross-sectional dimensions of the specimen.