CH. 5
STRESS-STRAIN-TEMPERATURE RELATIONS

5.1 Introduction

▶ The presence of only three equations of equilibrium for the six components of stress and the addition of three components (u, v, w in Ch. 4-10) of displacement in the six equations relating strain to displacement indicates;

→ Further relations are needed before the equations can be solved to determine the distributions of stress and strain in a body; i.e., the distribution of stress and strain will depend on the material behavior of the body.

▶ Two avenues of approach are suggested to investigate the relations between stress and strain;

i) Atomic level

→ Relations based on experimental evidence at the atomic level with theoretical extension to the macroscopic level

ii) Macroscopic level

→ Relations based on experimental evidence at the macroscopic level

▶ <u>Discussions in this chapter</u>

i) The stress-strain behavior of a wide variety of structural materials, including metals, wood, polymers, and composite materials

ii) Elastic, plastic, and viscoelastic behavior

iii) Various mathematical models are established to describe elasticity and plasticity.

iv) The theory of linear isotropic elasticity

v) Design criteria for, yielding of ductile materials, fracture of brittle materials, and fatigue under repeated loading

▶ Definitions

1 ▷ Elastic deformation

→ The deformation that returns to its origin shape on release of load

2 ▷ Plastic deformation

→ The deformation which depends on the applied load, is independent of time, and remains on release of load

3 ▷ Strain hardening

→ Increase in the load required for further plastic deformation

4 ▷ Ductile structure

→ Structure for which the plastic deformation before fracture is much larger than the elastic deformation

5 ▷ Brittle structure

→ Structure which exhibits little deformation before fracture

6 ▷ Fatigue

→ Progressive fracture under repeated load

7 ▷ Notch brittle

→ A larger part might have a large elastic region surrounding the plastic zone when the crack started to grow so that the overall deformation of the part would only be little more than the elastic deformation. Such a part or structure would be called notch-brittle.

8 ▷ Creep

→ Time-dependent part of the deformation

9 ▷ Elastic after effect or recovery

→ Elastic spring back followed by a relatively slow unfolding

10 ▷ Visco-elasticity

→ A mixture of creep and elastic aftereffects at room temperature (Ex: long chin polymers)

5.2 Tensile Test

▶ Tensile test

→ Test in which a relatively slender member is pulled in the direction of its axis

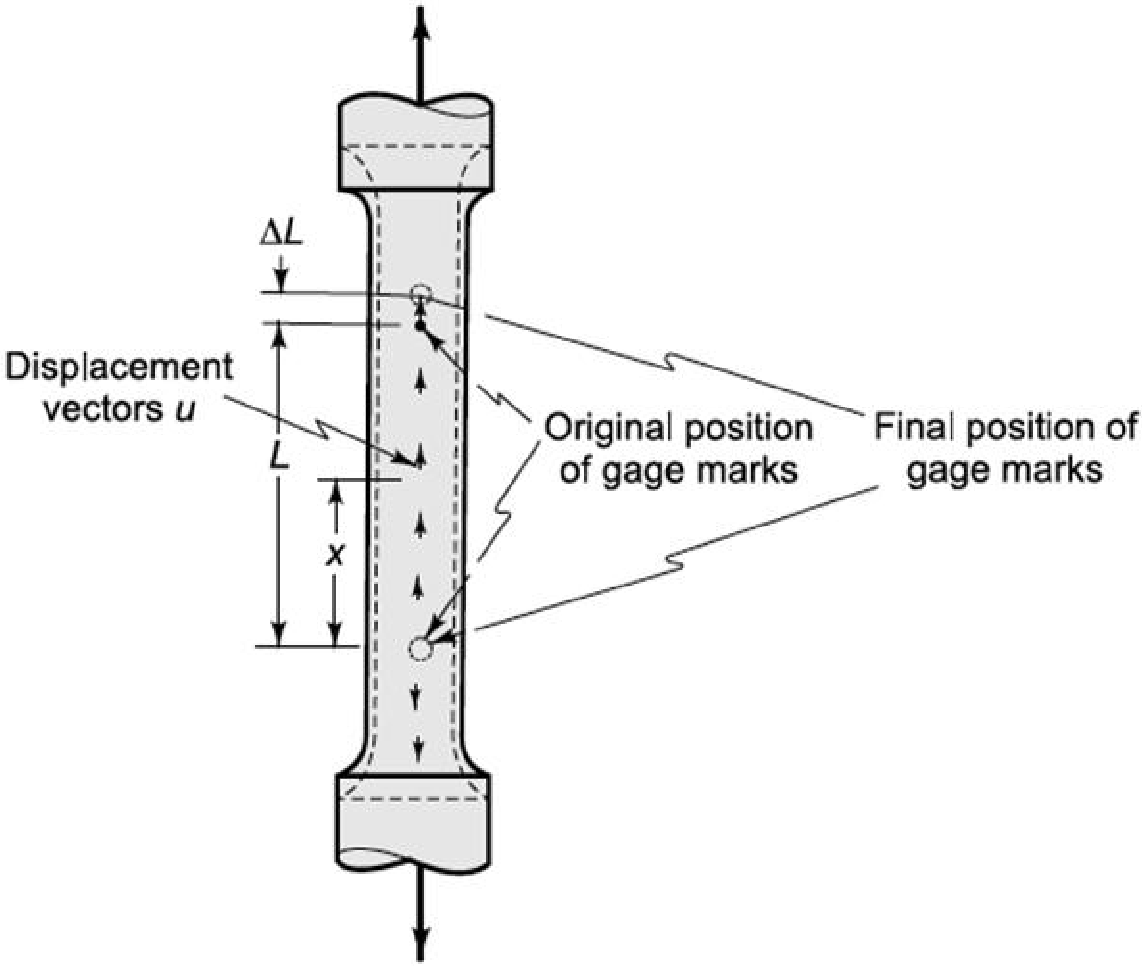


Fig. 5.4 *Displacements in a tensile test*

→ Our aim is to use tensile test data to formulate quantitative stress-strain relations which, when incorporated with equilibrium and compatibility requirements, will produce theoretical predictions in agreement with the experimental results in complicated situations.

→ The elongation and lateral contraction are also noted as the test proceeds.

▷ From Fig. 5.4, if the displacements vary uniformly over the gage length L,

$$u = (x/L)\Delta L$$

$$\therefore \varepsilon_x = \partial u/\partial x = \Delta L/L \tag{5.1}$$

▶ Stress-strain diagram & definitions

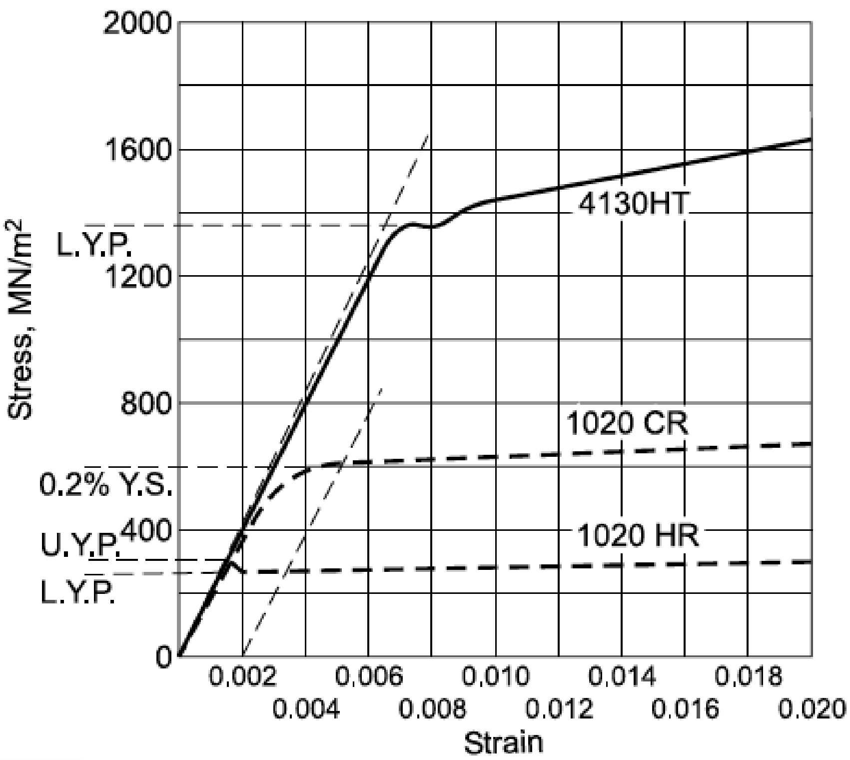


Fig. 5.5(a) *Stress-strain curves for three steels.*

- - - - Mild steel, hot-rolled (1020 HR)

– – Mild steel, cold-rolled (1020 CR)

—— 0.3% C, 0.5% Mn, 0.25% Si, 0.9% Cr, balance Fe (4130 HT)

Heat treatment: Oil quenched from 870°C, tempered at 315°C

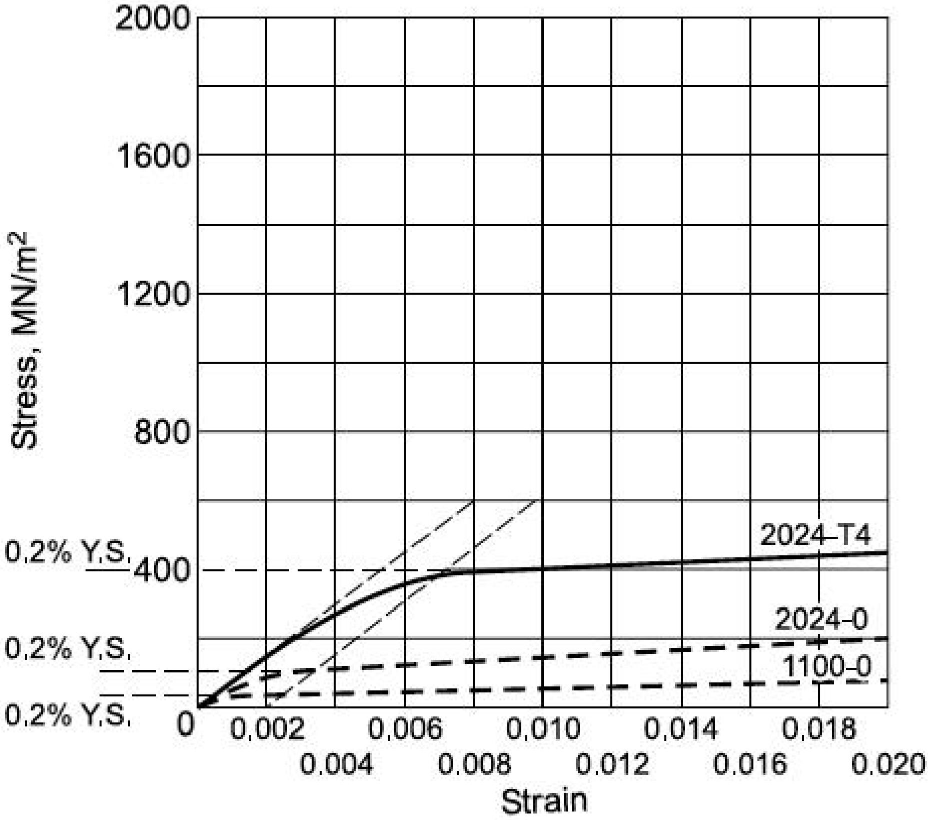


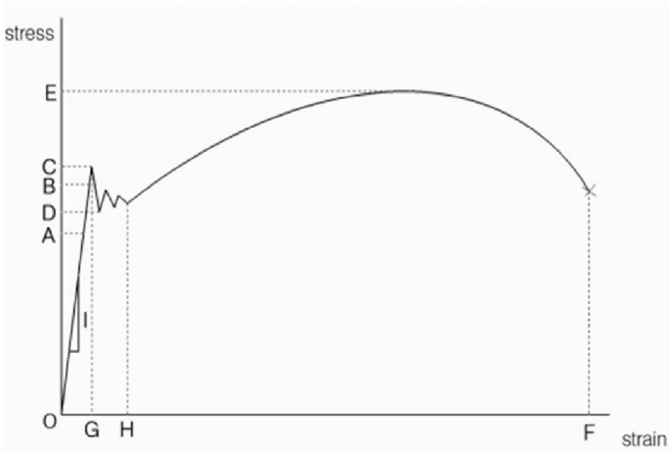
Fig. 5.5(b) *Stress-strain curves for aluminum and two aluminum alloys.*

- - - - Commercially pure aluminum, annealed (1100-0)

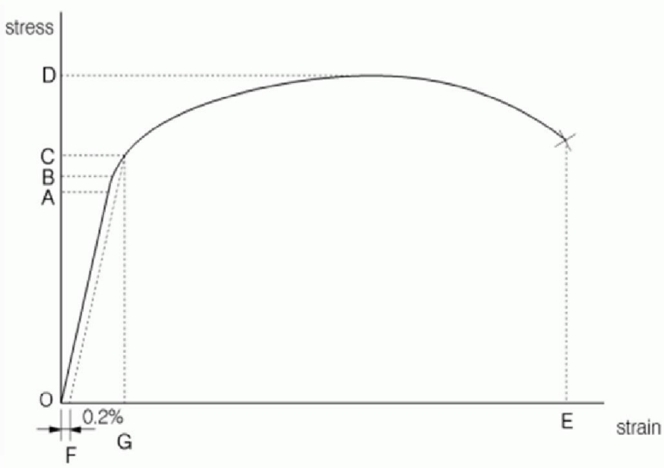
– – 4.6% Cu, 1.5% Mg, 0.7% Mn, balance Al, annealed (2024-0)

—— 4.6% Cu, 1.5% Mg, 0.7% Mn, balance Al (2024-T4)

Water quenched from 490°C, aged 24 hr at 120°C



A: Proportionality limit B: Elastic limit C: Upper yield point D: Lower yield point
E: Ultimate tensile strength F: Fracture O~G: Elastic region G~H: Yielding region
G~F : Plastic region I: Young's Modulus



A: Proportionality limit B: Elastic limit C: Yield point D: Ultimate tensile strength
E: Fracture F: 0.2% strain offset O~G: Elastic region G~E: Plastic region

1 ▷ Proportional limit

The greatest stress for which the stress is still proportional to the strain

2 ▷ Elastic limit

The greatest stress which can be applied without resulting in any permanent strain on release of stress

cf. For the materials shown in Fig. 5.5, the proportional and elastic limits coincide.

cf. Neither the proportional nor the elastic limits can be determined precisely, for they deal with the limiting cases of zero deviation from linearity and of no permanent set.

3 ▷ Yield strength

Standard practice to report a quantity called the yield strength, which is the stress required to produce a certain arbitrary plastic deformation

cf. Yield strength ≡ Offset yield stress (ductile material)

4 ▷ Upper yield point

Plastic deformation first begins

→ The upper yield point is very sensitive to rate of loading and accidental bending stresses or irregularities in the specimen. (from the Cottrell effect in the mechanics of materials)

5 ▷ Lower yield point

Subsequent plastic deformation which occurs at a lower stress

→ Because L.Y.P is the material property, so the lower yield point should be used for design purposes.

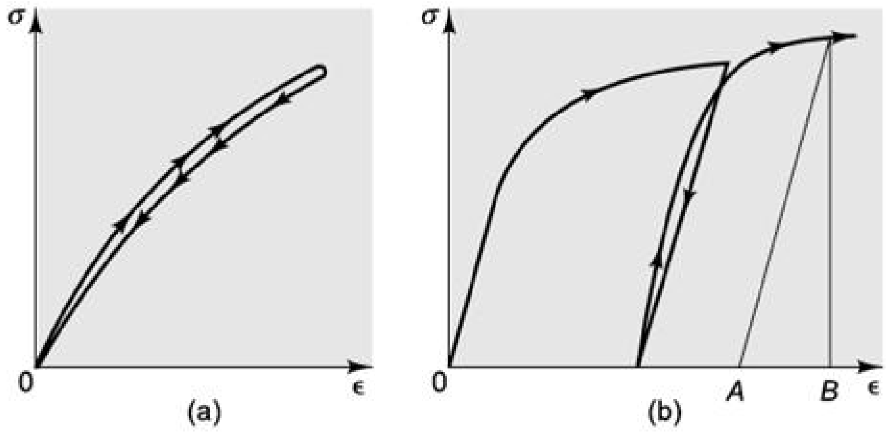


Fig. 5.6 *Effect of unloading and reloading. (a) Elastic; (b) plastic*

6 ▷ Flow stress(Strength)

As plastic deformation is continued, the stress required for further plastic flow, termed the flow strength, rises.

7 ▷ Strain hardening

The characteristic of the material in which further deformation requires an increase in the stress usually is referred to as strain-hardening of the material.

8 ▷ Brittle material

For glass, its behavior is entirely elastic and the stress at which fracture occurs is much greater in compression than in tension.

→ This is a usual characteristic of brittle materials.

9 ▷ From Fig. 5.6 (b)

The result of loading and unloading;

cf. In materials at room temperature, the strain rate change can be observed but is much less.

cf. For most ductile materials the stress-strain curves for tension and compression are nearly the same for strains small compared to unity, and in the following theoretical developments we shall assume that they are identical

5.3 Idealization of Stress-Strain Curve

→ Because we wish the mathematical part of our analysis to be as simple as possible, consistent with physical reality, we shall idealize the stress-strain curves of Fig. 5.5 into forms which can be described by simple equations.

→ The appropriateness of any such idealization will depend on the magnitude of the strains being considered, and this in turn will depend upon whatever practical problem is being studied at the moment.

▶ Fracture

→ Fracture is the most dangerous mode of failure.

i) Brittle structures are those that fracture with little plastic deformation compared with the elastic deformation.

→ We may base all our calculations for these materials on a linear relation between stress and strain.

ii) For ductile structures there is as yet no quantitative theory which will predict fracture.

① Our lack of knowledge of the distributions of stress and strain in the plastic region in front of a crack

② Our lack of knowledge of strain around the holes that grow from inclusions and coalesce to cause fracture

→ It will be necessary to have available stress-strain relations which are reasonable approximations for large plastic strains.

iii) Fatigue: the repetitions of stress eventually produce fine cracks which grow very slowly at first and then extend rapidly across the entire part.

→ Since fatigue can occur even if the stresses are below the yield strength, it is sufficient for most practical design purposes to know the relation between the stress and the strain within the elastic region.

cf. Safety factor "n"

$$n = \frac{\text{Actual strength (= Strength of material)}}{\text{Required strength (= Maximum allowable stress)}} > 1$$

▶ Other failures

i) A small corrosion pit will cause a local stress concentration which will in turn create an electromotive force between the highly stressed and the less stressed regions. This electromotive force in turn accelerates the corrosion, and the process can lead to the development of cracks and final fracture of the part.

→ As in fatigue, the phenomenon may occur when stresses are below the yield strength so that the elastic stress-strain assumptions are of practical use.

ii) The most common form of mechanical failure is by wear. The laws governing the overall friction and wear between two surfaces seem to depend primarily on the total force transmitted across the two surfaces rather than on the local distribution of the force.

→ The local distributions of stress and strain are unimportant, and one may assume that the two bodies in contact are perfectly rigid.

▶ Six ideal model

1 ▷ Rigid material

i) A rigid material is one which has no strain regardless of the applied stress.

ii) This idealization is useful in studying the gross motions and forces on machine parts to provide for adequate power and for resistance to wear.

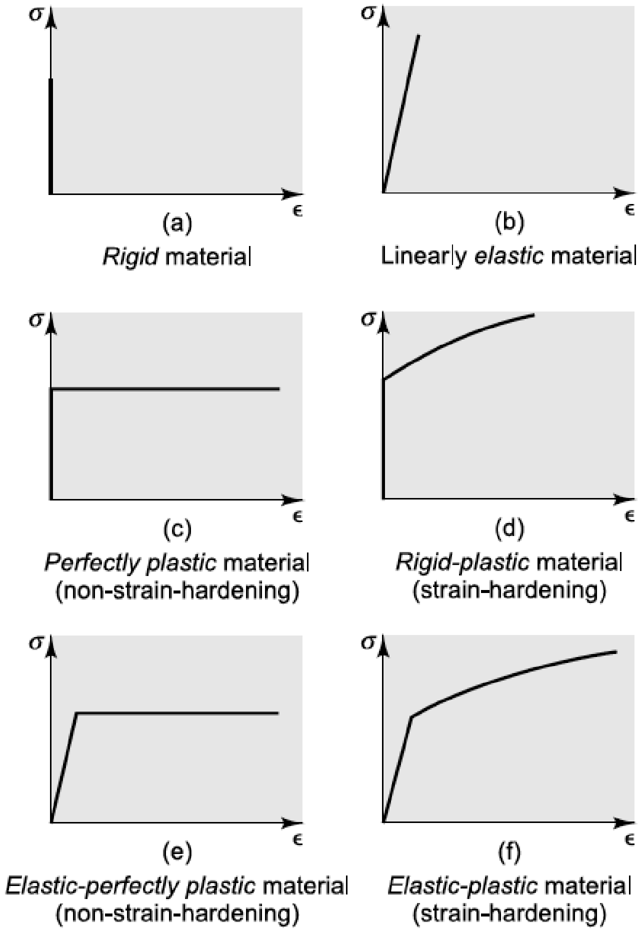


Fig. 5.7 *Idealized models of material behavior*

2 ▷ Linearly elastic material

i) A linearly elastic material is one in which the strain is proportional to the stress.

ii) This idealization is useful when we are designing for small deformations, for stiffness, or to prevent fatigue or fracture in brittle structures.

3 ▷ Rigid plastic material

i) A rigid-plastic material is one in which elastic and time-dependent deformations are neglected.

ii) Such idealizations are useful in designing structures for their maximum loads and in studying many machining and metal-forming problems, and in some detailed studies of fracture.

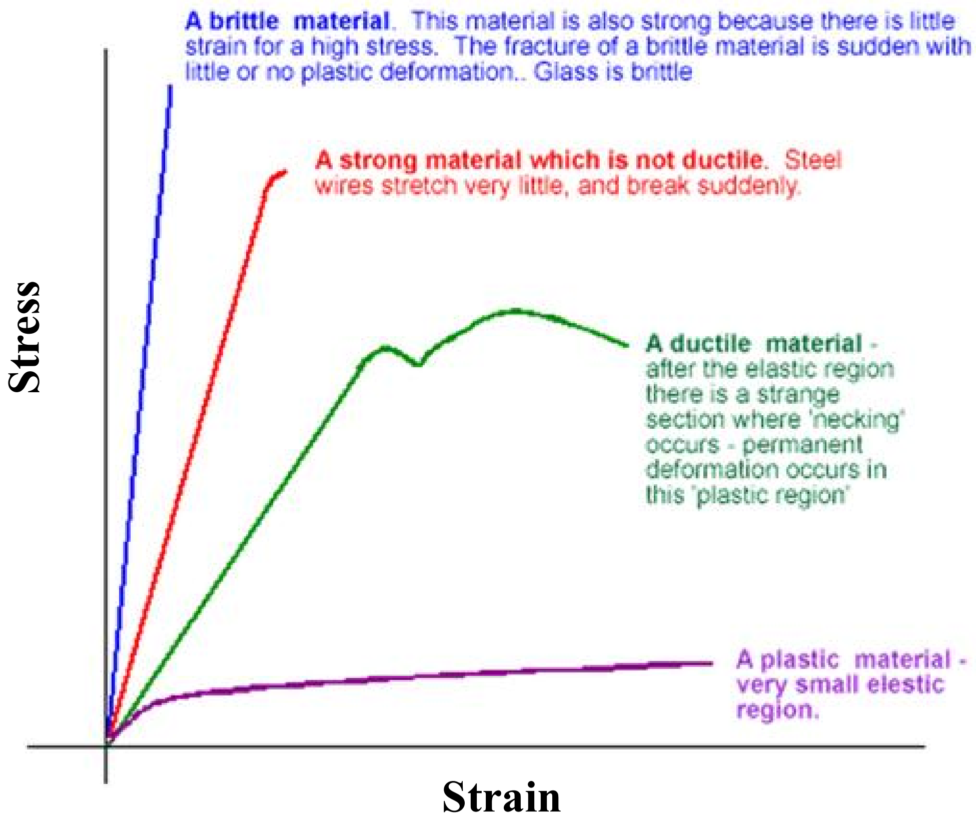
cf. The material that strain-hardening may be neglected (Fig. 5.7 (c)) is termed perfectly plastic material.

4 ▷ Elastic-plastic material

i) An elastic-plastic material is one in which both elastic and plastic strains are present; strain-hardening may or may not be assumed to be negligible (Figs. 5.7 (f) and (e)).

ii) These idealizations are useful in designing against moderate deformations when carrying out detailed studies of the mechanisms of fracture, wear, and friction.

▶ Examples of the behaviors of other materials



▶ <u>Example 5.1</u> Two coaxial tubes, the inner one of 1020 CR steel and cross-sectional area A_s, and the outer one of 2024-T4 aluminum alloy and of area A_a, are compressed between heavy, flat end plates, as shown in Fig 5.8. We wish to determine the load-deflection curve of the assembly as it is compressed into the plastic region by an axial force P.

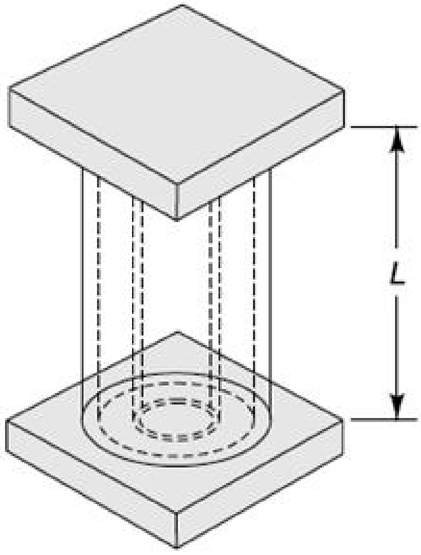


Fig. 5.8 Example 5.1

▷ Geometry

$$\epsilon_s = \epsilon_a = \epsilon = \delta / L$$

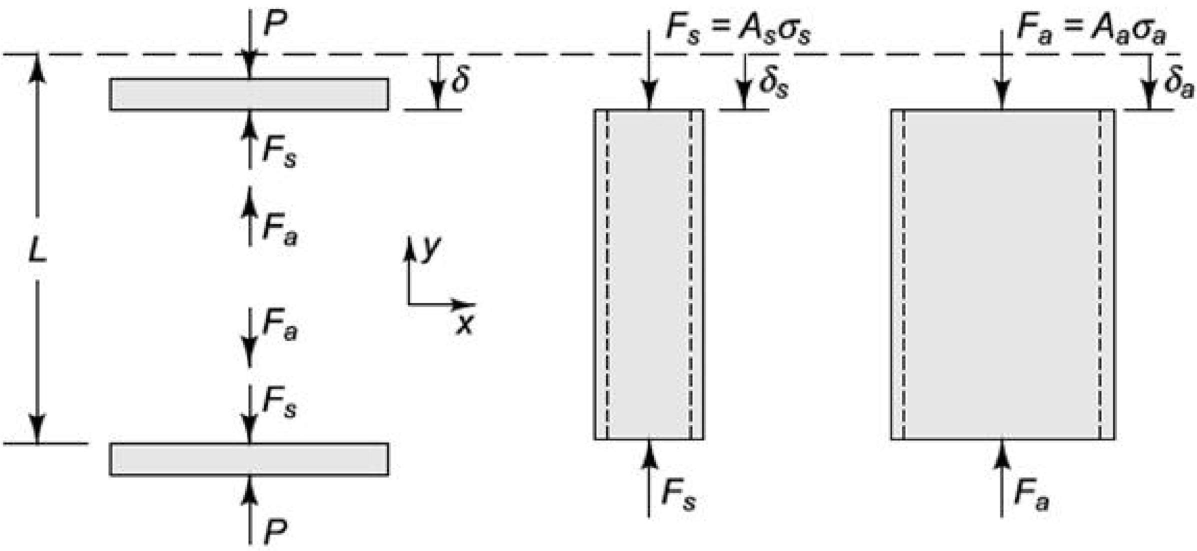


Fig. 5.9 Idealized model for Example 5.1

▷ Stress-strain relation

→ We can, with reasonable accuracy, idealize both these curves as being of the elastic-perfectly plastic type of Fig. 5.7 (e).

→ From Fig 5.10, we see that there are three regions of strain which are of interest as we compress the assembly.

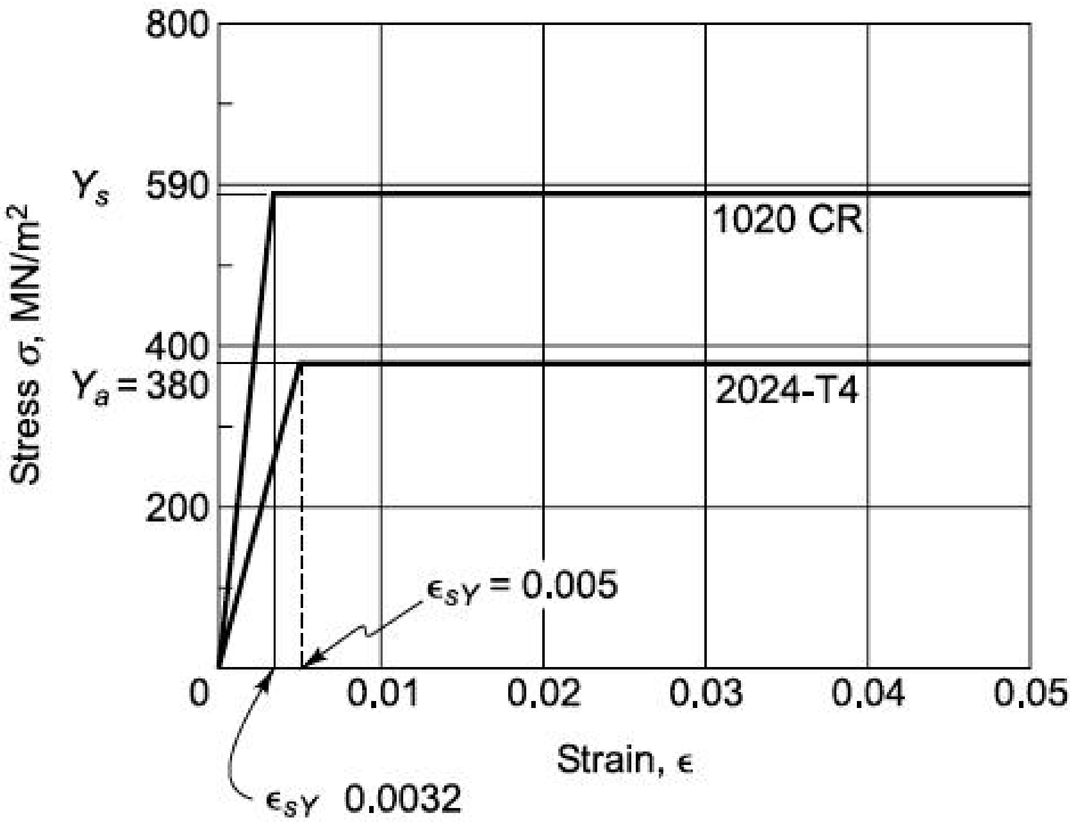


Fig. 5.10 *Idealized stress-strain curves for Example 5.1*

i) For $0.0000 \le \epsilon \le 0.0032$

$$\begin{cases} \sigma_s = E_s \epsilon_s = E_s \epsilon \\ \sigma_a = E_a \epsilon_a = E_a \epsilon \end{cases} \tag{b}$$

where, $\begin{cases} E_s = \frac{590}{0.0032} = 184 \text{ GN/m}^2 \\ E_a = \frac{380}{0.005} = 76 \text{ GN/m}^2 \end{cases}$

ii) For $0.0032 \le \epsilon \le 0.0050$

$$\begin{cases} \sigma_s = Y_s = 590 \text{ MN/m}^2 \\ \sigma_a = E_a \epsilon_a = E_a \epsilon \end{cases} \tag{c}$$

iii) For $0.0050 \le \epsilon$

$$\begin{cases} \sigma_s = Y_s = 590 \text{ MN/m}^2 \\ \sigma_a = Y_a = 380 \text{ MN/m}^2 \end{cases} \tag{d}$$

▷ Equilibrium

i) In Fig. 5.9 the top plate is in equilibrium when

$$\sum F_y = \sigma_s A_s + \sigma_a A_a - P = 0 \tag{e}$$

ii) Combining (e) with (b), (c), and (d) in succession, we obtain the load deformation curve of Fig. 5.11.

cf. We now turn to the generalization of these idealized uniaxial stress-strain relations for application to more general situations, where any or all components of stress and strain may be present.

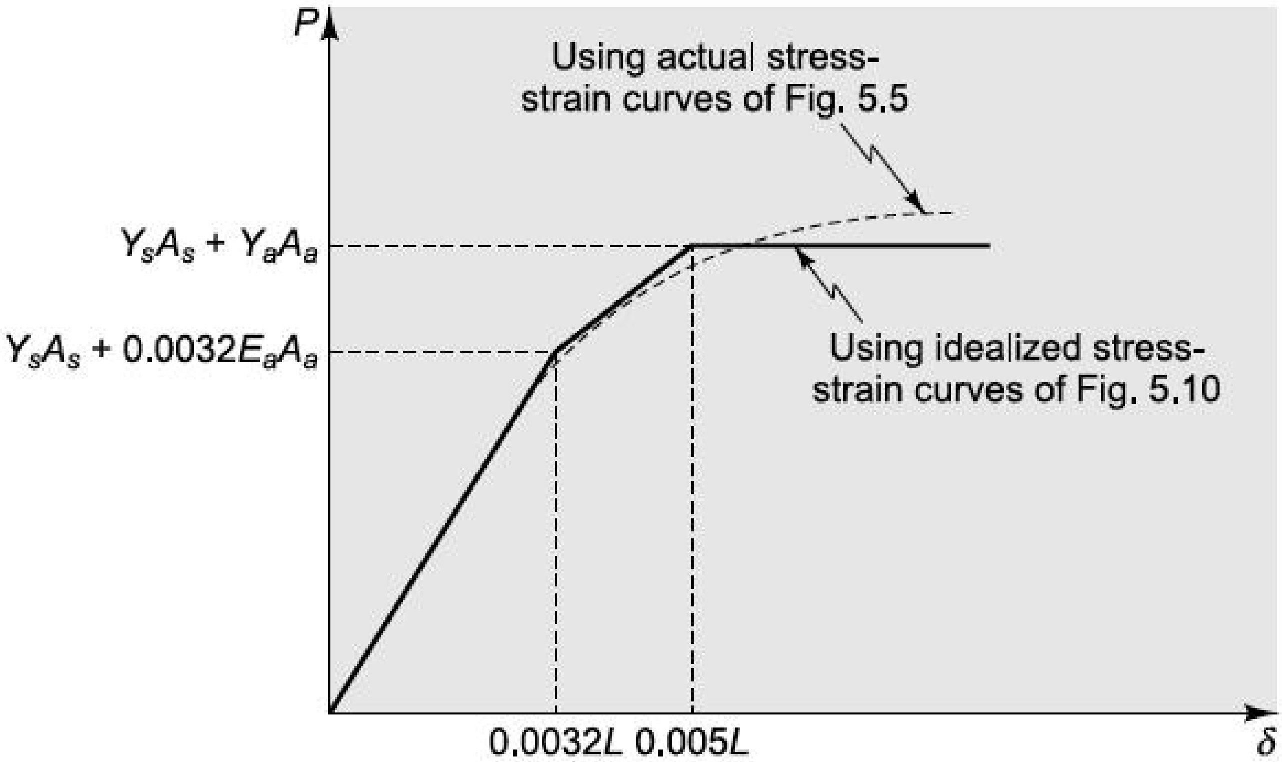


Fig. 5.11 *Load-deformation curve for Example 5.1*

5.4 Elastic Stress-strain Relations

▶ Assumptions in this section

i) We shall generalize the elastic behavior in the tension test to arrive at relations which connect all six components of stress with all six components of elastic strain.

ii) We shall restrict ourselves to materials which are linearly elastic. (linear elasticity)

iii) We also restrict ourselves to strains small compared to unity. (small strain)

iv) We shall consider the materials that are independent of orientation which is assumed to be isotropic. (isotropic)

▶ Definitions

$$\sigma_x = E\epsilon_x, \qquad \epsilon_x = \frac{\sigma_x}{E}$$

1. Young's modulus (or modulus of elasticity)

i) The modulus of elasticity E is numerically equal to the slope of the linear-elastic region in stress-strain curve and it is the material property.

ii) The modulus of elasticity at compression and extension is same.

iii) Unit: Because ϵ is a dimensionless number, it is homogeneous to stress σ.

$$\tau_{xy} = G\gamma_{xy}, \qquad \gamma_{xy} = \tau_{xy}/G$$

2. Shear modulus of elasticity G

i) Unit: $[G] = [E] = [\sigma] = [\tau]$

ii) The relation between G and E

$$G = \frac{E}{2(1+\nu)} \tag{5.3}$$

→ $E, G,$ and ν are dependent each other.

→ In common materials, $0 < \nu < 0.5$, so $\frac{E}{3} < G < \frac{E}{2}$.

3. Poisson's ratio

→ Tests in uniaxial compression show a lateral extensional strain which has the same fixed fraction to the longitudinal compressive strain.

$$\nu = -\frac{Lateral\ Strain}{Axial\ Strain}$$

i) Poisson's ratio is the example of non-stress strain and thermal strain.

ii) For isotropic, linear-elastic material

$$\epsilon_y = \epsilon_z = -\nu\epsilon_x = -\nu\sigma_x/E$$

cf. The conditions that lateral strain in proportional to axial strain in linear-elastic region

① Material has the same components in all regions.

→ Homogeneous

② Material properties are independent of orientation.

→ Isotropic

Meanwhile, the lumbers are not isotropic but homogeneous.

In general, the structural materials (i.e., steel) is satisfied with the above requirements.

▶ The conclusions obtained under the assumption that the material is isotropic

i) No shear-strain due to normal stress components.

ii) The principal axes of strain at a point of a stressed body coincide with the principal axes of stress at that point.

iii) Each shear stress component produces only its corresponding shear-strain component.

iv) No strain components other than γ_{zx}, can exist, singly or in combination, as a result of the shear-stress component τ_{zx}.

v) The thermal strain cannot produce the shear strain.

▶ The stress-strain relations of a linear-elastic isotropic material with all components of stress present

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\epsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right] \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \qquad (5.2)$$

$$\epsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

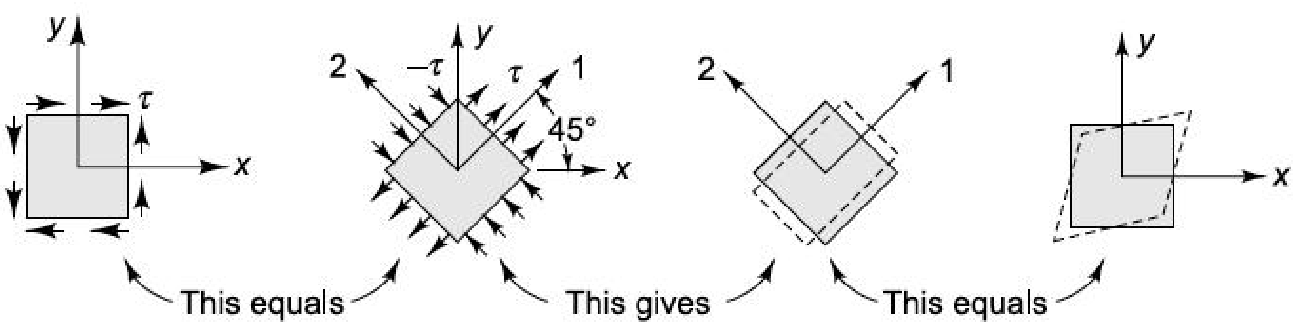


Fig. 5.16 *Equivalent states of stress and strain*

→ From Fig. 5.16,

$$\epsilon_1 = \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} = \frac{\tau(1+\nu)}{E}, \quad \epsilon_2 = \frac{\sigma_2}{E} - \nu\frac{\sigma_1}{E} = -\frac{\tau(1+\nu)}{E}$$

Meanwhile, upon use of the strain transformation formulas

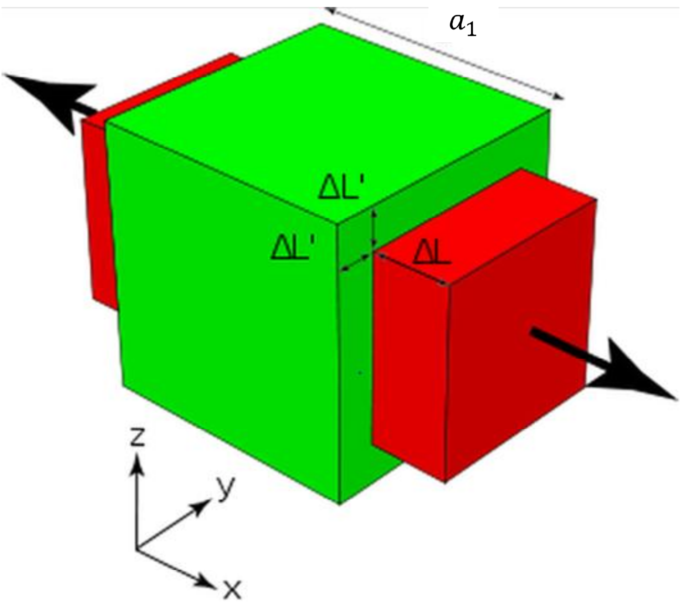
$$\gamma_{xy} = \epsilon_1 - \epsilon_2 = \frac{2(1+\nu)}{E}\tau$$

This equation and $\gamma_{xy} = \frac{\tau}{G}$ must be equal, so

$$G = \frac{E}{2(1+\nu)} \tag{5.3}$$

→ It is true, although it will not be proved here, that no other choice of coordinate axes gives any added information about the elastic constants, and thus for an isotropic material there are just two independent elastic constants.

▶ Volume change of the isotropic, linear-elastic material at extension



$$\Delta L = a_1\epsilon$$

$$\Delta L' = b_1\nu\epsilon = c_1\nu\epsilon$$

The lengths of each side after deformation are

$$\begin{cases} a_1(1+\epsilon) \\ b_1(1-\nu\epsilon) \\ c_1(1-\nu\epsilon) \end{cases}$$

$$\therefore\ v_f = a_1 b_1 c_1 (1+\epsilon)(1-\nu\epsilon)^2$$

$$= a_1 b_1 c_1 (1 - 2\nu\epsilon + \nu^2\epsilon^2 + \epsilon - 2\nu\epsilon^2 + \nu^2\epsilon^3)$$

$$v_f = a_1 b_1 c_1 (1 + \epsilon - 2\nu\epsilon)$$

$$\therefore\ e = \frac{\Delta V}{V_0} = \frac{V_f - V_0}{V_0} = \frac{a_1 b_1 c_1(\epsilon - 2\nu\epsilon)}{a_1 b_1 c_1}$$

$$= \epsilon(1-2\nu) = \frac{\sigma}{E}(1-2\nu)$$

→ Volume increase of a slender member in tensile test can be obtained when ϵ, ν are known.

cf. If $\nu > 0.5$, there is a contradiction that volume decreases when material is extended, so $\nu_{max} = 0.5$.

i) In linear-elastic region: $\frac{1}{4} \sim \frac{1}{3} \ \rightarrow\ \therefore 0.3\epsilon < e < 0.5\epsilon$

ii) In plastic region: in general, $\Delta V = 0$, so it is fine that $\nu = 0.5$.

▶ Unit volume change in three-axial stresses

→ Having unit length and $V_0 = 1$,

$$V_f = (1+\epsilon_x)(1+\epsilon_y)(1+\epsilon_z)$$

$$e = \frac{\Delta V}{V_0} = \frac{V_f - V_0}{V_0} = \frac{V_f}{V_0} - 1 \fallingdotseq \epsilon_x + \epsilon_y + \epsilon_z$$

$$= \frac{1-2\nu}{E}(\sigma_x + \sigma_y + \sigma_z)$$

cf. The shear-stress components cannot have an effect on the volume change.

$$e = -\frac{3(1-2\nu)}{E}p \qquad \frac{1}{\kappa} = \frac{E}{3(1-2\nu)}$$

κ : Bulk modulus or modulus of compression

5.5 Thermal strain

▶ In the elastic region the effect of temperature on strain appears in two ways.

i) By causing a modification in the values of the elastic constants

ii) By directly producing a strain even in the absence of stress

cf. For an isotropic material, symmetry arguments show that the thermal strain must be a pure expansion or contraction with no shear-strain components referred to any set of axes.

$$\begin{cases} \epsilon_x^t = \epsilon_y^t = \epsilon_z^t = \alpha(T - T_0) \\ \quad \gamma_{xy}^t = \gamma_{yz}^t = \gamma_{zx}^t = 0 \end{cases} \tag{5.4}$$

▶ Total strain ϵ

$$\epsilon = \epsilon^t + \epsilon^e \tag{5.5}$$

5.6 Complete equations of elasticity

→ The problem was outlined previously in broad generality by the three steps given in (2.1). For convenience we summarize below, under the three steps of (2.1), explicit equations which must be satisfied at each point of a nonaccelerating, isotropic, linear-elastic body subject to small strains.

▶ Equilibrium (3 equations; 6 unknowns)

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0 \tag{5.6}$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0$$

▶ Geometric Compatibility (6 equations and 9 unknowns)

$$\epsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\epsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial y} \tag{5.7}$$

$$\epsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

▶ Stress-strain-temperature relation (6 equations)

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] + \alpha(T - T_0) \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\epsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right] + \alpha(T - T_0) \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \tag{5.8}$$

$$\epsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] + \alpha(T - T_0) \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

→ The equilibrium equations (5.6), the strain-displacement equations (5.7), and the strain-stress-temperature relations (5.8) provide 15 equations for the six components of stress, the six components of strain, and the three components of displacement.

cf. The complete equations (5.6), (5.7), and (5.8) apply to deformations of isotropic, linearly elastic solids which involve small strains and for which it is acceptable to apply the equilibrium requirements in the undeformed configuration.

cf. We shall be primarily concerned with the three steps of (2.1), expressed not in the infinitesimal formulation of (5.6), (5.7), and (5.8) but expressed, instead, on a macroscopic level in terms of rods, shafts, and beams.

▶ Example 5.2 A long, thin plate of width b, thickness t, and length L is placed between two rigid walls a distance b apart and is acted on by an axial force P, as shown in Fig. 5.17 (a). We wish to find the deflection of the plate parallel to the force P. We idealize the situation in Fig 5.17 (b).

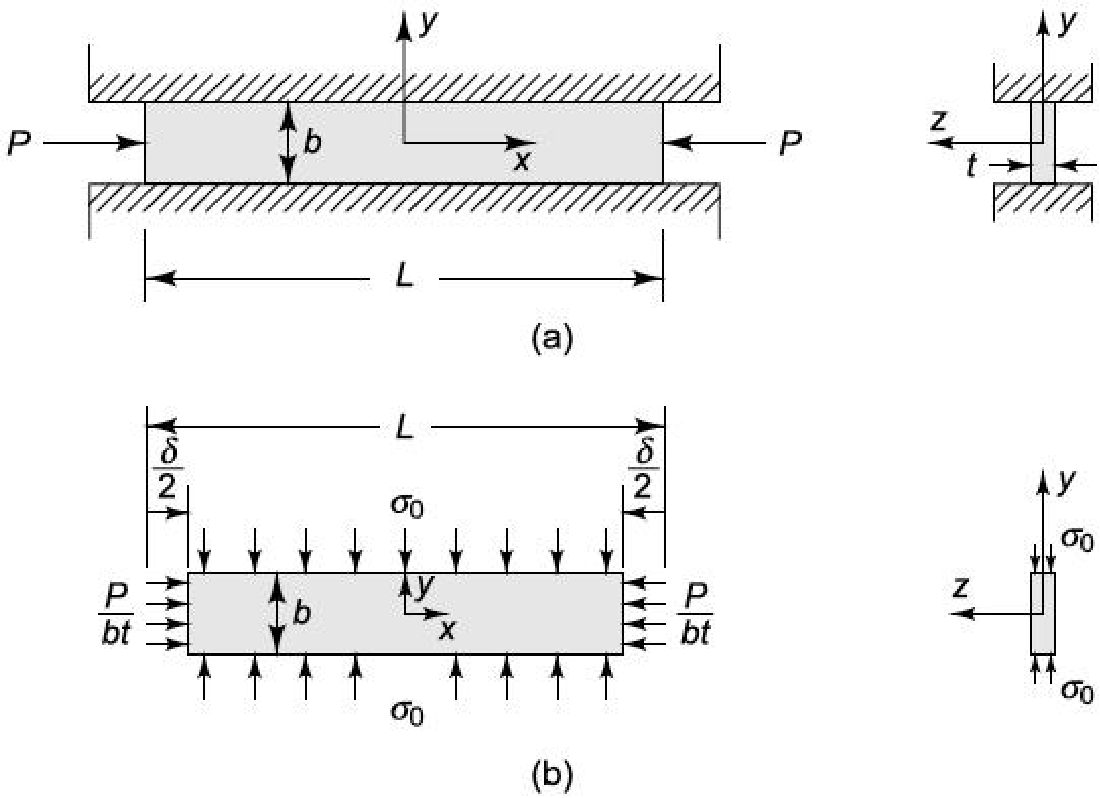


Fig. 5.17 *Example 5.2. (a) Actual problem; (b) idealized model*

▷ Assumptions

i) The axial force P results in an axial normal stress uniformly distributed over the plate area, including the end areas.

ii) There is no normal stress in the thin direction. (Note that this implies a case of plane stress in the xy plane.)

iii) There is no deformation in the y direction, that is, $\epsilon_y = 0$. (Note that this implies a case of plane strain in the xz plane.)

iv) There is no friction force at the walls (or, alternatively, it is small enough to be negligible).

v) The normal stress of contact between the plate and wall is uniform over the length and width of the plate. We now satisfy the requirements (2.1) for the idealized model of Fig. 5.17 (b).

▷ Equilibrium

$$\sigma_x = -\frac{P}{bt}, \quad \sigma_y = -\sigma_0, \qquad \sigma_z = 0 \tag{a}$$

$$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

These stresses also satisfy the equilibrium equations (5.6).

▷ Geometric compatibility

$$\epsilon_y = 0 \tag{b}$$

$$\epsilon_x = -\frac{\delta}{L} \tag{c}$$

▷ Stress-strain relation

→ eq. (5.8) is

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y), \qquad \epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x), \qquad \epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma y$$

$$\gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0 \tag{d}$$

→ Solving the system of equations (a), (b), (c), and (d)

$$\sigma_y = \nu\sigma_x = -\frac{\nu P}{bt}, \qquad \delta = \frac{(1-\nu^2)PL}{Ebt}$$

$$\epsilon_z = \frac{\nu(1+\nu)P}{Ebt} = \frac{\nu}{1-\nu}\frac{\delta}{L}$$

cf. We note that the presence of the rigid walls reduces the axial deflection of the plate by the factor $(1-\nu^2)$.

▷ Strain-displacement relation

$$u = -\frac{\delta}{L}x\,, \quad v = 0\,, \quad w = \frac{\nu}{1-\nu}\frac{\delta}{L}z$$

cf. It is relatively easy to get an exact or nearly exact solution to an idealized approximation of the real problem.

5.7 Complete Elastic Solution for a Thick-walled Cylinder

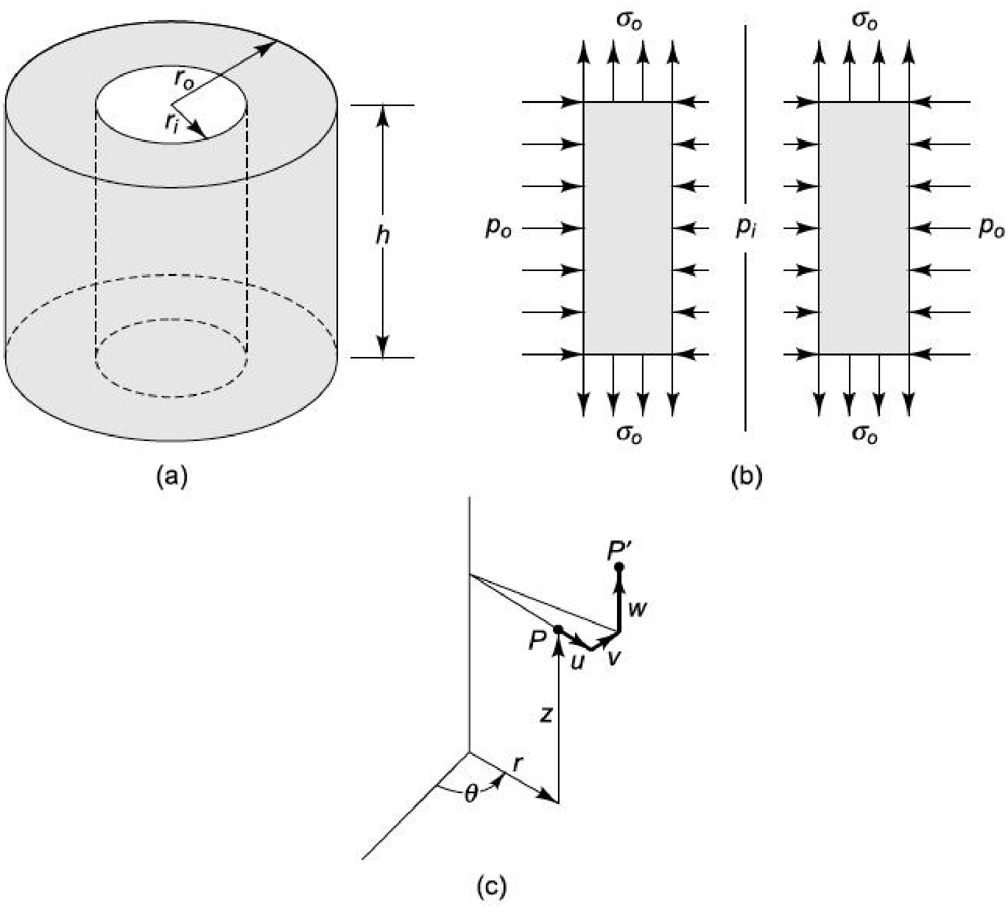


Fig. 5.18 *Thick-walled cylinder (a) subjected to inner and outer pressures and axial tension (b). Cylindrical coordinates and displacement components (c).*

→ There is uniform inner pressure p_i, uniform outer pressure p_o, and uniform axial tensile stress σ_o.

▶ Boundary condition

i) For $r = r_i$

$$\sigma_r = -p_i\,, \qquad \tau_{rz} = 0\,, \quad \tau_{r\theta} = 0 \qquad \text{(a)}$$

ii) For $r = r_0$

$$\sigma_r = -p_0\,, \qquad \tau_{rz} = 0\,, \quad \tau_{r\theta} = 0 \qquad \text{(b)}$$

iii) For $z = 0$ & $z = h$

$$\sigma_z = \sigma_0, \qquad \tau_{rz} = 0\,, \qquad \tau_{\theta z} = 0 \qquad \text{(c)}$$

▶ Geometric compatibility

→ Based on the uniformity of the axial loading, $\sigma_z = \sigma_0$ throughout the interior and that all stresses and strains are independent of z.

→ Based on symmetry, we shall look for a solution in which v and the θ component of displacement vanish everywhere and in which all stresses, strains, and displacements are independent of θ.

→ The shear stresses $\tau_{r\theta}, \tau_{\theta z}, \tau_{rz}$ and the corresponding strains $\gamma_{r\theta}, \gamma_{\theta z}, \gamma_{rz}$ vanish everywhere.

$$\tau_{r\theta} = \tau_{\theta z} = \tau_{rz} = \gamma_{r\theta} = \gamma_{\theta z} = \gamma_{rz} = 0$$

▶ Equilibrium equation for cylindrical coordinate system

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + 2\frac{\tau_{r\theta}}{r} = 0$$

$$\frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{zr}}{r} = 0$$

Remaining equilibrium equation

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \qquad \text{(d)}$$

▶ <u>Strain-displacement equations for cylindrical coordinate system</u>

$$\epsilon_r = \frac{\partial u}{\partial r} \qquad \epsilon_\theta = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} \qquad \epsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} \quad \gamma_{\theta z} = \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z} \quad \gamma_{zr} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

Remaining strain-displacement equations

$$\epsilon_r = \frac{du}{dr}, \qquad \epsilon_\theta = \frac{u}{r}, \qquad \epsilon_z = \frac{dw}{dz} \qquad \text{(e)}$$

▶ <u>Stress-strain relation for cylindrical coordinate system</u>

$$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_r + \nu(\epsilon_\theta + \epsilon_z)]$$

$$\sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_\theta + \nu(\epsilon_r + \epsilon_z)] \qquad \text{(f)}$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_z + \nu(\epsilon_r + \epsilon_\theta)]$$

Substitute (e) into (f) yields

$$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{du}{dr} + \nu\left(\frac{u}{r} + \epsilon_z\right)\right]$$

$$\sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{u}{r} + \nu\left(\frac{du}{dr} + \epsilon_z\right)\right]$$

Let k indicate constant in equation (f) and from equilibrium,

$$k = \frac{E}{(1+\nu)(1-2\nu)}$$

$$\frac{d\sigma_r}{dr} = k\left[\frac{(1-\nu)\, d^2u}{dr^2} + \nu\left(\frac{1}{r}\frac{du}{dr} - u\frac{1}{r^2}\right)\right]$$

$$\frac{\sigma_r - \sigma_\theta}{r} = k\left[(1-2\nu)\frac{1}{r}\frac{du}{dr} + 2\nu\frac{u}{r^2} - \frac{u}{r^2}\right]$$

$$\frac{d\sigma_r}{dr}+\frac{\sigma_r-\sigma_\theta}{r}=k(1-\nu)\left[\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}\right]=0$$ →Cauchy equation

$$\therefore\ u=Ar+\frac{B}{r}\qquad \epsilon_r=\frac{du}{dr}=A-\frac{B}{r^2}\qquad \epsilon_\theta=A+\frac{B}{r^2}$$

$$\therefore\ \sigma_r=k\left[A+\nu\epsilon_z-(1-2\nu)\frac{B}{r^2}\right]$$

$$\sigma_\theta=k\left[A+\nu\epsilon_z+(1-2\nu)\frac{B}{r^2}\right]$$

▶ Boundary condition

$$\sigma_{r_i}=k\left[A+\nu\epsilon_z-(1-2\nu)\frac{B}{r_i^2}\right]=-p_i$$

$$\sigma_{r_0}=k\left[A+\nu\epsilon_z-(1-2\nu)\frac{B}{r_0^2}\right]=-p_0$$

$$\begin{bmatrix}-\frac{p_i}{k}-\nu\epsilon_z\\ -\frac{p_0}{k}-\nu\epsilon_z\end{bmatrix}=\begin{bmatrix}1 & -\frac{1-2\nu}{r_i^2}\\ 1 & -\frac{1-2\nu}{r_0^2}\end{bmatrix}\begin{bmatrix}A\\ B\end{bmatrix}=\mathrm{X}\begin{bmatrix}A\\ B\end{bmatrix}$$

$$det(X)=\frac{(1-2\nu)(r_0^2-r_i^2)}{(r_0r_i)^2}$$

$$X^{-1}=\frac{1}{dex(X)}\begin{bmatrix}-\frac{1-2\nu}{r_0^2} & \frac{1-2\nu}{r_i^2}\\ -1 & 1\end{bmatrix}$$

$$\therefore\ \begin{bmatrix}A\\ B\end{bmatrix}=\frac{(r_0r_i)^2}{(1-2\nu)(r_0^2-r_i^2)}\begin{bmatrix}-\frac{1-2\nu}{r_0^2} & \frac{1-2\nu}{r_i^2}\\ -1 & 1\end{bmatrix}\begin{bmatrix}-\frac{p_i}{k}-\nu\epsilon_z\\ -\frac{p_0}{k}-\nu\epsilon_z\end{bmatrix}$$

$$\therefore\ A=\frac{1}{k}\frac{-p_0r_0^2+p_ir_i^2}{r_0^2-r_i^2}-\nu\epsilon_z$$

$$B=\frac{1}{(1-2\nu)k}(-p_0+p_i)\frac{(r_0r_i)^2}{r_0^2-r_i^2}$$

$$\therefore\ \sigma_r=k\left[\frac{1}{k}\frac{-p_0r_0^2+p_ir_i^2}{(r_0^2-r_i^2)}-(1-2\nu)\frac{1}{(1-2\nu)k}(-p_0+p_i)\frac{\left(\frac{r_0r_i}{r}\right)^2}{r_0^2-r_i^2}\right]$$

$$= -\frac{p_i[(r_0/r)^2-1]+p_0[(r_0/r_i)^2-(r_0/r)^2]}{(r_0/r_i)^2-1}$$

$$\sigma_\theta = k\left[\frac{1}{k}\frac{-p_0 r_0^2+p_i r_i^2}{(r_0^2-r_i^2)} + (1-2\nu)\frac{1}{(1-2\nu)k}(-p_0+p_i)\frac{\left(\frac{r_0 r_i}{r}\right)^2}{r_0^2-r_i^2}\right]$$

$$= \frac{p_i[(r_0/r)^2+1]-p_0[(r_0/r_i)^2+(r_0/r)^2]}{(r_0/r_i)^2-1}$$

$$\epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_r+\sigma_\theta)]$$

$$= \frac{\sigma_0}{E} - \frac{2\nu}{E}\frac{p_i r_i^2 - p_0 r_0^2}{r_0^2-r_i^2}$$

→ Note that ϵ_z is independent of position within the cylinder.

▶ Stress-strain equations (following textbook)

From generalized Hooke's law

$$\epsilon_r = \frac{1}{E}[\sigma_r - \nu(\sigma_\theta+\sigma_z)]$$

$$\epsilon_\theta = \frac{1}{E}[\sigma_\theta - \nu(\sigma_z+\sigma_r)] \qquad \text{(f)}$$

$$\epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_r+\sigma_\theta)]$$

i) From the first two equations of (f) we solve for the transverse stresses σ_r and σ_θ, in terms of ϵ_r and ϵ_θ and thus obtain the stresses also as functions of u.

ii) Finally, substituting the stresses into (d) leads to the following differential equation for $u(r)$

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \qquad \text{(h)}$$

$$\therefore\ r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u = 0$$

$$u = Ar + \frac{B}{r} \rightarrow \text{general solution} \tag{i}$$

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u = r^2m(m-1)r^{m-2} + rmr^{m-1} - r^m$$

$$\therefore\ (m(m-1) + m - 1)r^m = 0$$

$$\therefore\ u = Ar + Br^{-1}$$

Apply the boundary conditions

$$\begin{cases} \sigma_r = -\dfrac{p_i[(r_0/r)^2-1]+p_0[(r_0/r_i)^2-(r_0/r)^2]}{(r_0/r_i)^2-1} \\ \sigma_\theta = \dfrac{p[(r_0/r)^2+1]-p_0[(r_0/r_i)^2+(r_0/r)^2]}{(r_0/r_i)^2-1} \end{cases} \tag{5.9}$$

→ The axial strain is obtained by substituting these stresses together with $\sigma_z = \sigma_o$ into the third equation of (f).

$$\therefore\ \epsilon_z = \frac{\sigma_0}{E} - \frac{2\nu}{E}\frac{p_i r_i^2 - p_0 r_0^2}{r_0^2 - r_i^2} \tag{5.10}$$

→ Note that ϵ_z is independent of position within the cylinder.

▶ Analysis

i) The axial displacement w thus varies linearly with z.

ii) The transvers stresses $(\sigma_r, \sigma_\theta)$ are independent of σ_o and ϵ_z depends on the axial loading σ_o.

iii) When the inner and outer pressures are both equal (that is, $p_i = p_o = p$), we find that $\sigma_r = \sigma_\theta = -p$ throughout the interior.

iv) When the outer pressure is absent ($p_o = 0$), we note that an inner pressure p_i results in a compressive radial stress which varies from $\sigma_r = -p_i$ at the inner wall to $\sigma_r = 0$ at the outer wall.

v) Note that the numerically greatest stress in both Fig. 5.19 (a) and Fig. 5.19 (b) is the tangential stress σ_θ at the inner wall of the cylinder.

vi) When the cylinder wall-thickness $t = r_o - r_i$ becomes small in comparison with r_i, the solution (5.9) approaches the thin-walled-tube approximation of Prob. 4.10 (see also Prob. 5.47).

vii) When the axial stress vanishes $(\sigma_o = 0)$, the cylinder is said to be subject to a plane stress distribution. In this case the axial strain ϵ_z is generally not zero. ($\because$ plane stress distribution $\neq$ plane strain distribution)

viii) We can use the exact result (5.9) to illustrate the concept of stress concentration.

cf. A characteristic of the solution (5.9) is that, although it depends on the material's being homogeneous, isotropic, and linearly elastic, the stresses are independent of the actual magnitudes of the elastic parameters E and ν.

cf. Note that the results (5.9) and (5.10) involve the quite significance in engineering.

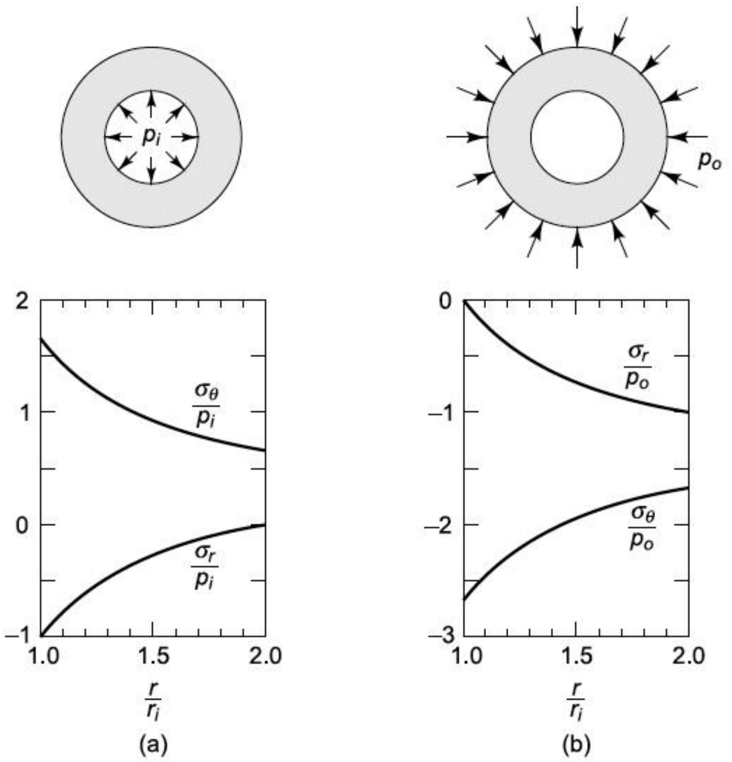


Fig. 5.19 *Distribution of radial stress $\sigma_r(r)$ and tangential stress $\sigma_\theta(r)$ in cylinder with $r_o = 2r_i$ due to (a) inner pressure p_i and (b) outer pressure p_o*

5.8 Strain Energy in an Elastic Body

→ In Sec. 2.6 the concept of elastic energy was introduced in terms of springs and uniaxial members. Here we extend the concept to arbitrary linearly elastic bodies subjected to small deformations.

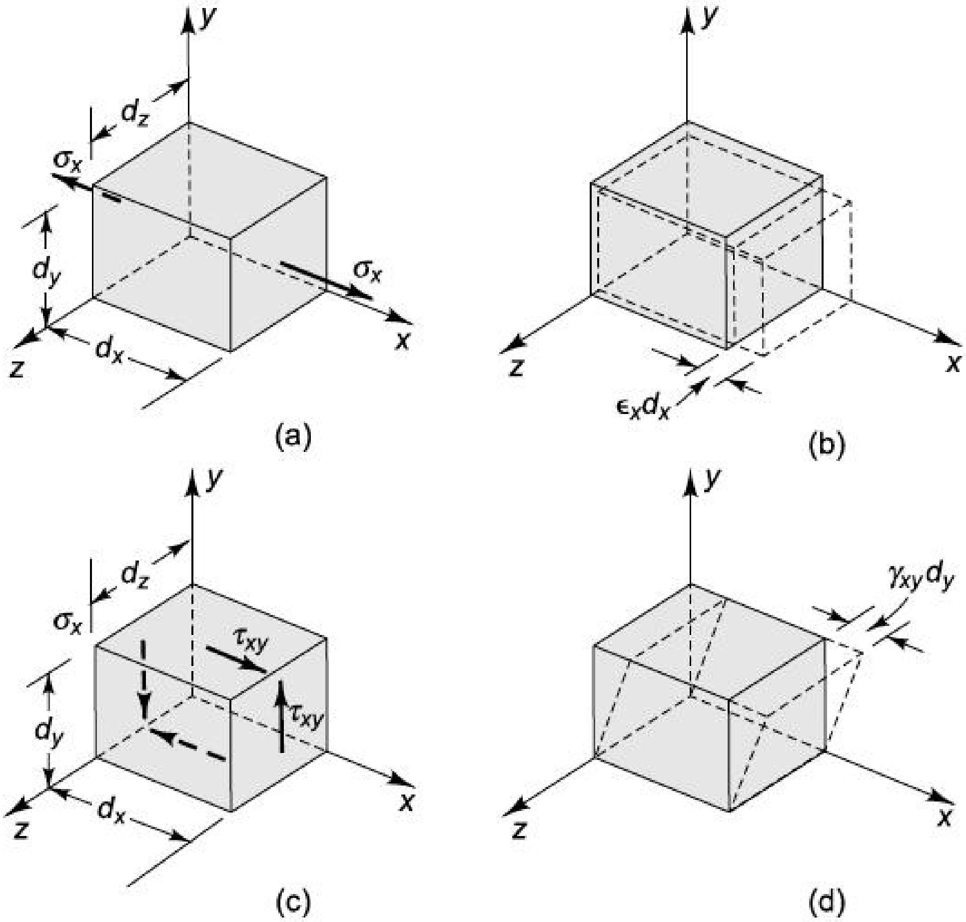


Fig. 5.20 *Infinitesimal element subjected to: uniaxial tension (a), with resulting deformation (b); pure shear (c), with resulting deformation (d)*

$$U = \frac{1}{2} P\delta \tag{5.11}$$

▶ <u>The strain energy stored in the element (in a linearly elastic material)</u>

<u>From Fig. 5.20 (a)</u>

$$dU = \frac{1}{2}\,(\sigma_x dydz)(\epsilon_x dx) = \frac{1}{2}\sigma_x \epsilon_x dV \tag{5.12}$$

$$\rightarrow U = \frac{1}{2}\int_v \sigma_x \epsilon_x \, dV \tag{5.13}$$

Since $\sigma_x = P/A, \quad \epsilon_x = \delta/L$;

$$U = \frac{1}{2}\left(\frac{P}{A}\right)\left(\frac{\delta}{L}\right)\int_v dV$$

$$= \frac{1}{2}P\delta \tag{5.14}$$

From Fig. 5.20 (c)

$$dU = \frac{1}{2}(\tau_{xy}dxdz)(\gamma_{xy}dy)$$

$$= \frac{1}{2}\tau_{xy}\gamma_{xy}dV \tag{5.15}$$

cf. $U = T\phi/2$

cf. The individual strain components may depend on more than one stress component, but we assume that the dependence is linear. Thus, if we imagine a gradual loading process in which all stress components maintain the same relative magnitudes as in the final stress state, the strain components will also grow in proportion, maintaining the same relative magnitudes as in the final strain state. During this process in which all stresses and strains are growing, a single stress component such as σ_x will do work only on the deformation due to its corresponding strain ϵ_x.

▶ The total stain energy stored in the element

$$dU = \frac{1}{2}(\sigma_x\epsilon_x + \sigma_y\epsilon_y + \sigma_z\epsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})dV \tag{5.16}$$

∴ In general, the final stresses and strains vary from point to point in the body. The strain energy stored in the entire body is obtained by integrating (5.16) over the volume of the body.

$$U = \frac{1}{2}\int_v(\sigma_x\epsilon_x + \sigma_y\epsilon_y + \sigma_z\epsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})dV \tag{5.17}$$

cf. In the case of plane stress or plane strain

$$U = \frac{1}{2}\int_v \left(\sigma_x \epsilon_x + \sigma_y \epsilon_y + \tau_{xy}\gamma_{xy}\right) dV \tag{5.18}$$

In Chapter 6 and 7 we shall use these results to develop special formulas for strain energy in torsion and bending.

Overall Summary

▶ Hooke's law

$$\begin{aligned}
\epsilon_x &= \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] + \alpha(T - T_0) \qquad \gamma_{xy} = \frac{\tau_{xy}}{G} \\
\epsilon_y &= \frac{1}{E}\left[\sigma_y - \nu(\sigma_z + \sigma_x)\right] + \alpha(T - T_0) \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \\
\epsilon_z &= \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] + \alpha(T - T_0) \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}
\end{aligned} \tag{5.8}$$

In case of statically determinate structure, the thermal strain does not generate the stress. But in the case of statically indeterminate structure, it generates the stress.

By strain-term

In 2D case

$$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y)$$

$$\sigma_y = \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x)$$

In 3D case

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left\{(1-\nu)\epsilon_x + \nu\left(\epsilon_y + \epsilon_z\right)\right\}$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left\{(1-\nu)\epsilon_y + \nu(\epsilon_z + \epsilon_x)\right\}$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left\{(1-\nu)\epsilon_z + \nu\left(\epsilon_x + \epsilon_y\right)\right\}$$

▶ Unit volume change

$$e = \frac{\Delta V}{V_0} = \epsilon_x + \epsilon_y + \epsilon_z = \frac{1-2\nu}{E}(\sigma_x + \sigma_y + \sigma_z)$$

Spherical stress : In the case of $\sigma_x = \sigma_y = \sigma_z = \sigma_o$ and shear stress components are absent. In addition, the Mohr's circle of stress and strain is indicated by a point.

$$\epsilon_x = \epsilon_y = \epsilon_z = \epsilon_0 = \frac{\sigma_0}{E}(1 - 2\nu)$$

$$e = \frac{\Delta V}{V_0} = \frac{3(1-2\nu)\sigma_0}{E} = 3\epsilon_0$$

∴ This stress distribution is called hydrostatic stress distribution.

▶ Relation between E and G

$$G = \frac{E}{2(1+\nu)} \qquad (5.3)$$

▶ Strain energy density $(u = \mathrm{U/V})$

By stress-term

$$u = \frac{1}{2}(\sigma_x\epsilon_x + \sigma_y\epsilon_y + \sigma_{xz}\epsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})$$

$$= \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 + \sigma_y^2) - \frac{\nu}{E}(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + \frac{1}{2G}(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)$$

By strain-term

$$u = \frac{E(1-\nu)}{2(1+\nu)(1-2\nu)}(\epsilon_x + \epsilon_y + \epsilon_z)^2 - \frac{E}{1+\nu}\{\epsilon_x\epsilon_y + \epsilon_y\epsilon_z + \epsilon_z\epsilon_x -$$
$$14\gamma xy2 + \gamma yx2 + \gamma zx2$$

$$= \frac{E}{2(1+\nu)(1-2\nu)}[(1-\nu)(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2) + 2\nu(\epsilon_x\epsilon_y + \epsilon_y\epsilon_z +$$

$\epsilon_z\epsilon_x)] - \frac{G}{2}(\gamma_{xy}^2 + \gamma_{yx}^2 + \gamma_{zx}^2)$

5.9 Stress Concentration

$$\begin{cases} \sigma_r = -\frac{p_i[(r_0/r)^2-1]+p_0[(r_0/r_i)^2-(r_0/r)^2]}{(r_0/r_i)^2-1} \\ \sigma_\theta = \frac{p_i[(r_0/r)^2+1]-p_0[(r_0/r_i)^2+(r_0/r)^2]}{(r_0/r_i)^2-1} \end{cases} \tag{5.9}$$

▶ Stress concentration

The local increase in stress caused by the irregularity in geometry

▶ Stress concentration factor

$K_t = \sigma_{max}/\sigma_{nom}$

σ_{max}: The maximum stress in the presence of a geometric irregularity or discontinuity.

σ_{nom}: The nominal stress which would exist at the point if the irregularity were not there.

→ The magnitude of this factor depends upon the particular geometry and loading involved, but factors of 2 or more are common.

cf. In case of plastic flow or ductile fracture, strain concentration might be more important than stress concentration.

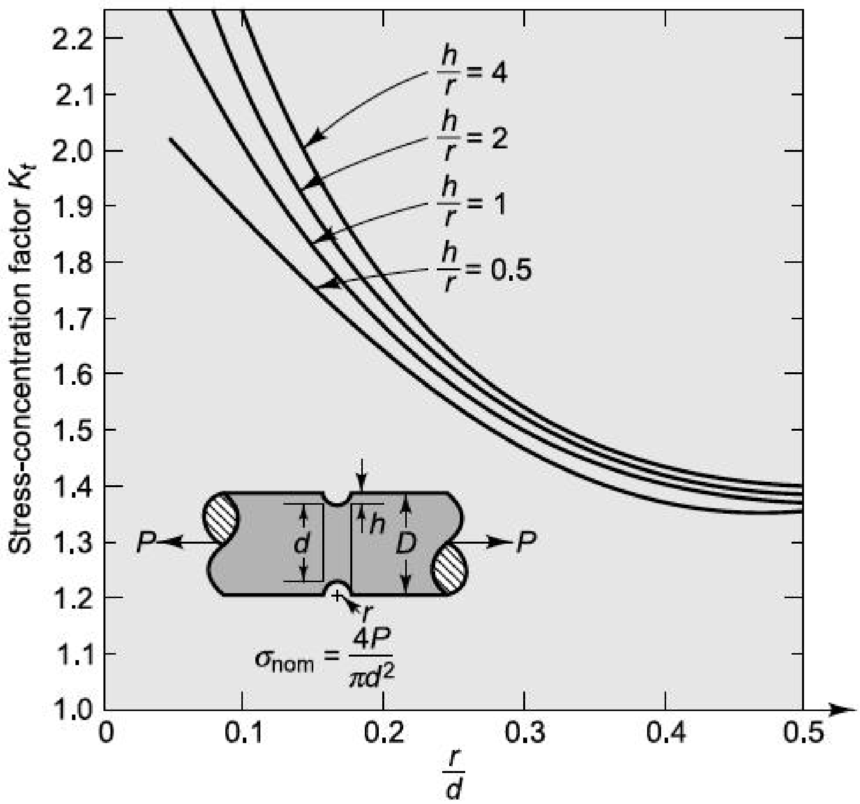


Fig. 5.21 *Stress concentration factor K_t for a circular groove in a solid circular shaft with tensile force P. (From C. Lipson and R. Juvinall, "Handbook of Stress and Strength," The Macmillan Company, New York, 1963)*

5.11 Criteria for Initial Yielding

We now turn to the problem of what happens when, in a general state of stress, the material is stressed to the point where it no longer behaves in a linearly elastic manner.

For most materials, including metals, the deviation from proportionality in a uniaxial tensile test is an indication of the beginning of plastic flow (yielding).

→ We shall restrict ourselves to polycrystalline materials which are at least statistically isotropic.

▶ Dislocation

i) During elastic deformation of a crystal, there is a uniform shifting of the whole planes of atoms relative to each other.

ii) Plastic deformation depends on the motion of individual imperfections in the crystal structure.

iii) Under the presence of a shear stress, one kind of imperfection called an edge dislocation will tend to migrate until there has been a displacement of the upper part of the crystal relative to the lower by approximately one atomic spacing.

iv) By a combination of such motions, plastic strain can be produced.

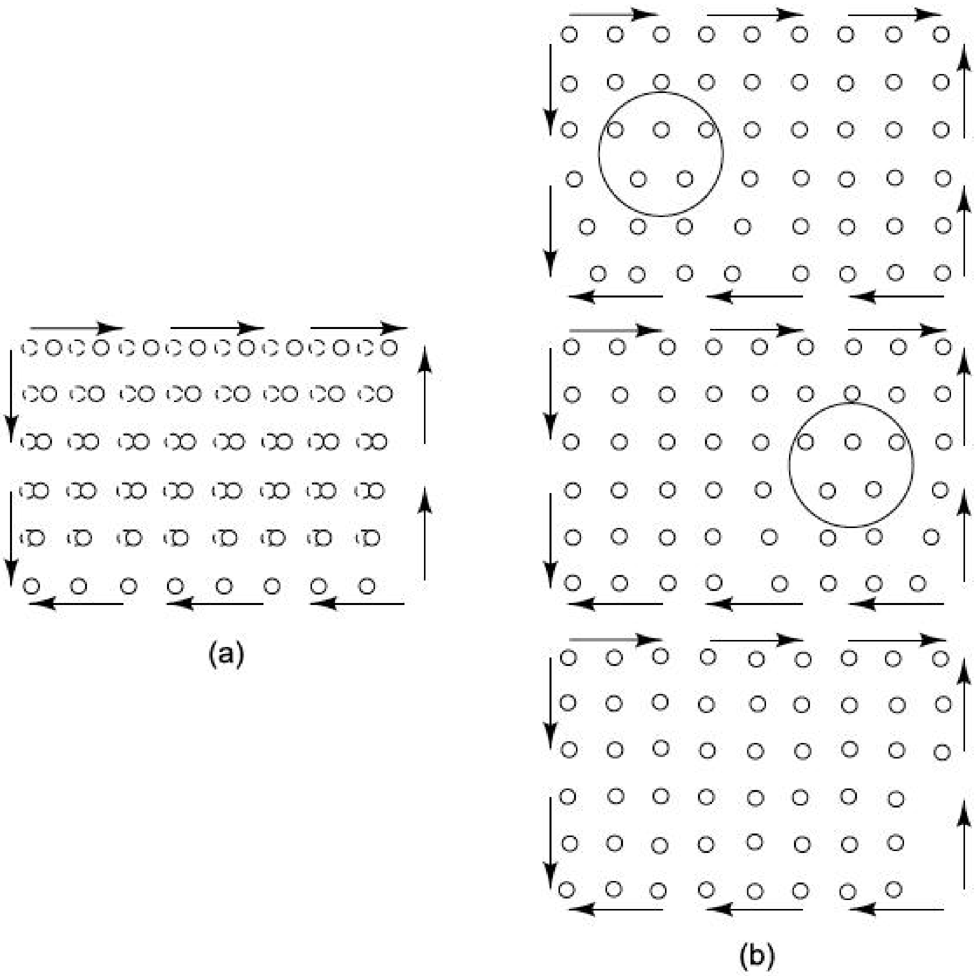


Fig. 5.27 *Deformation of a crystal lattice. (a) Elastic deformation; (b) plastic deformation*

→ It is important to note that a consequence of this simple model is that shear stress is the dominant agent in the migration of these dislocations.

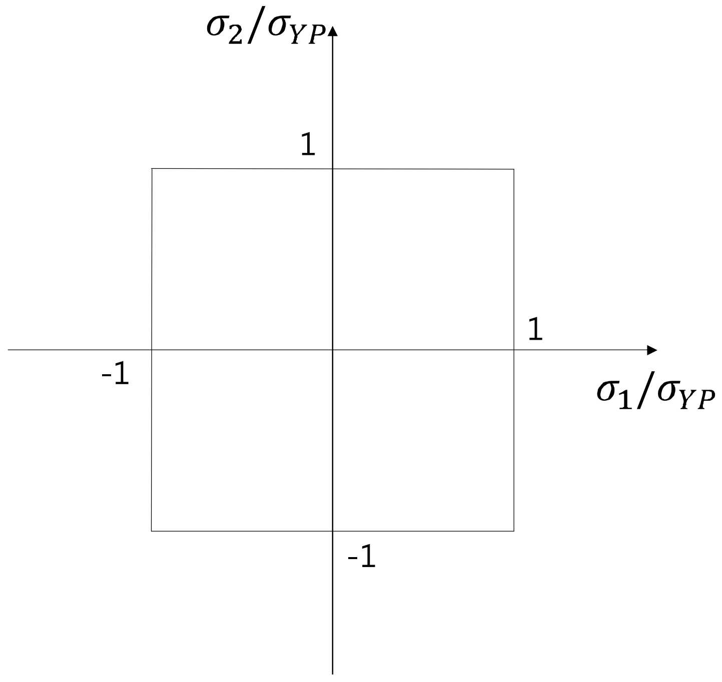
▶ Yielding Criteria

i) The state of stress can be described completely by giving the magnitude and orientation of the principal stresses.

ii) Since we are considering only isotropic materials, the orientation of the principal stresses is unimportant, thus the criteria for yielding are based only on the magnitude of the principal stresses.

iii) Since experimental work that a hydrostatic state of stress does not affect yielding, above two criteria are based not on the absolute magnitude of the principal stresses but rather on the magnitude of the differences between the principal stresses.

▶ Maximum Stress Theory



Yielding can occur when the any principal stress at arbitrary point reaches the same value which the stress has when yielding occurs in the tensile test

$\therefore$ $(\sigma_1)_{YP} = \sigma_{YP}$ or $|(\sigma_2)_{YP}| = |\sigma_{YP}|$

cf. Limitations: 1) $(\sigma_{YP})_{Tensile} \neq (\sigma_{YP})_{Compression}$,

2) $(\tau_{max})_{YP}$ differs for different materials

▶ Von Mises Criterion

→ It is also called the maximum distortion-energy theory and applied to the ductile materials.

Yielding condition

Yielding can occur in a three-dimensional state of stress when the root mean square of the differences between the principal stresses reaches the same value which it has when yielding occurs in the tensile test.

Since $\sigma_1 = Y,\ \ \sigma_2 = \sigma_3 = 0,$ the yielding occurs when the stress condition is satisfied.

$$\sqrt{\frac{1}{3}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]}$$

$$= \sqrt{\frac{1}{3}[(Y-0)^2+(0-0)^2+(0-Y)^2} = \sqrt{2/3}\,Y$$

→ For general stress state, we can derive

$$\sqrt{\frac{1}{2}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]} = Y \qquad (5.23)$$

→ In case of non-principal stress axis, we can derive

$$\sqrt{\frac{1}{2}\left[\left(\sigma_x-\sigma_y\right)^2+\left(\sigma_y-\sigma_z\right)^2+(\sigma_z-\sigma_x)^2\right]+3\tau_{xy}^2+3\tau_{yz}^2+\tau_{zx}^2}$$

$$=Y$$

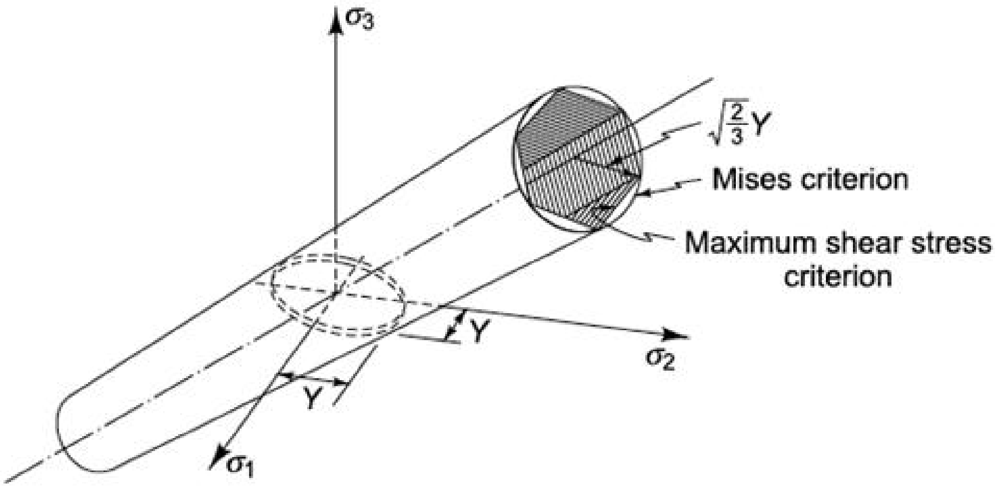


Fig. 5.30 *Geometrical representation in principal stress space of the Mises and maximum shear-stress yield criteria*

cf. The criterion (5.23) then is represented in this space by a right-circular cylinder of radius $\sqrt{\frac{2}{3}}\boldsymbol{Y}$ whose axis makes equal angles with the $\boldsymbol{\sigma_1, \sigma_2}$ ***and*** $\boldsymbol{\sigma_3}$ coordinate axes, as illustrated in Fig. 5.30. Yielding occurs for any state of stress which lies on the surface of this circular cylinder.

Yielding condition in Plain stress

$$\left(\frac{\sigma_1}{Y}\right)^2-\frac{\sigma_1\sigma_2}{Y^2}+\left(\frac{\sigma_2}{Y}\right)^2=1$$

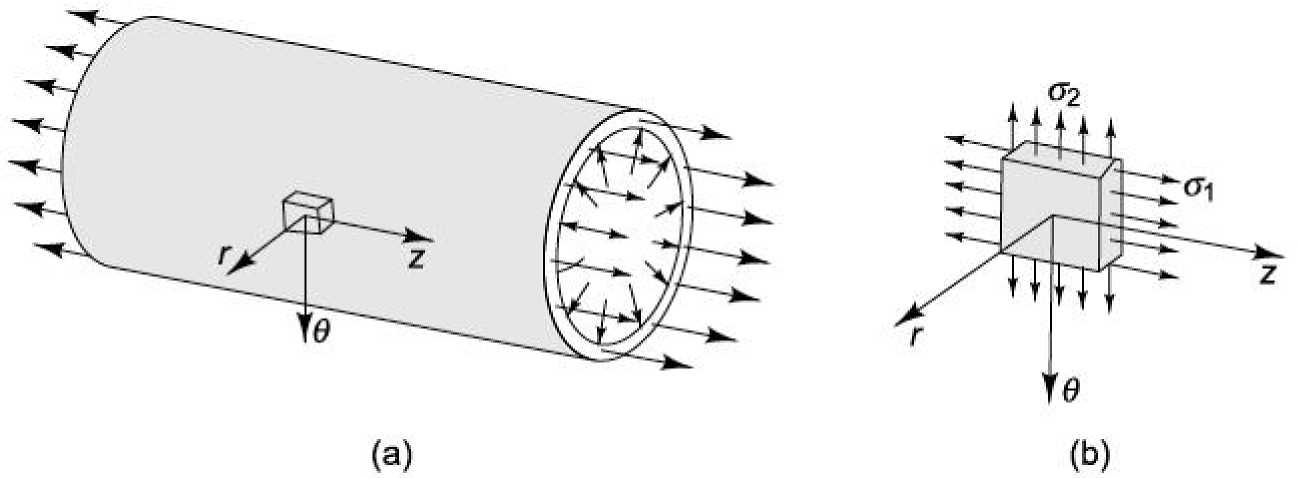


Fig. 5.28 *Example of biaxial stress in a thin-walled cylinder*

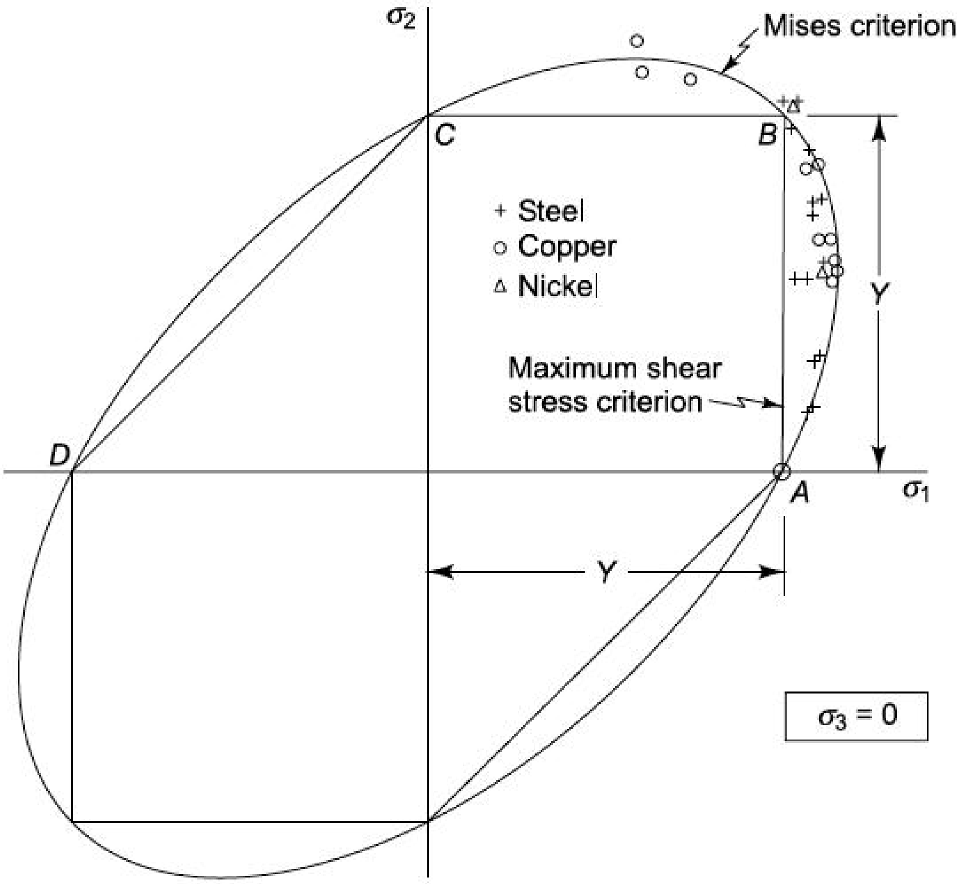


Fig. 5.29 *Yielding of thin-walled tubes under combined stress. (From W. Lode, Versuche uber den Einfluss der mittleren Hauptspannung auf das Fliessen der Metalle Eisen, Kupfer, und Nickel, Z. Physik, vol. 36, pp. 913–939, 1926)*

▶ Tresca Criterion

→ It is also called the maximum shear-stress criterion and applied to the elastic materials.

Yielding condition

Yielding occurs whenever the maximum shear stress reaches the value it has when yielding occurs in the tensile test.

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{Y}{2} \tag{5.25}$$

cf. The criterion (5.25) can be represented by a hexagonal cylinder inscribed within the right-circular cylinder of the Von Mises criterion.

Yielding condition in Plain stress

Refer to Fig. 5.29

Application of the Tresca Criterion (see Fig. 5.28, 5.29)

i) When only internal pressure (σ_2) increases, it corresponds to proceeding along the straight line from A toward B. (Fig. 5.29)

cf. Further increasing the inner pressure (σ_2), the axial load or σ_1 no more influence on yielding condition, and thus

$$\tau_{max} = 1/2(\sigma_z - \sigma_r) = 1/2(\sigma_1 - \sigma_3)$$

cf. In case of $\sigma_\theta = \sigma_z$ ($\therefore \sigma_1 = \sigma_2$), it corresponds to the point B.

ii) If the axial load (σ_1) decreases, it corresponds to proceeding along the straight line from B toward C.

cf. If axial load (σ_1) changes from tensile to compressive, it corresponds to proceeding along the straight line from C toward D.

→ This means that the internal pressure (σ_2) must be decreased in order to avoid the yielding.

In this case, as σ_{min}, σ_{max} are important, it's not possible to apply the Von Mises criterion to this situation directly.

→ Check the figures below

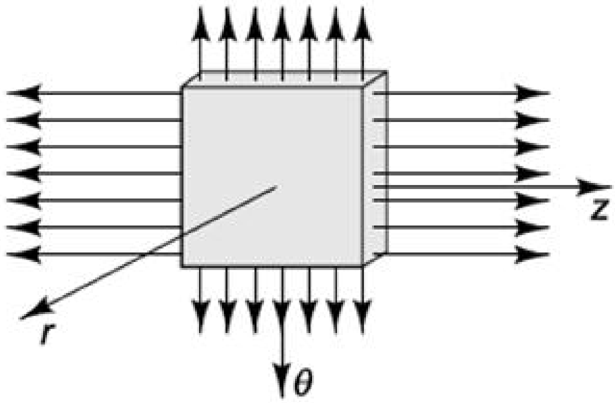


Fig. 5.31 *State of stress in cylinder wall of Fig. 5.28(a) when σ_z and σ_r determine the maximum shear stress*

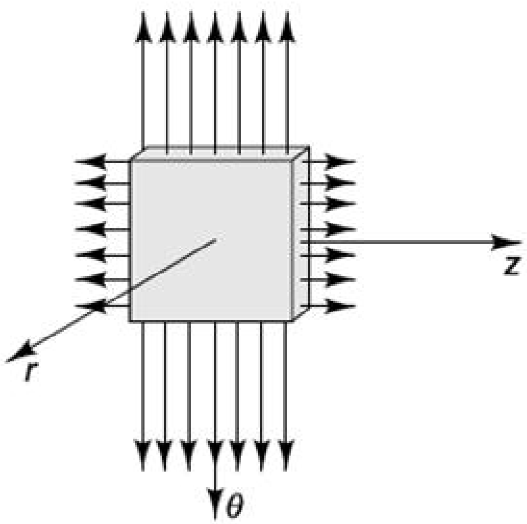


Fig. 5.32 *State of stress in cylinder wall of Fig. 5.28(a) when σ_θ and σ_r determine the maximum shear stress*

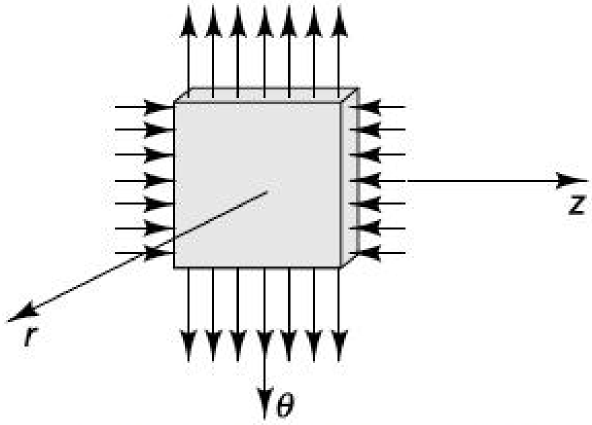


Fig. 5.33 *State of stress in cylinder wall of Fig. 5.28(a) when σ_θ and σ_z determine the maximum shear stress*

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▶ <u>Comparison of the criteria</u>

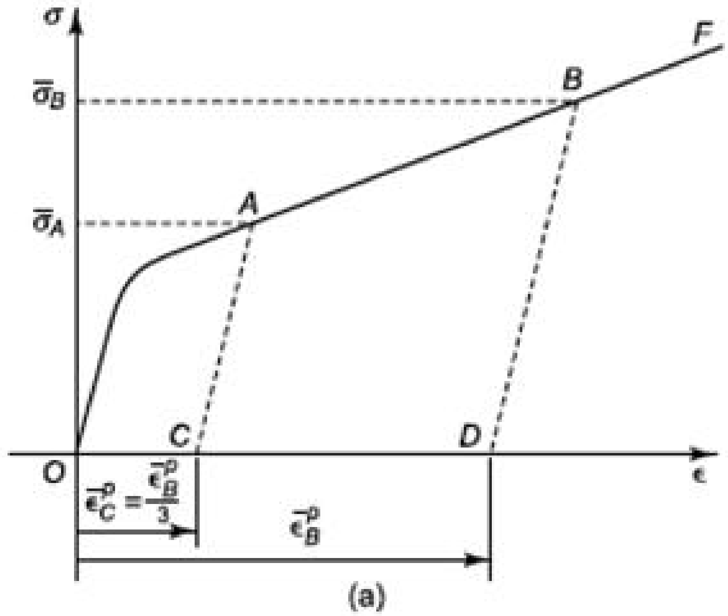
These criteria are identical in case of uniaxial stress.

Thus, one of the principal stress at arbitrary point is greater than the others, these criteria have identical values in the majority of case.

On the other hands, in case that the absolute value of principal stress is same, these criteria have distinguished difference.

5.12 Behavior Beyond Initial Yielding in the Tensile Test

→ The following description is an idealized description of the behavior of a real material during loading and unloading beyond initial yielding.



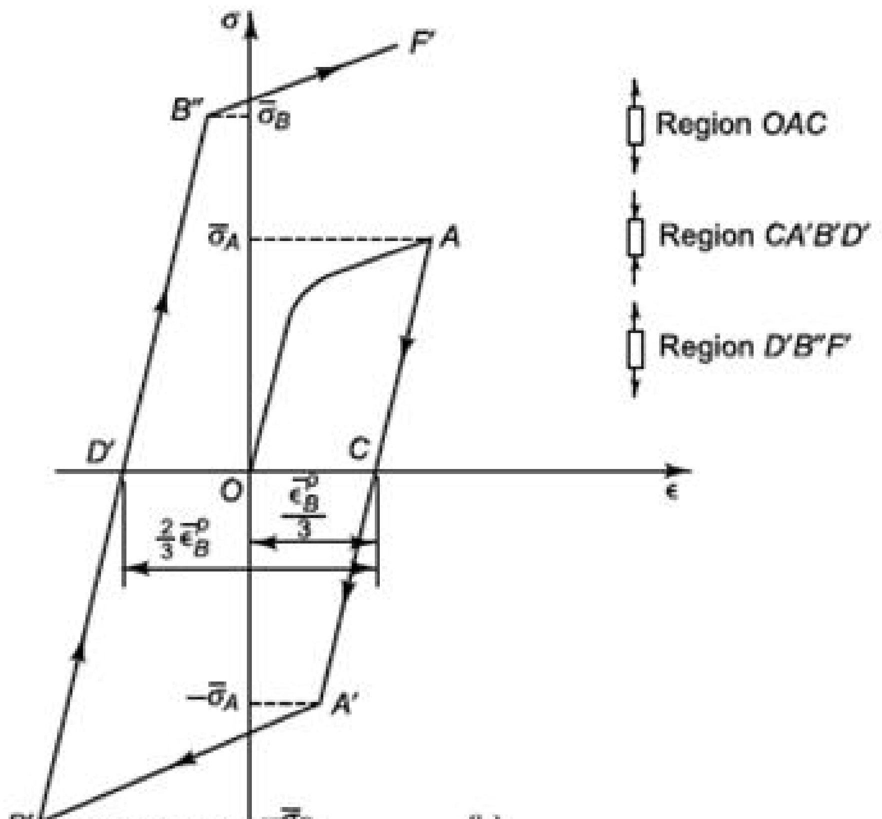


Fig. 5.34 *Example of simple loading path. (a) Stress-strain curve in uniaxial tensile test; (b) stress-strain behavior in alternate uniaxial tension and compression*

▶ <u>For Fig. 5.34 (b)</u>

i) A fresh specimen of the material is stretched in tension to point A, where the plastic extensional strain is $\frac{1}{3}\bar{\epsilon}_B^P$ and the stress is $\bar{\sigma}_A$.

ii) The load is released, bringing the specimen to point C, and then reapplied as compression.

iii) Further yielding begins when the stress $-\bar{\sigma}_A$ is reached at point A'.

iv) As the compressive load is increased, yielding continues along the curve $A'B'$, which has the same shape as the curve AB in Fig. 5.34 (a).

v) When the point B' is reached, a compressive plastic strain of $\frac{2}{3}\bar{\epsilon}_B^P$ has occurred between A' and B', and the stress required to cause further yielding has reached the value $-\bar{\sigma}_B$.

vi) If the load is now released, the material returns to D'.

vii) A reapplication of the tensile load will cause the material to move along the curve $D'B'F'$, which is identical with the curve DBF in Fig. 5.34 (a).

cf. All the plastic-strain increments along the loading path have contributed in a positive manner to the strain-hardening so that the material in state D' has been strain-hardened the same amount as the material in state D in Fig. 5.34 (a).

▶ <u>Example 5.3</u> Returning to Example 5.1, we ask, what will happen if we remove the load P after we have strained the

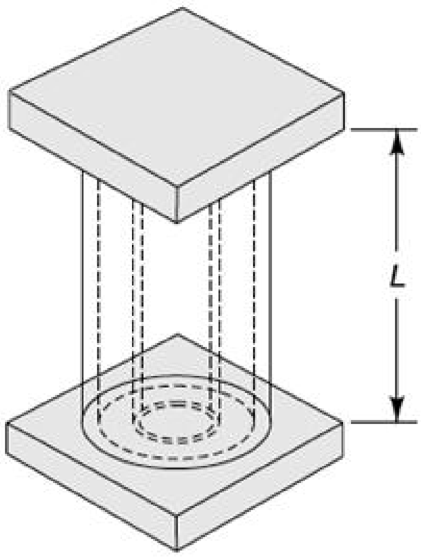


Fig. 5.8 Example 5.1

combined assembly so that both the steel and the aluminum are in the plastic range, that is, beyond a strain of 0.005?

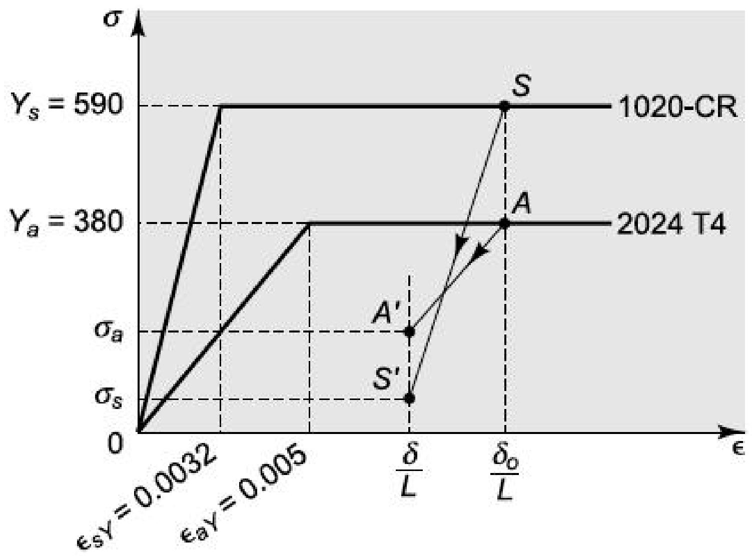


Fig. 5.35 *Example 5.3. Stress-strain behavior of the assembly of Example 5.1 when the load P is decreased after both the steel and aluminum alloy have been strained plastically*

We can again use the model of Fig. 5.9, and the equilibrium relation (e) and geometric compatibility relation (a) still remain valid. We need new stress-strain relations which will be valid during unloading.

▷ Stress-strain relation

δ_0: deflection when the assembly is loaded by P

δ: deflection after the load has been decreased somewhat

Then,

$$\begin{cases} \sigma_s = Y_s - E_s \dfrac{(\delta_0 - \delta)}{L} \\ \sigma_a = Y_a - E_a \dfrac{(\delta_0 - \delta)}{L} \end{cases} \qquad \text{(f)}$$

Substituting (f) into Eq. (e) of Example 5.1 and setting $P = 0$, we obtain

$$\sum F_y = \sigma_s A_s + \sigma_a A_a - P = 0 \tag{e}$$

$$A_s\left(Y_s - E_s \frac{\delta_0-\delta}{L}\right) + A_a\left(Y_a - E_a \frac{\delta_0-\delta}{L}\right) = 0 \tag{g}$$

$$\therefore \frac{\delta_0-\delta}{L} = \frac{A_s Y_s + A_a Y_a}{(A_s E_s + A_a E_a)} \tag{h}$$

Substituting (h) into (f), we find the residual stresses which remain in the assembly after the load has been removed

$$\begin{aligned}(\sigma_s)_{RESIDUAL} &= Y_s \frac{1-\frac{Y_a/E_a}{Y_s/E_s}}{1+\frac{E_s A_s}{E_a A_a}} = Y_s \frac{1-\frac{\epsilon_{aY}}{\epsilon_{sY}}}{1+\frac{E_s A_s}{E_a A_a}} \\ (\sigma_a)_{RESIDUAL} &= Y_a \frac{1-\frac{Y_s/E_s}{Y_a/E_a}}{1+\frac{E_a A_a}{E_s A_s}} = Y_s \frac{1-\frac{\epsilon_{sY}}{\epsilon_{aY}}}{1+\frac{E_a A_a}{E_s A_s}}\end{aligned} \tag{i}$$

→ Since in the present case $\epsilon_{aY} > \epsilon_{sY}$, the Eq. (i) show that the steel will be in compression and the aluminum in tension.

cf. The residual stresses will be zero only when the initial yield strains $\epsilon_{sY} = Y_s/E_s$ and $\epsilon_{aY} = Y_a/E_a$ are equal.

▶ Engineering stress-strain

1 ▷ Engineering stress

$$\sigma_E = \frac{load}{original(before\ loading)\ area}$$

→ The maximum value of the engineering stress is termed the tensile strength.

2 ▷ Engineering strain

$$\varepsilon_E = \Delta L/L_0 = (L_f - L_0)/L_0 \tag{5.26}$$

where L_0: original length between two dots of specimen,

L_f: length between two dots of specimen after loading.

▶ True stress-strain

1 ▷ True stress

$$\sigma_T = \frac{load}{actual(under\ loading)\ area}$$

→ Even when the axial strain has reached the relatively large (for engineering purposes) value of 0.05, the true stress is only about 5 percent greater than the engineering stress.

2 ▷ True strain

The strain, obtained by adding up the increments of strain which are based on the current dimensions, is called a true strain. Sometimes it is called logarithmic strain or natural strain.

$$\epsilon_T = \int_{L_0}^{L_f} (1/L)\, dL = ln\left(L_f/L_0\right) \tag{5.27}$$

→ For very small strain, assume that $A_0 L_0 = A_f L_f$.

$$\epsilon_T = ln\frac{A_o}{A_f} = 2\,ln\frac{D_0}{D_f} \tag{5.28}$$

▷ ▷ Confer

1) Most of the dislocation processes are more conveniently described by an incremental concept of strain.

2) When a ductile metal is tested both in tension and in compression, the true-stress and true strain curves practically coincide, whereas the two curves are quite different when engineering strain is used.

∴ When we decide which definition of strain to use in describing the behavior beyond initial yielding in the tensile test, the balance is in favor of using true strain.

▶ Necking

→ It is difficult to decide the time when necking starts.

cf. Details about necking will be discussed in Ch. 9-7.

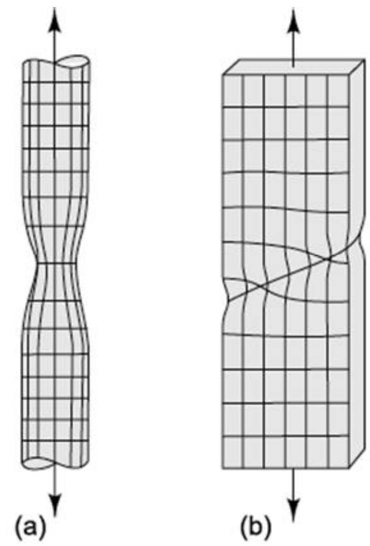


Fig. 5.37 *Examples of necking*

▶ Reduction of area (R.A.)

$$R.A. = (A_0 - A_f)/A_0 = 1 - A_f/A_0 = 1 - e^{-\epsilon_f}$$

→ The ductility of a material can be described by the reduction of area (R.A.).

▶ Elongation

$$Elongation = \Delta L/L_0 = (L_f - L_0)/L_0$$

→ Elongation is defined as the change in gage length to final fracture divided by the original gage length (i.e., the engineering strain at fracture).

→ As a measure of ductility of the material, the elongation has the disadvantage that it is an engineering, rather than a true, strain.

→ It is very dependent on the length as well as on the cross-sectional dimensions of the specimen.