Mechanics and Design

Chapter 5. FEM - Truss

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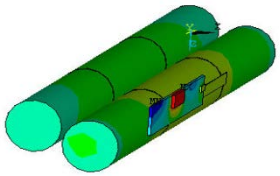
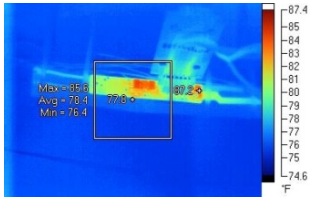
CONTENTS

Introduction

- The finite element method (FEM) is a technique for analyzing the behavior of engineered structures subjected to a variety of loads. FEM is the most widely applied computer simulation method in engineering.
- The basic idea is to divide a complicated structure into small and manageable pieces (*discretization*) and solve the algebraic equation.

Applications of FEM in Engineering				
Simulation				
Experiment				
Engineered system Predicted measure	Vehicle Safety	Tire Cornering force	Cellular phone Reliability	Li-ion battery system Thermal behavior

Introduction

Available Commercial FEM Software

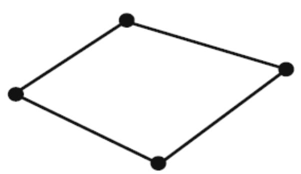
- Midas (General purpose)
- ANSYS (General purpose)
- I-DEAS (Complete CAD/CAM/CAE package)
- HyperMesh (Pre/Post Processor)
- ABAQUS (Nonlinear and dynamic analyses)
- PATRAN (Pre/Post Processor)
- NASTRAN (General purpose)
- COSNOS (General purpose)
- Dyna-3D (Crash/impact analysis)

Types of Finite Elements

1-D (Line) Element

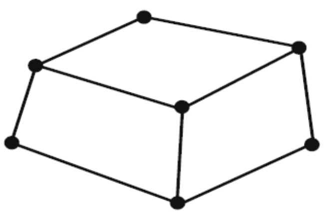
(Spring, truss, beam, pipe, etc.)

2-D (Plane) Element



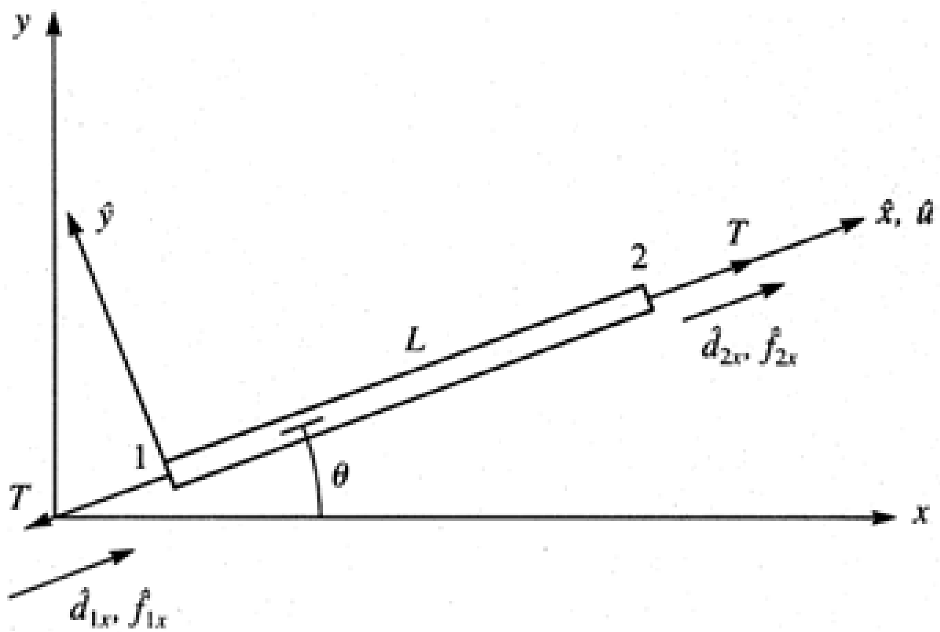
(Membrane, plate, shell, etc.)

3-D (Solid) Element



(3-D fields – temperature, displacement, stress, flow velocity, etc.)

Truss Element in 1D Space



x, y : global coordination system

$\hat{x}, \hat{y}$: local coordination system

$\hat{d}$: displacement

$\hat{f}$: force

$\hat{u}$: deformation

L : length

Truss Element in 1D Space

Linear Static Analysis

Most structural analysis problems can be treated as linear static problems, based on the following assumptions.

1. Small deformations (loading pattern is not changed due to the deformed shape)
2. Static loads (the load is applied to the structure in slow or steady fashion)
3. Elastic materials (no plasticity or failures)

Hooke’s Law and Deformation Equations

$$\sigma = E\varepsilon \qquad \varepsilon = \frac{d\hat{u}}{d\hat{x}}$$

Equilibriums

$$A\sigma_x = T = \text{Constant} \qquad \frac{d}{d\hat{x}}\left(AE\frac{d\hat{u}}{d\hat{x}}\right) = 0$$

Assumptions

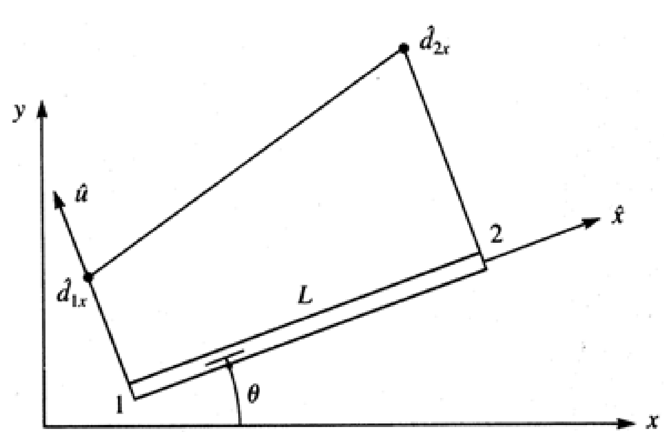
1. Truss cannot support shear force. $\hat{f}_{1y} = 0 \quad \hat{f}_{2y} = 0$
2. Ignore the effect of lateral deformation

Truss Element in 1D Space

Direct Stiffness Method

DSM is an approach to calculate a stiffness matrix for a system by directly superposing the stiffness matrixes of all elements. DSM is beneficial to get the stiffness matrix of relatively simple structures consisting of several trusses or beams.

Step 1: Determination of element type



Step 2: Determination of displacement function

Assume a linear deformation function as

$$\hat{u} = a_1 + a_2\hat{x} = \left(\frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}\right)\hat{x} + \hat{d}_{1x}$$

In vector form

$$\hat{u} = [N_1 \;\; N_2]\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

$$N_1 = 1 - \frac{\hat{x}}{L}, \qquad N_2 = \frac{\hat{x}}{L}$$

N_1, N_2: shape function

Truss Element in 1D Space

Step 3: Strain and stress calculation

Deformation rate - strain

$$\varepsilon_x = \frac{d\hat{u}}{d\hat{x}} = \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}$$

Stress - strain

$$\sigma_x = E\varepsilon_x$$

Step 4: Derivation of element stiffness matrix

$$T = A\sigma_x = AE\left(\frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}\right)$$

$$\hat{f}_{1x} = -T = \frac{AE}{L}(\hat{d}_{1x} - \hat{d}_{2x})$$

$$\hat{f}_{2x} = T = \frac{AE}{L}(\hat{d}_{2x} - \hat{d}_{1x})$$

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

$$\underline{\hat{k}} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

NOTE: In a truss element, stiffness (spring constant, *k*) is equivalent to *AE/L*

Truss Element in 1D Space

Step 5: Constitution of global stiffness matrix

Construct a global stiffness matrix in a global coordinate system

$$\underline{K} = [K] = \sum_{e=1}^{N} \underline{k}^{(e)} \qquad \underline{F} = \{F\} = \sum_{e=1}^{N} \underline{f}^{(e)}$$

Step 6: Calculation of nodal displacement

- Use boundary conditions
- Solve the system of linear algebraic equations $\underline{F} = \underline{K}\,\underline{d}$ to calculate the nodal deformation in a truss structure

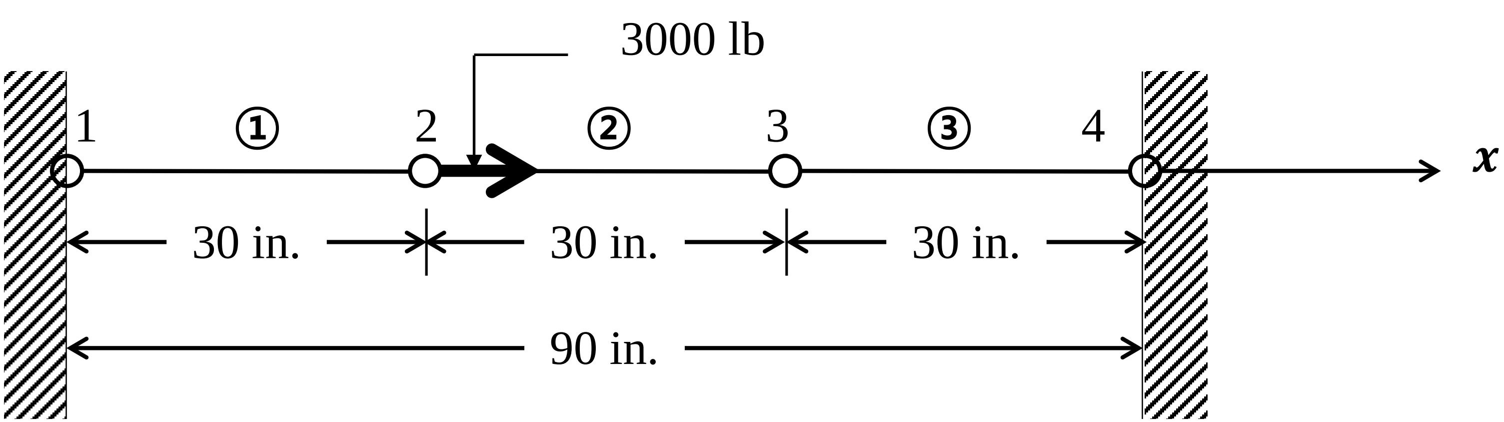
Step 7: Calculation of stress and strain in an element

Compute a strain and stress at any point within an element

Example (Truss Element in 1D Space)

A structure consists of three beams (see below figure). Find (a) the global stiffness matrix in a global coordinate system, (b) the displacements at the node 2 and 3, (c) the reaction forces at the node 2 and 3. 3000lb loading acts in the positive x-direction at node 2. The length of the elements is 30in.

- Young's modulus for the elements 1 and 2: $E = 30\times10^6$ psi
- Cross-sectional area for the elements 1 and 2: $A = 1\text{in}^2$
- Young's modulus for the elements 3: $E = 15\times10^6$ psi
- Cross-sectional area for the elements 3 : $A = 2\text{in}^2$



Example (Truss Element in 1D Space)

Step 4: Derivation of element stiffness matrix

$$\hat{\underline{k}} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

element #

node #

$$\underline{k}^{(1)} = \underline{k}^{(2)} = \frac{(1)(30\times10^6)}{30}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^6 \begin{array}{c} \begin{array}{cc} 1 & 2 \\ 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$\underline{k}^{(3)} = \frac{(2)(15\times10^6)}{30}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^6 \begin{array}{c} \begin{array}{cc} 3 & 4 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

Step 5: Constitution of global stiffness matrix

node #

$$\underline{K} = 10^6 \begin{array}{c} \begin{array}{cccc} d_{1x} & d_{2x} & d_{3x} & d_{4x} \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1+1 & -1 & 0 \\ 0 & -1 & 1+1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

Answer of (a)

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = 10^6 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \\ d_{4x} \end{Bmatrix}$$

Example (Truss Element in 1D Space)

Step 6: Calculation of nodal displacement

Using the boundary conditions $d_{1x} = d_{4x} = 0$

$$\begin{Bmatrix} 3000 \\ 0 \end{Bmatrix} = 10^6 \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix}$$

So, the displacements for this system are

$$d_{2x} = 0.002in. \quad d_{3x} = 0.001in.$$

Answer of (b)

Substituting the displacements to the equation, the load can be obtained as follows.

Answer of (c)

$$F_{1x} = 10^6(d_{1x} - d_{2x}) = 10^6(0 - 0.002) = -2000lb$$

$$F_{2x} = 10^6(-d_{1x} + 2d_{2x} - d_{3x}) = 10^6[0 + 2(0.002) - 0.001] = 3000lb$$

$$F_{3x} = 10^6(-d_{2x} + 2d_{3x} - d_{4x}) = 10^6[-0.002 + 2(0.001) - 0] = 0$$

$$F_{4x} = 10^6(-d_{3x} + d_{4x}) = 10^6(-0.001 + 0) = -1000lb$$

Step 7: Calculation of stress and strain in an element

$$\sigma_x = \frac{F}{A}$$

Truss Element in 1D Space

Selection of Deformation Function

1. The deformation function needs to be continuous within an element.
2. The deformation function needs to provide continuity among elements.

1-D linear deformation function satisfies (1) and (2). → *compatibility*

3. The deformation function needs to express the displacement of a rigid body and the constant strain in element.

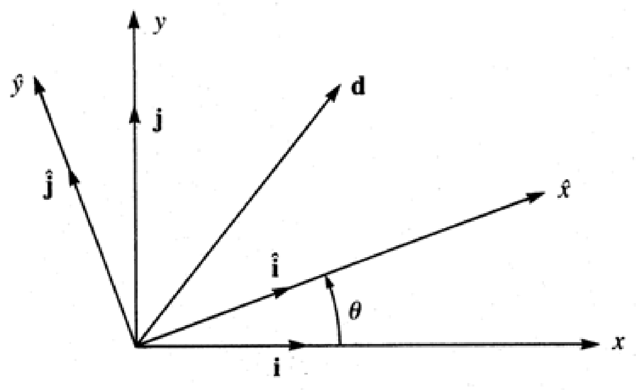
$\hat{u} = a_1 + a_2\hat{x}$: 1-D linear deformation function → *completeness*

a_1 considers the motion of a rigid body.

$a_2\hat{x}$ considers the constant strain ($\varepsilon_x = d\hat{u}/d\hat{x} = a_2$)

Truss Element in 2D Space

Vector transformation in 2-D



Displacement vector **d** in global coordinate and local coordinate systems.

$$\mathbf{d} = d_x\mathbf{i} + d_y\mathbf{j} = \hat{d}_x\hat{\mathbf{i}} + \hat{d}_y\hat{\mathbf{j}}$$

Global *Local*

Relation between two coordination systems

$$\begin{Bmatrix} \hat{d}_x \\ \hat{d}_y \end{Bmatrix} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{Bmatrix} d_x \\ d_y \end{Bmatrix} \quad C = \cos\theta \quad S = \sin\theta$$

Eq. (1)

$\begin{bmatrix} C & S \\ -S & C \end{bmatrix}$: Transformation Matrix

Truss Element in 2D Space

Stiffness matrix in global coordinate system

The stiffness matrix of truss element in local coordinate system

$$\begin{Bmatrix} \hat{f}_1 \\ \hat{f}_2 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix} \qquad \underline{\hat{f}} = \underline{\hat{k}}\underline{\hat{d}}$$

: Step 4 on page 6

The stiffness matrix of truss element in local coordinate system

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \underline{k}\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix} \qquad \underline{f} = \underline{k}\underline{d}$$

: What we want to construct in 2-D space

Let's calculate the relation between $\underline{\hat{k}}$ and $\underline{k}$.

Truss Element in 2D Space

From *Eq.(1)*,

$$\hat{d}_{1x} = d_{1x}\cos\theta + d_{1y}\sin\theta$$
$$\hat{d}_{2x} = d_{2x}\cos\theta + d_{2y}\sin\theta$$

$$\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}$$

Simply,

$$\boxed{\underline{\hat{d}} = \underline{T}^* \underline{d}}$$ ***Eq. (2)***

$$\underline{T}^* = \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix}$$

Transform the loading in a same way,

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix} \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix}$$

Simply,

$$\boxed{\underline{\hat{f}} = \underline{T}^* \underline{f}}$$ ***Eq. (3)***

Truss Element in 2D Space

From *Eq.(2)* and *(3)*

$$\hat{\underline{f}} = \hat{\underline{k}}\hat{\underline{d}}$$

$$\underline{T}^*\underline{f} = \hat{\underline{k}}\underline{T}^*\underline{d}$$

where, $\underline{T}^* = \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix}$

We needs the inverse matrix of T*; however, because T* is not the square matrix, it cannot be immediately transformed.

Invite $(\hat{f}_{1y}, \hat{d}_{1y})$ and $(\hat{f}_{2y}, \hat{d}_{2y})$

$$\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{d}_{2x} \\ \hat{d}_{2y} \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}$$

$$\hat{\underline{d}} = \underline{T}\underline{d}$$

Transformation Matrix

In a same way,

$$\hat{\underline{f}} = \underline{T}\underline{f}$$

Truss Element in 2D Space

Expanded 4×4 matrix $\underline{\hat{k}}$

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{f}_{2x} \\ \hat{f}_{2y} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{d}_{2x} \\ \hat{d}_{2y} \end{Bmatrix}$$

The stiffness matrix of truss elements in global coordinate system

$$\underline{T}\underline{f} = \underline{\hat{k}}\underline{T}\underline{d} \qquad \underline{f} = \underline{T}^{-1}\underline{\hat{k}}\underline{T}\underline{d} = \underline{T}^{T}\underline{\hat{k}}\underline{T}\underline{d}$$

T is an orthogonal matrix ($T^T = T^{-1}$).

$$\underline{k} = \underline{T}^{T}\underline{\hat{k}}\underline{T}$$

Truss Element in 2D Space

So,

$$\underline{k} = \frac{AE}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ & S^2 & -CS & -S^2 \\ & & C^2 & CS \\ Symmetry & & & S^2 \end{bmatrix} \quad \textbf{\textit{Eq. (4)}}$$

$$\underline{k} = \underline{T}^T \underline{\hat{k}} \underline{T}$$

$$\underline{T} = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix}$$

$$\underline{\hat{k}} = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Assemble the stiffness matrix for the whole system

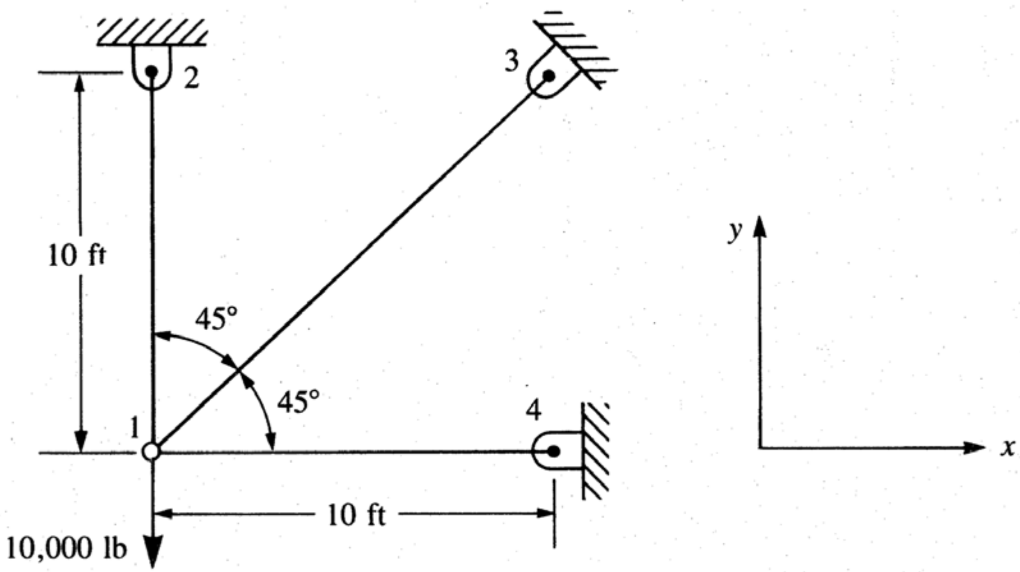
$$\sum_{e=1}^{N} \underline{k}^{(e)} = \underline{K} \qquad \sum_{e=1}^{N} \underline{f}^{(e)} = \underline{F}$$

The relation between the nodal loading and displacement in global coordinate system

$$\underline{F} = \underline{K}\underline{d}$$

Example (Truss Element in 2D Space)

Three truss elements are assembled and 10,000lb loading is applied at node 1 as shown in figure. Find the displacements at node 1. For all elements, Young's modulus: $E = 30 \times 10^6 psi$, cross-sectional area: $A = 2in.^2$



Example (Truss Element in 2D Space)

Data for stiffness matrix calculation

Node	Element	$\theta°$	C	S	C^2	S^2	CS
1 & 2	1	90°	0	1	0	1	0
1 & 3	2	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1 & 4	3	0°	1	0	1	0	0

Step 4: Derivation of element stiffness matrix (from *Eq. (4)*)

$$\underline{k}^{(1)} = \frac{(30\times 10^6)(2)}{120} \begin{array}{c} \begin{array}{cccc} d_{1x} & d_{1y} & d_{2x} & d_{2y} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

$$\underline{k}^{(2)} = \frac{(30\times 10^6)(2)}{120\times\sqrt{2}} \begin{array}{c} \begin{array}{cccc} d_{1x} & d_{1y} & d_{3x} & d_{3y} \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \end{array}$$

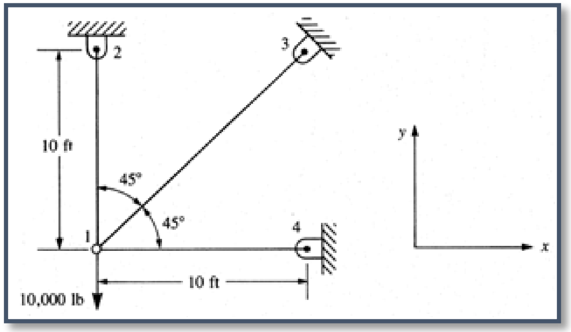
$$\underline{k}^{(3)} = \frac{(30\times 10^6)(2)}{120} \begin{array}{c} \begin{array}{cccc} d_{1x} & d_{1y} & d_{4x} & d_{4y} \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Example (Truss Element in 2D Space)

Step 5: Constitution of global stiffness matrix

$$\underline{K} = (500{,}000)\begin{bmatrix} 1.354 & 0.354 & 0 & 0 & -0.354 & -0.354 & -1 & 0 \\ 0.354 & 1.354 & 0 & -1 & -0.354 & -0.354 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example (Truss Element in 2D Space)



Step 6: Calculation of nodal displacement

Using boundary condition at node 2,3, and 4.

$$\begin{Bmatrix} 0 \\ -10{,}000 \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{Bmatrix} = (500{,}000)\begin{bmatrix} 1.354 & 0.354 & 0 & 0 & -0.354 & -0.354 & -1 & 0 \\ 0.354 & 1.354 & 0 & -1 & -0.354 & -0.354 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} = 0 \\ d_{2y} = 0 \\ d_{3x} = 0 \\ d_{3y} = 0 \\ d_{4x} = 0 \\ d_{4y} = 0 \end{Bmatrix}$$

Calculate unknown displacement

$$\begin{Bmatrix} 0 \\ -10{,}000 \end{Bmatrix} = (500{,}000)\begin{bmatrix} 1.354 & 0.354 \\ 0.354 & 1.354 \end{bmatrix}\begin{Bmatrix} d_{1x} \\ d_{1y} \end{Bmatrix}$$

$$d_{1x} = 0.414 \times 10^{-2} \ in. \qquad d_{1y} = -1.59 \times 10^{-2} \ in.$$

Example (Truss Element in 2D Space)

Step 7: Calculation of stress and strain in an element ($\sigma = E\varepsilon = \dfrac{F}{A}$)

$$\sigma^{(1)} = \frac{30\times 10^6}{120}\begin{bmatrix}0 & -1 & 0 & 1\end{bmatrix}\begin{Bmatrix} d_{1x} = 0.414\times 10^{-2} \\ d_{1y} = -1.59\times 10^{-2} \\ d_{2x} = 0 \\ d_{2y} = 0 \end{Bmatrix} = 3965psi \qquad \sigma^{(1)} = \frac{F_{2y}}{A}$$

$$\sigma^{(2)} = \frac{30\times 10^6}{120\sqrt{2}}\begin{bmatrix}\frac{-\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}\begin{Bmatrix} d_{1x} = 0.414\times 10^{-2} \\ d_{1y} = -1.59\times 10^{-2} \\ d_{3x} = 0 \\ d_{3y} = 0 \end{Bmatrix} = 1471psi \qquad \sigma^{(2)} = \frac{\sqrt{2}F_{3x}}{A}$$

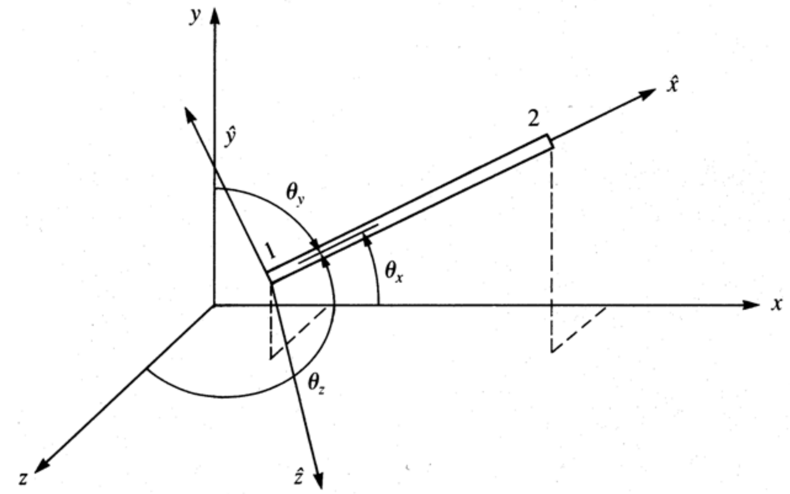
$$\sigma^{(3)} = \frac{30\times 10^6}{120}\begin{bmatrix}-1 & 0 & 1 & 0\end{bmatrix}\begin{Bmatrix} d_{1x} = 0.414\times 10^{-2} \\ d_{1y} = -1.59\times 10^{-2} \\ d_{4x} = 0 \\ d_{4y} = 0 \end{Bmatrix} = -1035psi \qquad \sigma^{(3)} = \frac{F_{4x}}{A}$$

Step 8: Confirmation the loading equilibrium at node 1

$$\sum F_x = 0 \qquad (1471psi)(2in.^2)\frac{\sqrt{2}}{2} - (1035psi)(2in.^2) = 0$$

$$\sum F_y = 0 \qquad (3695psi)(2in.^2) + (1471psi)(2in.^2)\frac{\sqrt{2}}{2} - 10{,}000 = 0$$

Truss Element in 3D Space



A truss in 3-D space

NOTE: $\hat{d}_x\hat{\mathrm{i}} + \hat{d}_y\hat{\mathrm{j}} + \hat{d}_z\hat{\mathrm{k}} = d_x\mathrm{i} + d_y\mathrm{j} + d_z\mathrm{k}$

$$\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1x} \end{Bmatrix} = \begin{bmatrix} C_x & C_y & C_z & 0 & 0 & 0 \\ 0 & 0 & 0 & C_x & C_y & C_z \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{1z} \\ d_{2x} \\ d_{2y} \\ d_{2z} \end{Bmatrix}$$

$$\underline{T}^* = \begin{bmatrix} C_x & C_y & C_z & 0 & 0 & 0 \\ 0 & 0 & 0 & C_x & C_y & C_z \end{bmatrix}$$

We can get k using $\underline{T}^*$ instead of $\underline{T}$ ($\underline{k} = T^T\underline{\hat{k}T}$)

Truss Element in 3D Space

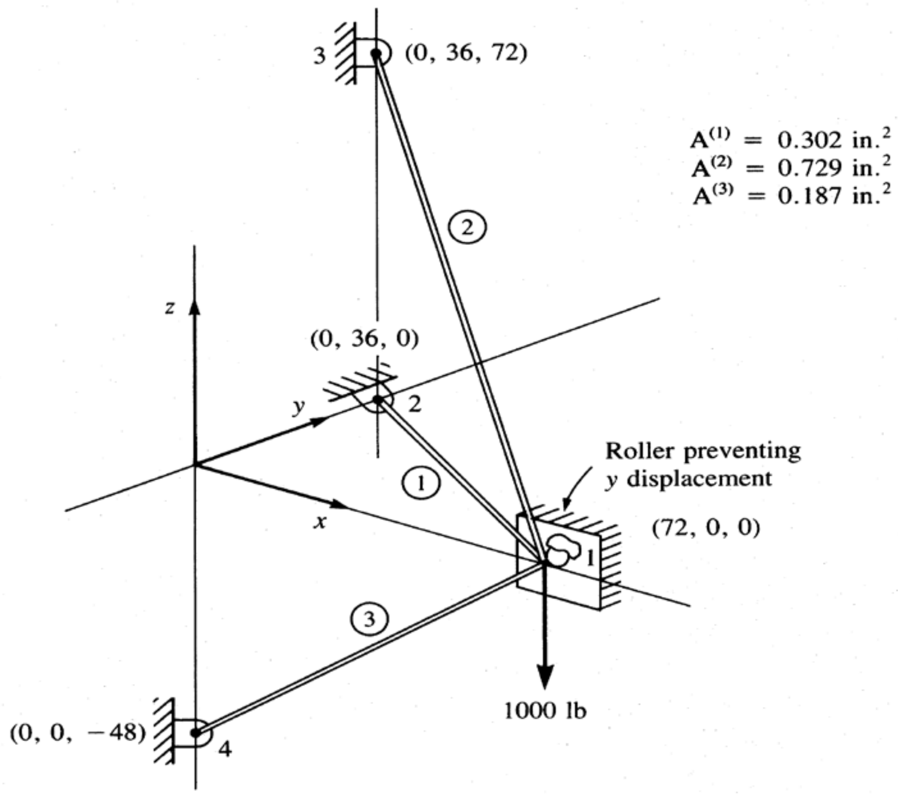
$\underline{k}$ is calculated by using $\underline{k} = (T^*)^T \underline{\hat{k}} T^*$ as below.

$$\underline{k} = \begin{bmatrix} C_x & 0 \\ C_y & 0 \\ C_z & 0 \\ 0 & C_x \\ 0 & C_y \\ 0 & C_z \end{bmatrix} \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} C_x & C_y & C_z & 0 & 0 & 0 \\ 0 & 0 & 0 & C_x & C_y & C_z \end{bmatrix}$$

$$\underline{k} = \frac{AE}{L} = \begin{bmatrix} C_x^2 & C_xC_y & C_xC_z & -C_x^2 & -C_xC_y & -C_xC_z \\ & C_y^2 & C_yC_z & -C_xC_y & -C_y^2 & -C_yC_z \\ & & C_z^2 & -C_xC_z & -C_yC_z & -C_z^2 \\ & & & C_x^2 & C_xC_y & C_xC_z \\ & & & & C_y^2 & C_yC_z \\ symmetry & & & & & C_z^2 \end{bmatrix}$$

Truss Element in 3D Space

Solve the 3-D truss problem as shown in below figure. The elastic moduli for all elements are $E = 1.2 \times 10^6 psi$.1000lb loading is applied in the negative z-direction at node 1. Node 2, 3, and 4 are supported on sockets with balls (*x*, *y*, and *z* direction movements are constrained). The movement along the y direction is constrained by a roller at Node 1.



Truss Element in 3D Space

Step 4: Derivation of element stiffness matrix

Stiffness matrix of the elements $\underline{k}$:

$$\underline{k} = \frac{AE}{L}\begin{bmatrix} \underline{\lambda} & -\underline{\lambda} \\ -\underline{\lambda} & \underline{\lambda} \end{bmatrix}$$

where the submatrix $\underline{\lambda}$:

$$\underline{\lambda} = \begin{bmatrix} C_x^2 & C_xC_y & C_xC_z \\ C_yC_x & C_y^2 & C_yC_z \\ C_zC_x & C_zC_y & C_z^2 \end{bmatrix}$$

So, $\underline{k}$ is defined as $\underline{\lambda}$ is defined.

(For element 3)

Directional cosine:

$$C_x = \frac{x_4 - x_1}{L^{(3)}} \qquad C_y = \frac{y_4 - y_1}{L^{(3)}} \qquad C_z = \frac{z_4 - z_1}{L^{(3)}}$$

Where length of the element:

$$L^{(3)} = [(-72.0)^2 + (-48.0)^2]^{1/2} = 86.5in.$$

$$\boxed{\begin{aligned} C_x &= \frac{-72.0}{86.5} = -0.833 \\ C_y &= 0 \\ C_z &= \frac{-48.5}{86.5} = -0.550 \end{aligned}}$$

Truss Element in 3D Space

Submatrix $\underline{\lambda}$

$$\underline{\lambda} = \begin{bmatrix} 0.69 & 0 & 0.46 \\ 0 & 0 & 0 \\ 0.46 & 0 & 0.30 \end{bmatrix}$$

Stiffness matrix of the element $\underline{k}$

$$\underline{k}^{(3)} = \frac{(0.187)(1.2 \times 10^6)}{86.5} \begin{array}{c} d_{1x} d_{1y} d_{1z} d_{4x} d_{4y} d_{4z} \\ \begin{bmatrix} \underline{\lambda} & -\underline{\lambda} \\ -\underline{\lambda} & \underline{\lambda} \end{bmatrix} \end{array}$$

(For element 1)

$$L^{(1)} = 80.5in. \qquad C_x = -0.89 \qquad C_y = 0.45 \qquad C_z = 0$$

$$\underline{\lambda} = \begin{bmatrix} 0.79 & -0.40 & 0 \\ -0.40 & 0.20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \underline{k}^{(1)} = \frac{(0.320)(1.2 \times 10^6)}{80.5} \begin{array}{c} d_{1x} d_{1y} d_{1z} d_{2x} d_{2y} d_{2z} \\ \begin{bmatrix} \underline{\lambda} & -\underline{\lambda} \\ -\underline{\lambda} & \underline{\lambda} \end{bmatrix} \end{array}$$

Truss Element in 3D Space

(For element 2)

$$L^{(2)} = 108in. \qquad C_x = -0.667 \qquad C_y = 0.33 \qquad C_z = 0.667$$

$$\underline{\lambda} = \begin{bmatrix} 0.45 & -0.22 & -0.45 \\ -0.22 & 0.11 & 0.45 \\ -0.45 & 0.45 & 0.45 \end{bmatrix} \qquad \underline{k}^{(2)} = \frac{(0.729)(1.2\times 10^6)}{108} \begin{array}{c} d_{1x} d_{1y} d_{1z} d_{3x} d_{3y} d_{3z} \\ \begin{bmatrix} \underline{\lambda} & -\underline{\lambda} \\ -\underline{\lambda} & \underline{\lambda} \end{bmatrix} \end{array}$$

Step 5, 6, 7

Using the boundary conditions

$$d_{1y} = 0, d_{2x} = d_{2y} = d_{2z} = 0, d_{3x} = d_{3y} = d_{3z} = 0, d_{4x} = d_{4y} = d_{4z} = 0$$

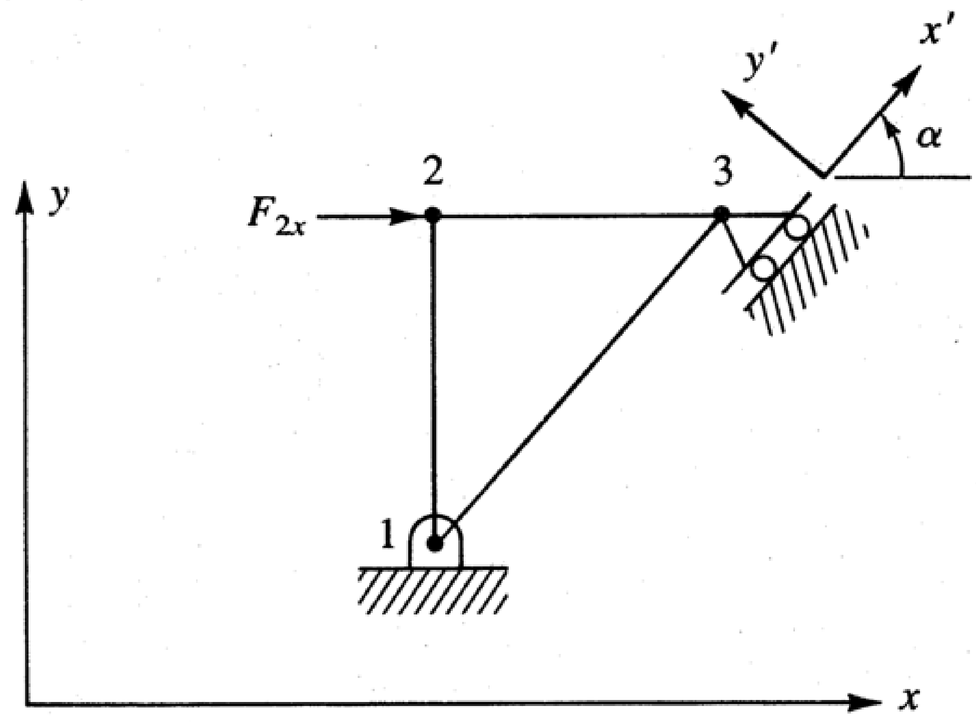
$$\underline{K} = \begin{array}{c} \begin{array}{cc} d_{1x} & d_{1z} \end{array} \\ \begin{bmatrix} 9000 & -2450 \\ -2450 & 4450 \end{bmatrix} \end{array}$$ Stiffness matrix

$$\begin{Bmatrix} 0 \\ -1000 \end{Bmatrix} = \begin{bmatrix} 9000 & -2450 \\ -2450 & 4550 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1z} \end{Bmatrix}$$ Stiffness equation in global coordinate system

$$d_{1x} = -0.072in.$$
$$d_{1z} = -0.264in.$$

Calculated displacement (The negative signs mean that the displacements are in the negative x and z direction)

Inclined or Skewed and Support



To be convenient, apply the boundary condition d'_{3y} to the local coordinate $x' - y'$

System Health & Risk Management

Inclined or Skewed and Support

Displacement vector transformation at node 3:

$$\begin{Bmatrix} d'_{3x} \\ d'_{3y} \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} d_{3x} \\ d_{3y} \end{Bmatrix}$$

or

$$\{d'_3\} = [t_3]\{d_3\}$$

Displacement vector transformation at all nodes:

$$\{d'\} = [T_1]\{d\}$$
$$\{d\} = [T_1]^T\{d'\}$$

or

$$\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \end{Bmatrix} = [T_1]^T\{d'\} = \begin{bmatrix} [I] & [0] & [0] \\ [0] & [I] & [0] \\ [0] & [0] & [t_3]^T \end{bmatrix} \begin{Bmatrix} d'_{1x} \\ d'_{1y} \\ d'_{2x} \\ d'_{2y} \\ d'_{3x} \\ d'_{3y} \end{Bmatrix}$$

Inclined or Skewed and Support

Loading vector transformation :

In local coordinate system $\{f'_3\} = [t_3]\{f_3\}$

In global coordinate system

$$\{f'\} = [T_1]\{f\} \qquad \text{or} \qquad \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f'_{3x} \\ f'_{3y} \end{Bmatrix} = \begin{bmatrix} [I] & [0] & [0] \\ [0] & [I] & [0] \\ [0] & [0] & [t_3] \end{bmatrix} \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{Bmatrix}$$

Note: $f_{1x} = fn_{1x}, f_{1y} = fn_{1y}, f_{2x} = fn_{2x}, f_{2y} = fn_{2y}$

$d_{1x} = d'_{1x}, d_{1y} = d'_{1y}, d_{2x} = d'_{2x}, d_{2y} = d'_{2y}$

Inclined or Skewed and Support

Transformation of the finite element formulation in global coordinates :

$$\{f\} = [K]\{d\}$$

$$\rightarrow \quad [T_1]\{f\} = [T_1][K]\{d\}$$

$$\rightarrow \quad [T_1]\{f\} = [T_1][K][T_1]^T\{d'\}$$

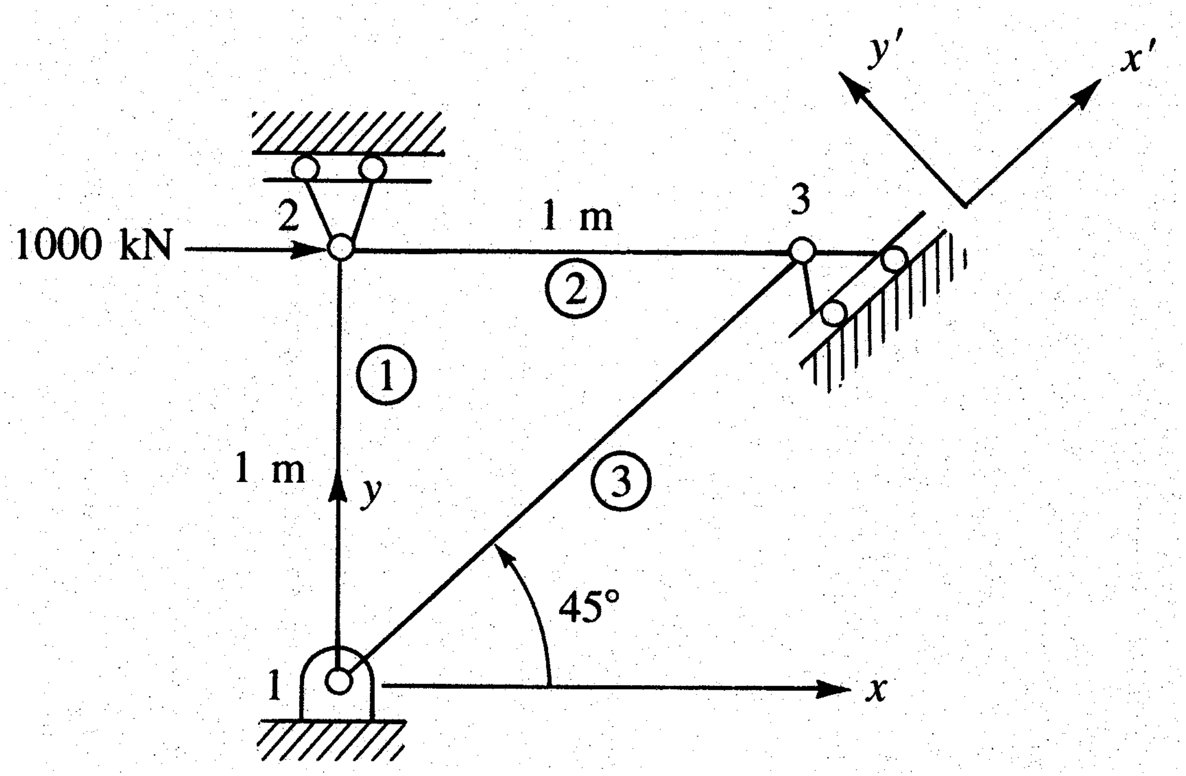
$$\rightarrow \quad \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f'_{3x} \\ f'_{3y} \end{Bmatrix} = [T_1][K][T_1]^T \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d'_{3x} \\ d'_{3y} \end{Bmatrix}$$

Boundary condition: $d_{1x} = 0, d_{1y} = 0, d'_{3y} = 0$ $\quad F_{2y} = 0, F'_{3x} = 0$ $\quad F_{2x}$

Calculation of displacement and loading: unknown displacements d_{2x}, d_{2y}, d'_{3x} reaction forces F_{1x} , F_{1y} and that of inclined roller F'_{3y}

Example (Truss element in inclined support)

Find displacements and reacting forces for the plane trusses in the figure. $E = 210GPa$
And $A = 6.00 \times 10^{-4} m^2$ for nodes 1 and 2, and $A = 6\sqrt{2} \times 10^{-4} m^2$ for node 3.



Example (Truss element in inclined support)

Element 1

$$\cos\theta = 0 \qquad \sin\theta = 1$$

$$\underline{k}^{(1)} = \frac{(6.0\times10^{-4})(210\times10^{9})}{1m} \begin{array}{c} \begin{array}{cccc} d_{1x} & d_{1y} & d_{2x} & d_{2y} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 1 & 0 & -1 \\ & & 0 & \\ Symmetry & & & 1 \end{bmatrix} \end{array}$$

Element 2

$$\cos\theta = 1 \qquad \sin\theta = 0$$

$$\underline{k}^{(2)} = \frac{(6.0\times10^{-4})(210\times10^{9})}{1m} \begin{array}{c} \begin{array}{cccc} d_{2x} & d_{2y} & d_{3x} & d_{3y} \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ & 0 & 0 & 0 \\ & & 1 & 0 \\ Symmetry & & & 0 \end{bmatrix} \end{array}$$

Example (Truss element in inclined support)

Element 3

$$\cos\theta = \frac{\sqrt{2}}{2} \qquad \sin\theta = \frac{\sqrt{2}}{2}$$

$$\underline{k}^{(3)} = \frac{(6.0\times 10^{-4})(210\times 10^{9})}{1m} \begin{array}{cccc} d_{1x} & d_{1y} & d_{3x} & d_{3y} \\ \end{array} \begin{bmatrix} 0.5 & 0.5 & -0.5 & 0.5 \\ & 0.5 & -0.5 & -0.5 \\ & & 0.5 & 0.5 \\ Symmetry & & & 0.5 \end{bmatrix}$$

Calculation of the matrix $\underline{K}$ in global coordinates using direct stiffness method:

$$\underline{K} = 1260\times 10^{5}\,N/m \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.5 & -0.5 \\ & 1.5 & 0 & -1 & -0.5 & -0.5 \\ & & 1 & 0 & -1 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1.5 & 0.5 \\ Symmetry & & & & & 0.5 \end{bmatrix}$$

Example (Truss element in inclined support)

Matrix $\underline{T}_1$, which transforms the displacement at node 3 in global coordinates to the local coordinates $x' - y'$:

$$[T_1] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & 0 & 0 & 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

Calculation of $\underline{K}^* = \underline{T}_1 \underline{K}\, \underline{T}_1^T$:

$$\underline{T}_1 \underline{K} = 1260 \times 10^5 \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.5 & -0.5 \\ 0.5 & 1.5 & 0 & 0 & -0.5 & -0.5 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -0.707 & -0.707 & -0.707 & 0 & 1.414 & 0.707 \\ 0 & 0 & 0.707 & 0 & -0.707 & 0 \end{bmatrix}$$

Example (Truss element in inclined support)

$$\underline{T}_1 \underline{K} \underline{T}_1^T = 1260 \times 10^5 \begin{array}{c} \begin{array}{cccccc} d_{1x} & d_{1y} & d_{2x} & d_{2y} & d'_{3x} & d'_{3y} \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.707 & 0 \\ 0.5 & 1.5 & 0 & -1 & -0.707 & 0 \\ 0 & 0 & 1 & 0 & -0.707 & 0.707 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -0.707 & -0.707 & -0.707 & 0 & 1.500 & -0.500 \\ 0 & 0 & 0.707 & 0 & -0.500 & 0.500 \end{bmatrix} \end{array}$$

Substituting the boundary conditions $d_{1x} = d_{1y} = d_{2y} = d'_{3y} = 0$ to the above equation,

$$\begin{Bmatrix} F_{2x} = 1000kN \\ F'_{3x} = 0 \end{Bmatrix} = (1260 \times 10^5) \begin{bmatrix} 1 & -0.707 \\ -0.707 & 1.50 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d'_{3x} \end{Bmatrix}$$

Calculating displacements,

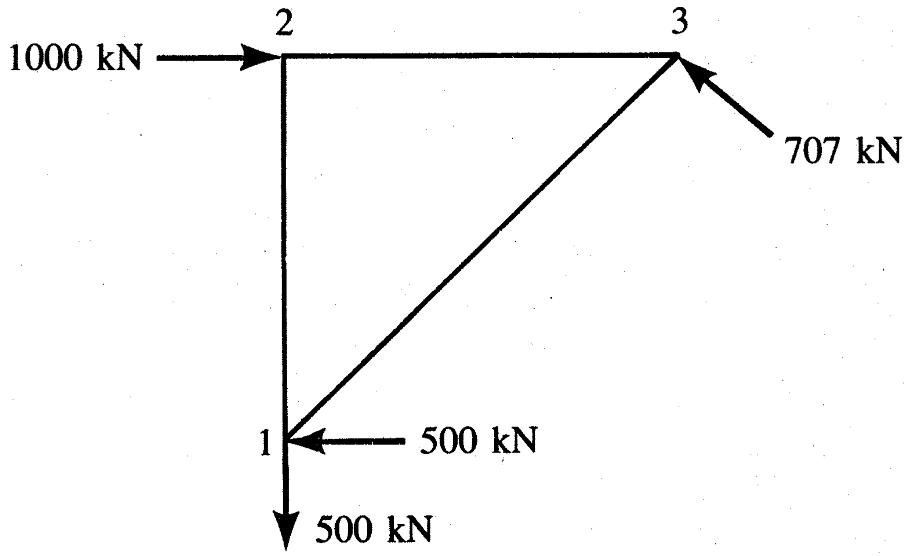
$$d_{2x} = 11.91mm$$

$$d'_{3x} = 5.613mm$$

Example (Truss element in inclined support)

Calculating loadings,

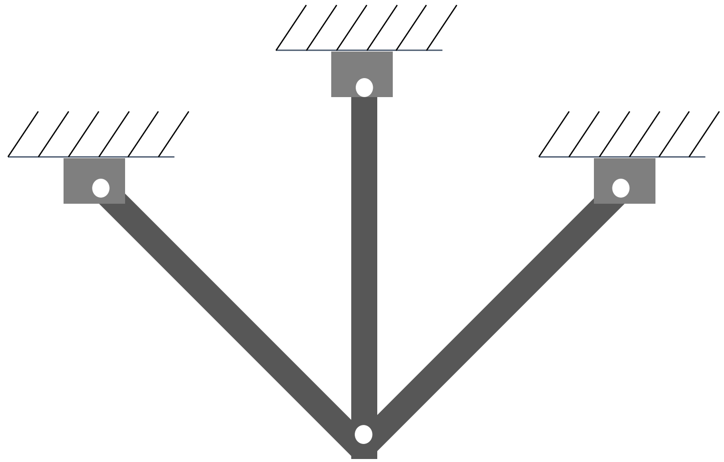
$$F_{1x} = -500kN$$
$$F_{1y} = -500kN$$
$$F_{2y} = 0$$
$$F'_{3y} = 707kN$$

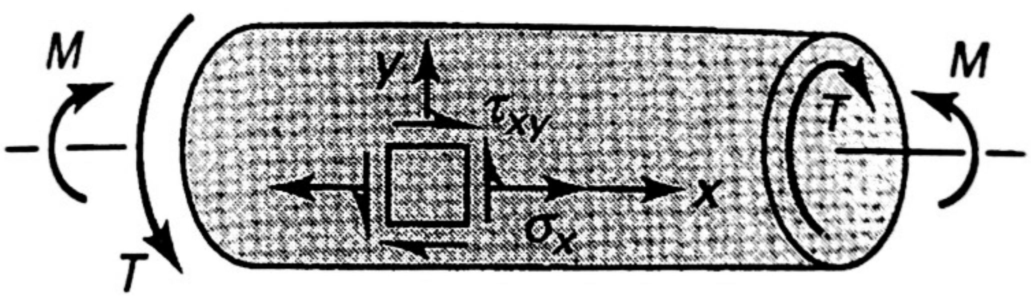


Free body diagram of truss structure

1. **Finite element solution coincides with the exact solution at nodes.** The reason why it coincides with the exact solution at nodes is because it calculates the element nodal loading energy-equivalent based on the linearly assumed displacement form at each element.
2. Although the nodal value for displacement coincides with the exact solution, the values between nodes might be inaccurate in the case of using few elements due to using linear displacement function at each element. However, **when the number of elements increases, the solution of finite element converges on the exact solution**.
3. Stress is derived from the slope of displacement curve as $\sigma = E\varepsilon = E(du/dx)$. Axial stress is constant at each element, for u of each element in the solution of finite element is a linear function. **More elements are needed for accurately modeling the first derivative of the displacement function like the modeling of axial stress**.
4. **The most approximate value of the stress appears not at nodes, but at the central point.** It is because the derivative of displacement is calculated more accurate between nodes than at nodes.
5. **Stress is not continuous over the element boundaries.** Therefore the equilibrium is not satisfied over the element boundaries. Also, the equilibrium at each element is usually not satisfied.

Example (Truss element in inclined support)





THANK YOU FOR LISTENING