Mechanics and Design

Chapter 6. FEM – Beam Element

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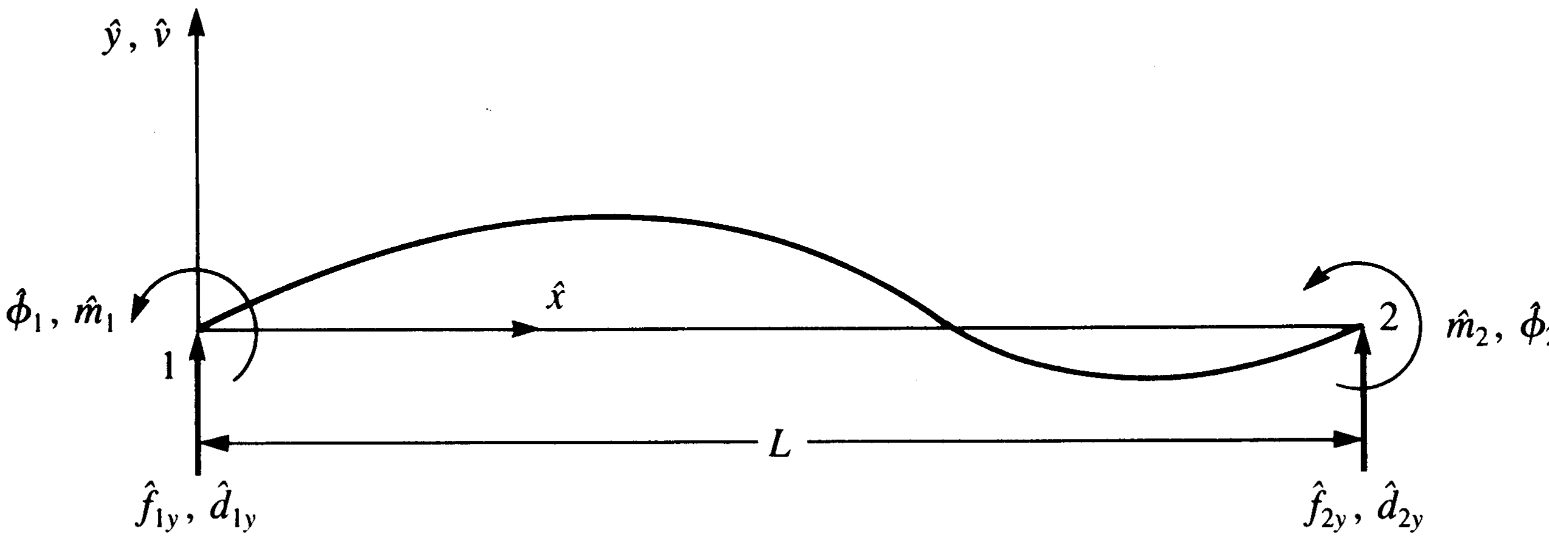
CONTENTS

Beam Element

Beam

A long thin structure that is subject to the vertical loads. A beam shows more evident bending deformation than the torsion and/or axial deformation.

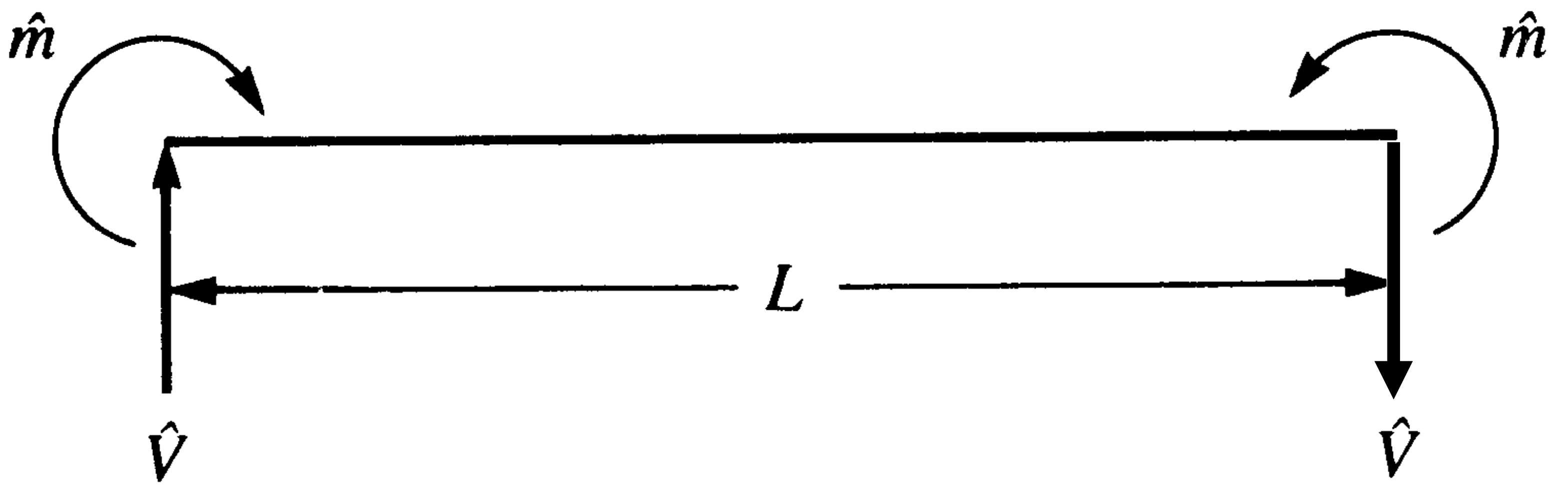
Bending strain is measured by the lateral deflection and the rotation → Lateral deflection and rotation determine the number of DoF (Degree of Freedom).



Beam Element

Sign Convention

1. Positive bending moment: Anti-clock wise rotation.
2. Positive load: $\hat{y}\ - direction$
3. Positive displacement: $\hat{y}\ - direction$

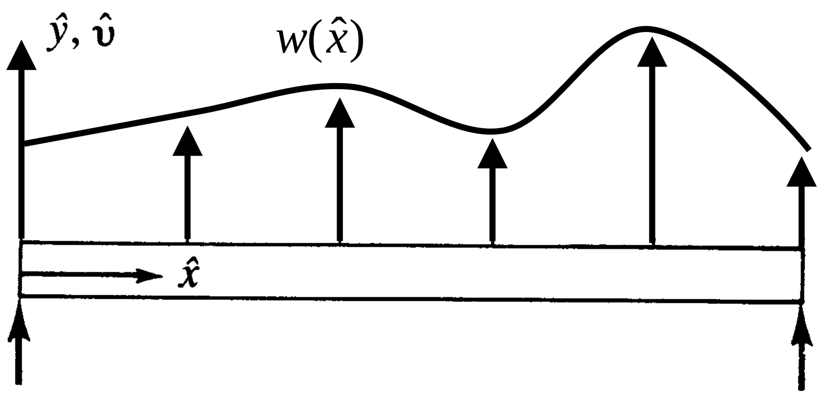


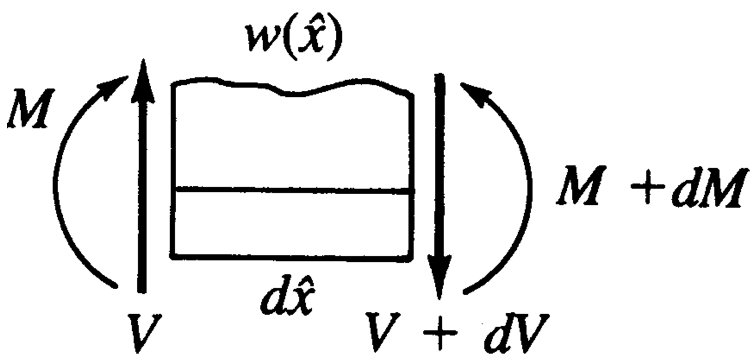
Beam Element

The Governing Equation

$$wdx + dV = 0 \ or \ w + \frac{dV}{dx} = 0$$

$$Vdx + dM = 0 \ or \ V + \frac{dM}{dx} = 0$$





Beam Element

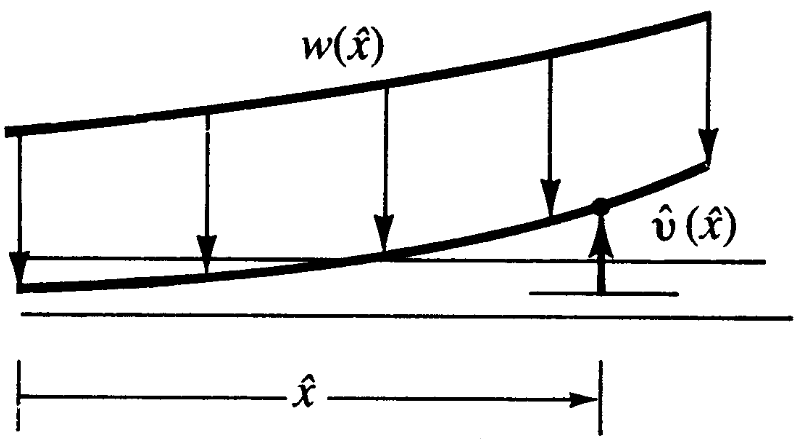
The Governing Equation

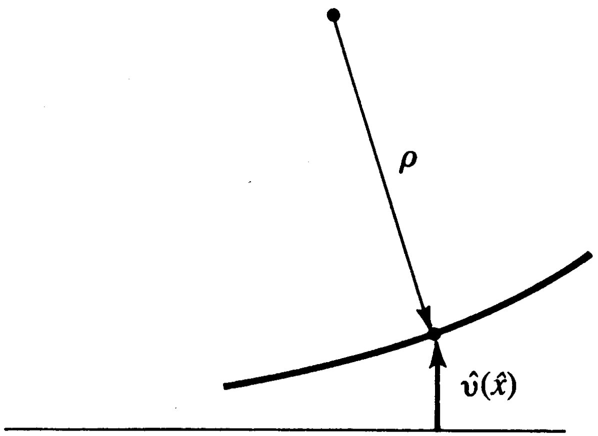
Relation between beam curvature(κ) and bending moment

$$\kappa = \frac{1}{\rho} = \frac{M}{EI} \qquad or \qquad \frac{d^2\hat{v}}{d\hat{x}^2} = \frac{M}{EI}$$

Curvature for small slope ($\theta = d\hat{v}/d\hat{x}$): $\kappa = \frac{d^2\hat{v}}{d\hat{x}^2}$

ρ : Radius of the deflection curve
$\hat{v}$: Lateral displacement function along the $\hat{y}$-axis direction
E : Stiffness coefficient, I : Moment of inertia along the $\hat{z}$-axis direction





Beam Element

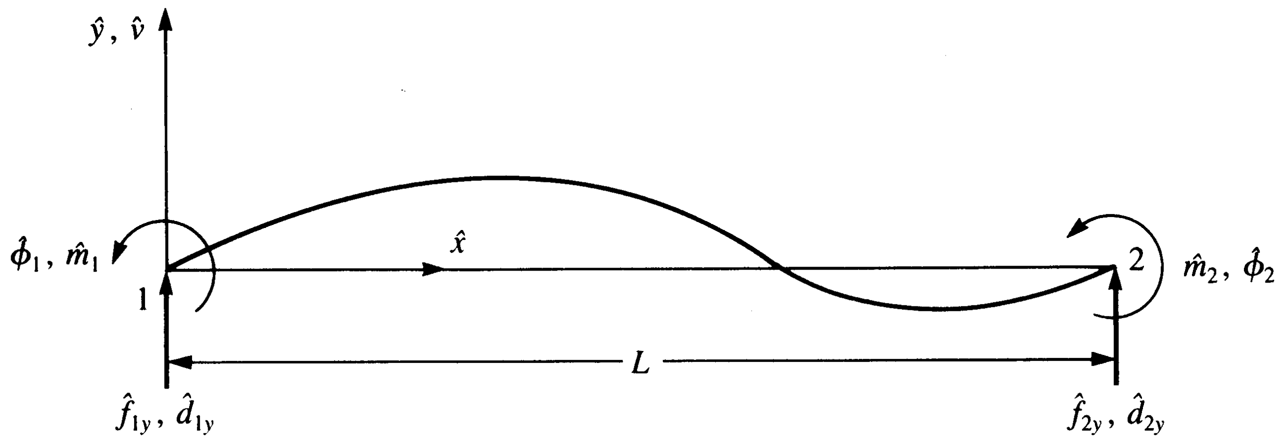
The Governing Equation

Solving the equation with M.

$$\frac{d}{d\hat{x}^2}\left(EI\frac{dv}{d\hat{x}^2}\right) = w(\hat{x})$$

When EI is constant, and force and moment are only applied at nodes,

$$EI\frac{d^4\hat{v}}{d\hat{x}^4} = 0$$



Beam Element

Procedure

Step1: To select beam element type

Step2: To select displacement function

Assumption of lateral displacement

$$\hat{v}(\hat{x}) = a_1\hat{x}^3 + a_2\hat{x}^2 + a_3\hat{x} + a_4$$

- Complete 3-order displacement function is suitable because it has four degree of freedom (one lateral displacement and one small rotation at each node)
- The function is proper, because it satisfies the fundamental differential equation of a beam.
- The function satisfies continuity of both displacement and slope at each node.

Representing $\hat{v}$ with function of $\hat{d}_{1y}, \hat{d}_{2y}, \hat{\phi}_1, \hat{\phi}_2$

$$\hat{v}(0) = \hat{d}_{1y} = a_4 \qquad \hat{v}(L) = \hat{d}_{2y} = a_1L^3 + a_2L^2 + a_3L + a_4$$

$$\frac{d\hat{v}(0)}{d\hat{v}} = \hat{\phi}_1 = a_3 \qquad \frac{d\hat{v}(L)}{d\hat{x}} = \hat{\phi}_2 = 3a_1L^2 + 2a_2L + a_3$$

Beam Element

Procedure

Replacing $a_1 - a_4$ with $\hat{d}_{1y}, \hat{d}_{2y}, \hat{\phi}_1, \hat{\phi}_2$

$$\hat{v} = \left[\frac{2}{L^3}(\hat{d}_{1y} - \hat{d}_{2y}) + \frac{1}{L^2}(\hat{\phi}_1 + \hat{\phi}_2)\right]\hat{x}^3$$
$$+\left[-\frac{3}{L}(\hat{d}_{1y} - \hat{d}_{2y}) - \frac{1}{L}(2\hat{\phi}_1 + \hat{\phi}_2)\right]\hat{x}^2 + \hat{\phi}_1\hat{x} + \hat{d}_{1y}$$

Representing it in matrix form, $\hat{v} = [N]\{\hat{d}\}$

$$\{\hat{d}\} = \begin{Bmatrix} \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix} \qquad [N] = [N_1 \quad N_2 \quad N_3 \quad N_4]$$

$$N_1 = \frac{1}{L^3}\left(2\hat{x}^3 - 3\hat{x}^2 L + L^3\right) \qquad N_2 = \frac{1}{L^3}\left(\hat{x}^3 L - 2\hat{x}^2 L^2 + \hat{x} L^3\right)$$

$$N_3 = \frac{1}{L^3}\left(-2\hat{x}^3 + 3\hat{x}^2 L\right) \qquad N_4 = \frac{1}{L^3}\left(\hat{x}^3 L - \hat{x}^2 L^2\right)$$

where N_1, N_2, N_3, N_4 : shape functions of the beam element.

Beam Element

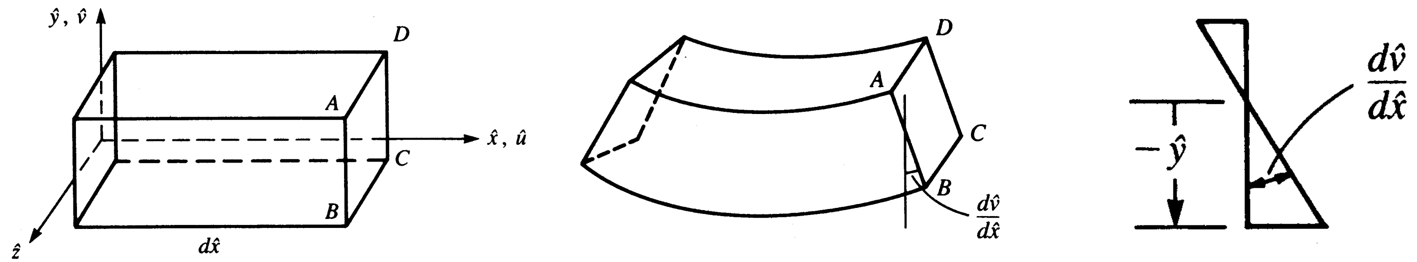
Procedure

Step3: To define strain-displacement relation and stress-strain relation

Assume that the equation of strain-displacement relation is valid

$$\varepsilon_x(\hat{x},\hat{y}) = \frac{d\hat{u}}{d\hat{x}}, \qquad \hat{u} = -\hat{y}\frac{d\hat{v}}{d\hat{x}} \qquad \Rightarrow \qquad \varepsilon_x(\hat{x},\hat{y}) = -\hat{y}\frac{d^2\hat{v}}{d\hat{x}^2}$$

Basic assumption: Cross-section of the beam sustains its shape after deformation by bending, and generally rotates by degree of $(d\hat{v}/d\hat{x})$.



Bending moment-lateral displacement relation and shear force-lateral displacement relation

$$\hat{m}(\hat{x}) = EI\frac{d^2\hat{v}}{d\hat{x}^2}, \qquad \hat{V} = EI\frac{d^3\hat{v}}{d\hat{x}^3}$$

Beam Element

Procedure

Step4: To derive an element stiffness matrix and governing equation by direct stiffness method

Element stiffness matrix and governing equations

$$\hat{f}_{1y} = \hat{V} = EI\frac{d^3\hat{v}(0)}{d\hat{x}^3} = \frac{EI}{L^3}\left(12\hat{d}_{1y} + 6L\hat{\phi}_1 - 12\hat{d}_{2y} + 6L\hat{\phi}_2\right)$$

$$\hat{m}_1 = -\hat{m} = -EI\frac{d^2\hat{v}(0)}{d\hat{x}^2} = \frac{EI}{L^3}\left(6L\hat{d}_{1y} + 4L^2\hat{\phi}_1 - 6L\hat{d}_{2y} + 2L^2\hat{\phi}_2\right)$$

$$\hat{f}_{2y} = -\hat{V} = -EI\frac{d^3\hat{v}(L)}{d\hat{x}^3} = \frac{EI}{L^3}\left(-12\hat{d}_{1y} - 6L\hat{\phi}_1 + 12\hat{d}_{2y} - 6L\hat{\phi}_2\right)$$

$$\hat{m}_2 = \hat{m} = EI\frac{d^2\hat{v}(L)}{d\hat{x}^2} = \frac{EI}{L^3}\left(6L\hat{d}_{1y} + 2L^2\hat{\phi}_1 - 6L\hat{d}_{2y} + 4L^2\hat{\phi}_2\right)$$

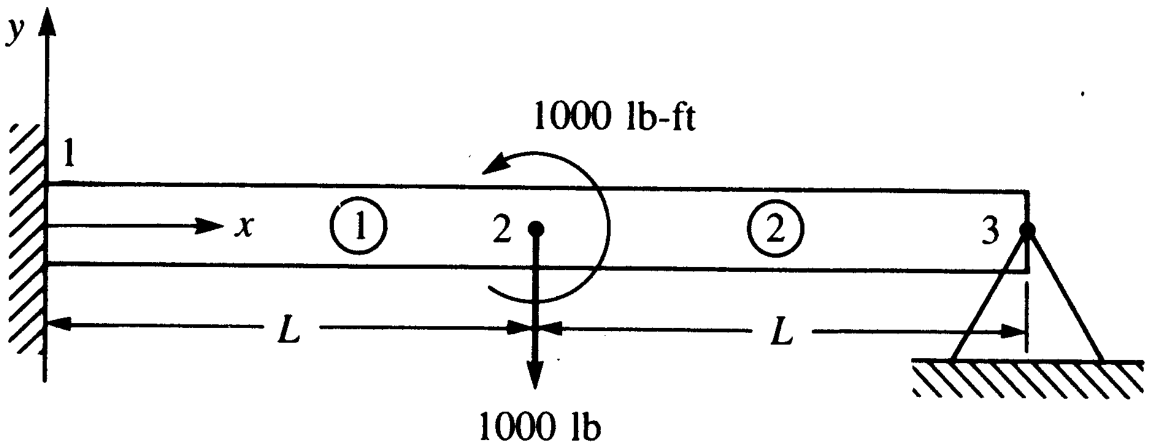
- Matrix Form

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix} \qquad \underline{\hat{k}} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Beam Element

Procedure

Step5: To constitute a global stiffness matrix using boundary conditions



<Assemble Example>

- Assume EI of the beam element is constant.
- 1000 lb load and 1000 lb-ft moment are applied at the center of the beam.
- Assume load and moment were only applied at nodes.
- Left end of the beam is fixed and right end is pin-connected.
- The beam is divided into two elements (node 1, 2, and 3 as shown above figure).

Beam Element

Procedure

Step5: To constitute a global stiffness matrix using boundary conditions

$$\underline{k}^{(1)} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \quad (d_{1y},\ \phi_1,\ d_{2y},\ \phi_2)$$

$$\underline{k}^{(2)} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \quad (d_{2y},\ \phi_2,\ d_{3y},\ \phi_3)$$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 12+12 & -6L+6L & -12 & 6L \\ 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix}$$

Beam Element

Procedure

Step5: To constitute a global stiffness matrix using boundary conditions

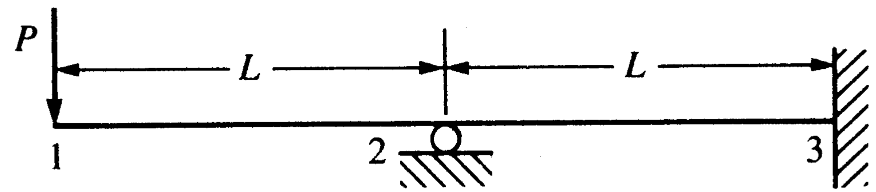
- Boundary conditions and constrains at the node 1(fixed) and node 3(pin-connected) are

$$\phi_1 = 0 \qquad d_{1y} = 0 \qquad d_{3y} = 0$$

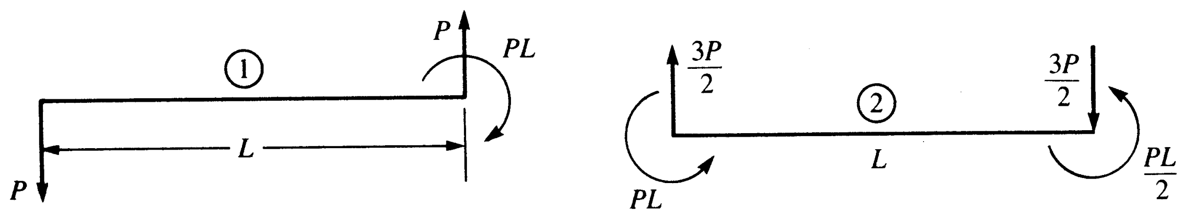
$$\begin{Bmatrix} -1000 \\ -1000 \\ 0 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 24 & 0 & 6L \\ 0 & 8L^2 & 2L^2 \\ 6L & 2L^2 & 4L^2 \end{bmatrix} \begin{Bmatrix} d_{2y} \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

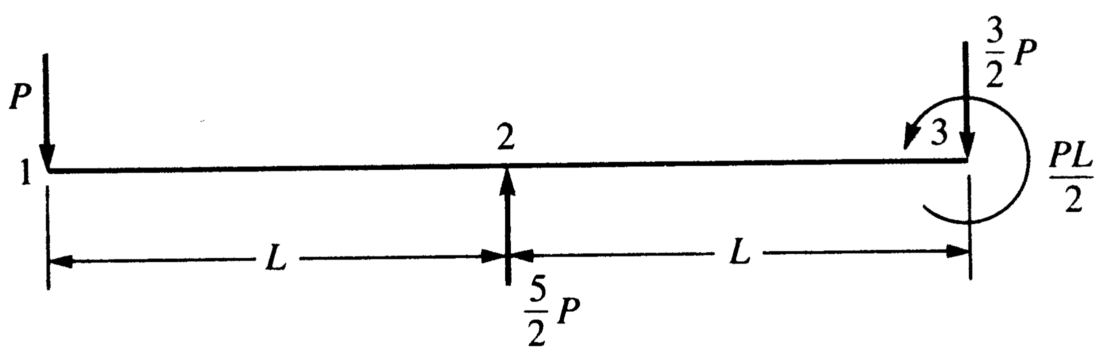
Beam Element

Example: Beam analysis using direct stiffness method



Cantilever beam supported by roller at the center





Beam Element

Example: Beam analysis using direct stiffness method

- Load P is applied at node 1.
- Length: $2L$, Stiffness: EI
- Constrains: (1) Roller at node 2, (2) Fixed at node 3.

Global stiffness matrix

$$\underline{K} = \frac{EI}{L^3}\begin{bmatrix} \overset{d_{1y}}{12} & \overset{\phi_1}{6L} & \overset{d_{2y}}{-12} & \overset{\phi_2}{6L} & \overset{d_{3y}}{0} & \overset{\phi_3}{0} \\ & 4L^2 & -6L & 2L^2 & 0 & 0 \\ & & 12+12 & -6L+6L & -12 & 6L \\ & & & 4L^2+4L^2 & -6L & 2L^2 \\ & & & & 12 & -6L \\ & \text{Symmetry} & & & & 4L^2 \end{bmatrix}$$

Beam Element

Example: Beam analysis using direct stiffness method

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix}$$

Boundary conditions: $d_{2y} = 0,\ d_{3y} = 0,\ \phi_3 = 0$

$$\begin{Bmatrix} -P \\ 0 \\ 0 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & 6L \\ 6L & 4L^2 & 2L^2 \\ 6L & 2L^2 & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ \phi_2 \end{Bmatrix}$$

$$d_{1y} = -\frac{7PL^3}{12EI}, \qquad \phi_1 = \frac{3PL^2}{4EI}, \qquad \phi_2 = \frac{PL^2}{4EI}$$

Beam Element

Example: Beam analysis using direct stiffness method

Substituting the obtained values to the final equation,

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} -\frac{7PL^3}{12EI} \\ \frac{3PL^2}{4EI} \\ 0 \\ \frac{PL^2}{4EI} \\ 0 \\ 0 \end{Bmatrix}$$

$$F_{1y} = -P \qquad M_1 = 0 \qquad F_{2y} = \frac{5}{2}P$$

$$M_2 = 0 \qquad F_{3y} = -\frac{3}{2}P \qquad M_3 = \frac{1}{2}PL$$

$F_{1y} = -P$: Force at node 1.

F_{2y}, F_{3y}, M_3 : Reacting forces and moment at nodes.

M_1, M_2 : Zero.

Beam Element

Example: Beam analysis using direct stiffness method

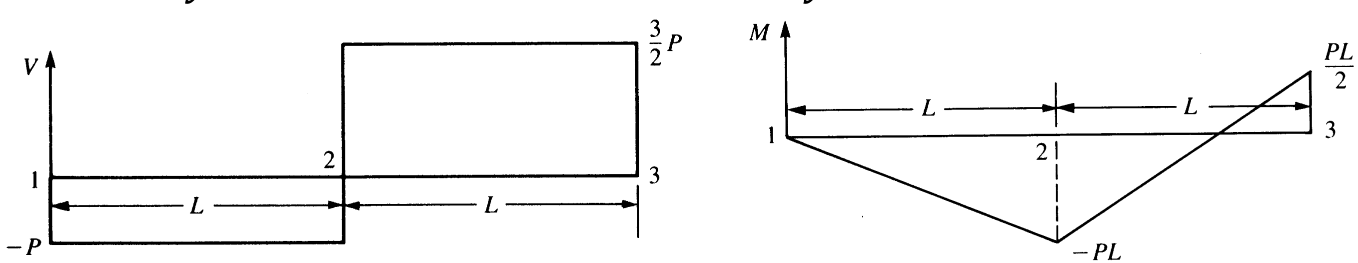
Calculating local nodal loads

Force at the element 1.

when $\underline{\hat{f}} = \underline{\hat{k}}\,\underline{\hat{d}}$

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} -\dfrac{7PL^3}{12EI} \\ -\dfrac{3PL^2}{4EI} \\ 0 \\ \dfrac{PL^2}{4EI} \end{Bmatrix}$$

$$\hat{f}_{1y} = -P, \qquad \hat{m}_1 = 0, \qquad \hat{f}_{2y} = P, \qquad \hat{m}_2 = -PL$$

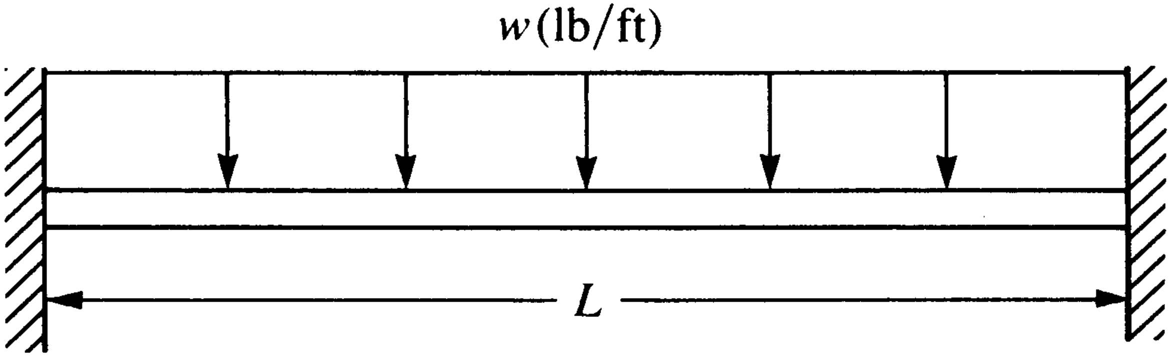


Shear moment curve

Beam Element

Homework: Distributed load

Equivalent force



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