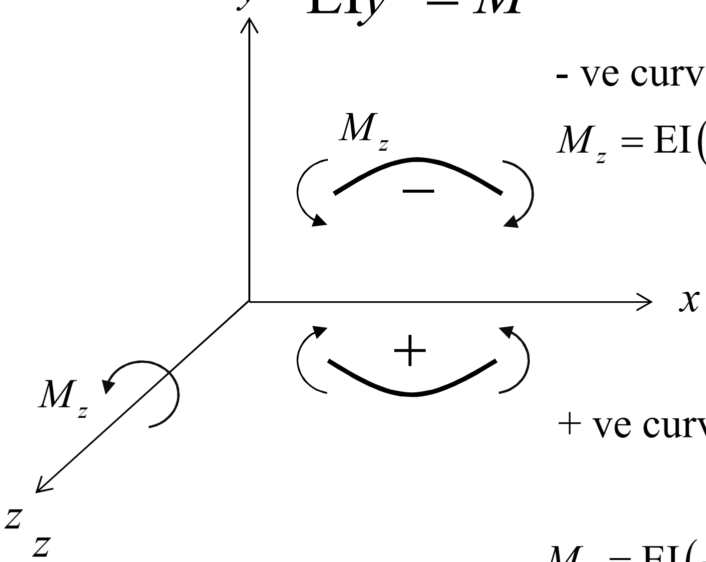
Chap 7.
First-order Hinge-by Hinge Analysis

7.1 Introduction

1) Elastic analysis of original structure. Find max. moment
2) Replace the max moment location by real hinge
3) Elastic analysis for new structure to find max moment
4) Perform the analysis until sufficient PH

Sign convention for differential equations

$$\mathrm{EI}y'' = M$$

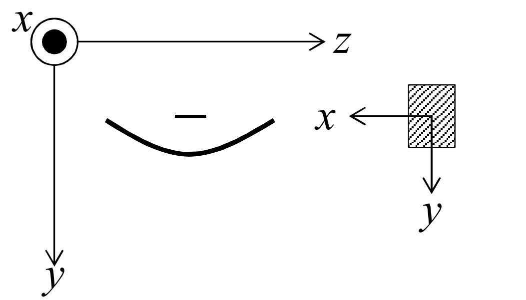


- ve curvature

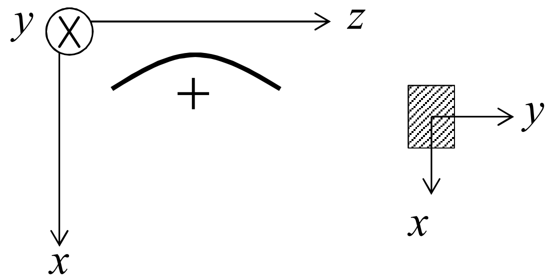
$$M_z = \mathrm{EI}\left(-y''\right)$$

+ ve curvature

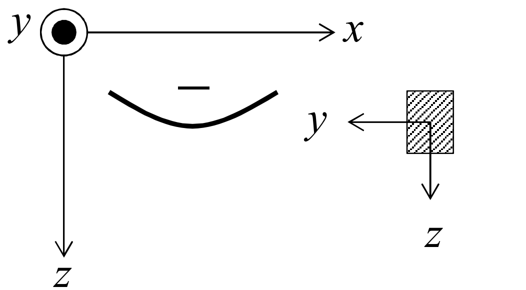
$$M_z = \mathrm{EI}\left(+y''\right)$$



$$M_x = \mathrm{EI}_x\left(-\frac{d^2v}{dz^2}\right)$$

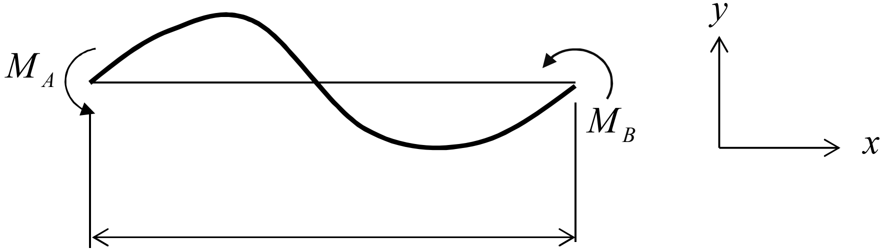


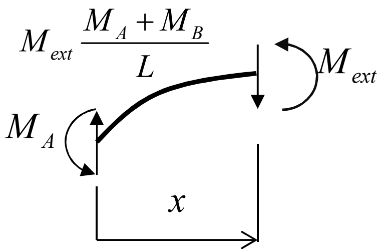
$$M_y = \mathrm{EI}_x\left(\frac{d^2u}{dz^2}\right)$$



$$M_y = \mathrm{EI}_y\left(-\frac{d^2w}{dx^2}\right)$$

7.2 Stiffness matrix





$$\begin{cases} M_{ext} = M_A - \dfrac{M_A + M_B}{L} x \\ M_{\text{int}} = -EIy'' \end{cases}$$

$$y'' = -\frac{M_A}{EI} + \frac{M_A + M_B}{EIL} x$$

$$y' = -\frac{M_A}{EI} x + \frac{M_A + M_B}{EIL} \frac{x^2}{2} + C_1$$

$$y = -\frac{M_A}{EI} \frac{x^2}{2} + \frac{M_A + M_B}{EIL} \frac{x^3}{6} + C_1 x + C_2$$

Since $y(0) = y(L) = 0$

$$C_2 = 0$$

$$C_1 = \frac{M_A}{EI} \frac{L}{2} - \frac{M_A + M_B}{EI} \frac{L}{6}$$

$$y' = -\frac{M_A}{EI} x + \frac{M_A + M_B}{EIL} \frac{x^2}{2} + \frac{2M_A L - M_B L}{6EI}$$

Since $\theta_A = y'(0), \quad \theta_B = y'(L)$

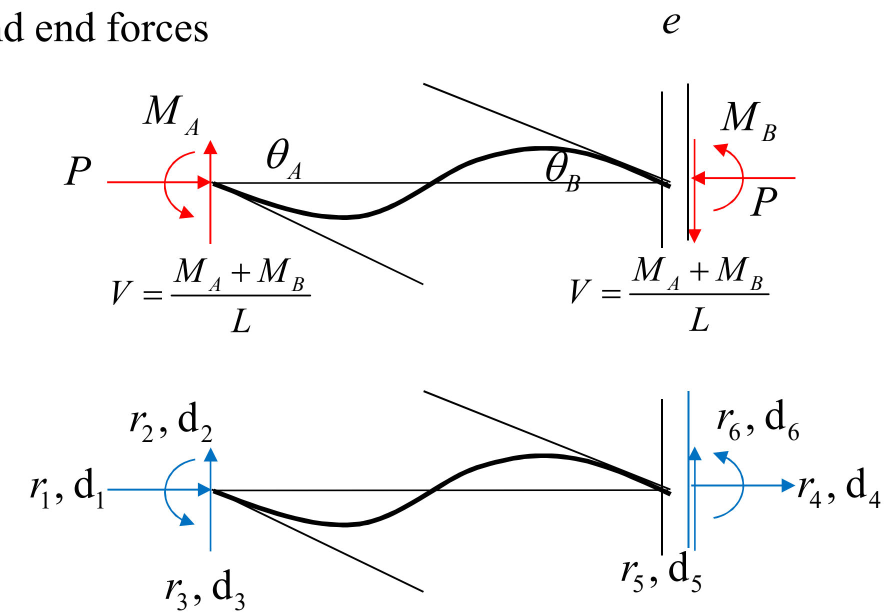
$$\begin{cases} \theta_A = \dfrac{M_A}{EI} \dfrac{L}{3} - \dfrac{M_B}{EI} \dfrac{L}{6} \\ \theta_B = -\dfrac{M_A}{EI} \dfrac{L}{6} + \dfrac{M_B}{EI} \dfrac{L}{3} \end{cases}$$

$$F = K\Delta$$

$$\begin{Bmatrix} M_A \\ M_B \end{Bmatrix} = \frac{EI}{L} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{Bmatrix} \theta_A \\ \theta_B \end{Bmatrix}$$

Equilibrium relationship between node forces and end forces

$$\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/L & 1/L \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1/L & -1/L \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} P \\ M_A \\ M_B \end{Bmatrix}$$



Kinematic relationship between node displacement and slope-deflection

$$\begin{Bmatrix} e \\ \theta_A \\ \theta_B \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1/L & 1 & 0 & -1/L & 0 \\ 0 & 1/L & 0 & 0 & -1/L & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix}$$

Nodal forces and node displacement

$$\begin{Bmatrix} P \\ M_A \\ M_B \end{Bmatrix} = \frac{EI}{L} \begin{bmatrix} \frac{A}{I} & & \\ & 4 & 2 \\ & 2 & 4 \end{bmatrix} \begin{Bmatrix} e \\ \theta_A \\ \theta_B \end{Bmatrix}$$

$$\begin{Bmatrix} e \\ \theta_A \\ \theta_B \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1/L & 1 & 0 & -1/L & 0 \\ 0 & 1/L & 0 & 0 & -1/L & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix}$$

$$\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/L & 1/L \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1/L & -1/L \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} P \\ M_A \\ M_B \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/L & 1/L \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1/L & -1/L \\ 0 & 0 & 1 \end{bmatrix} \frac{EI}{L} \begin{bmatrix} \frac{A}{I} & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1/L & 1 & 0 & -1/L & 0 \\ 0 & 1/L & 0 & 0 & -1/L & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix}$$

Element stiffness matrix in terms of nodal forces and displacements

$$\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{Bmatrix} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^3} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^3} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix}$$

7.3 Stiffness Matrix with a Plastic hinge at End A

Moment-slope relationship

$$\begin{Bmatrix} \dot{M}_A \\ \dot{M}_B \end{Bmatrix} = \frac{EI}{L}\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\begin{Bmatrix} \dot{\theta}_A \\ \dot{\theta}_B \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{P} \\ \dot{M}_A \\ \dot{M}_B \end{Bmatrix} = \frac{EI}{L}\begin{bmatrix} \frac{A}{I} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{Bmatrix} \dot{e} \\ \dot{\theta}_A \\ \dot{\theta}_B \end{Bmatrix}$$

Equilibrium

$$\begin{Bmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \\ \dot{r}_4 \\ \dot{r}_5 \\ \dot{r}_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/L & 1/L \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1/L & -1/L \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \dot{P} \\ \dot{M}_A \\ \dot{M}_B \end{Bmatrix}$$

Relationship between node displacement and node forces

$$\begin{Bmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \\ \dot{r}_4 \\ \dot{r}_5 \\ \dot{r}_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/L & 1/L \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1/L & -1/L \\ 0 & 0 & 1 \end{bmatrix}\frac{EI}{L}\begin{bmatrix} \frac{A}{I} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1/L & 1 & 0 & -1/L & 0 \\ 0 & 1/L & 0 & 0 & -1/L & 1 \end{bmatrix}\begin{Bmatrix} \dot{d}_1 \\ \dot{d}_2 \\ \dot{d}_3 \\ \dot{d}_4 \\ \dot{d}_5 \\ \dot{d}_6 \end{Bmatrix}$$

Incremental stiffness matrix

$$\begin{Bmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \\ \dot{r}_4 \\ \dot{r}_5 \\ \dot{r}_6 \end{Bmatrix} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{3EI}{L^3} & 0 & 0 & -\frac{3EI}{L^3} & \frac{3EI}{L^2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{3EI}{L^3} & 0 & 0 & \frac{3EI}{L^3} & -\frac{3EI}{L^2} \\ 0 & \frac{3EI}{L^2} & 0 & 0 & -\frac{3EI}{L^2} & \frac{3EI}{L} \end{bmatrix} \begin{Bmatrix} \dot{d}_1 \\ \dot{d}_2 \\ \dot{d}_3 \\ \dot{d}_4 \\ \dot{d}_5 \\ \dot{d}_6 \end{Bmatrix}$$

7.4 Stiffness Matrix with a Plastic hinge at End B

Moment-slope relationship

$$\begin{Bmatrix} \dot{M}_A \\ \dot{M}_B \end{Bmatrix} = \frac{EI}{L}\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{\theta}_A \\ \dot{\theta}_B \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{P} \\ \dot{M}_A \\ \dot{M}_B \end{Bmatrix} = \frac{EI}{L}\begin{bmatrix} \frac{A}{I} & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{e} \\ \dot{\theta}_A \\ \dot{\theta}_B \end{Bmatrix}$$

Equilibrium

$$\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/L & 1/L \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1/L & -1/L \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} P \\ M_A \\ M_B \end{Bmatrix}$$

Relationship between node displacement and node forces

$$\begin{Bmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \\ \dot{r}_4 \\ \dot{r}_5 \\ \dot{r}_6 \end{Bmatrix} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{3EI}{L^3} & \frac{3EI}{L^2} & 0 & -\frac{3EI}{L^3} & 0 \\ 0 & \frac{3EI}{L^2} & \frac{3EI}{L^2} & 0 & -\frac{3EI}{L^2} & 0 \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{3EI}{L^3} & -\frac{3EI}{L^2} & 0 & \frac{3EI}{L^3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{d}_1 \\ \dot{d}_2 \\ \dot{d}_3 \\ \dot{d}_4 \\ \dot{d}_5 \\ \dot{d}_6 \end{Bmatrix}$$

7.5Stiffness Matrix with a Plastic hinge at Ends A and B

Moment-slope relationship

$$\begin{Bmatrix} \dot{M}_A \\ \dot{M}_B \end{Bmatrix} = \frac{EI}{L}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{\theta}_A \\ \dot{\theta}_B \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{P} \\ \dot{M}_A \\ \dot{M}_B \end{Bmatrix} = \frac{EI}{L}\begin{bmatrix} \frac{A}{I} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{e} \\ \dot{\theta}_A \\ \dot{\theta}_B \end{Bmatrix}$$

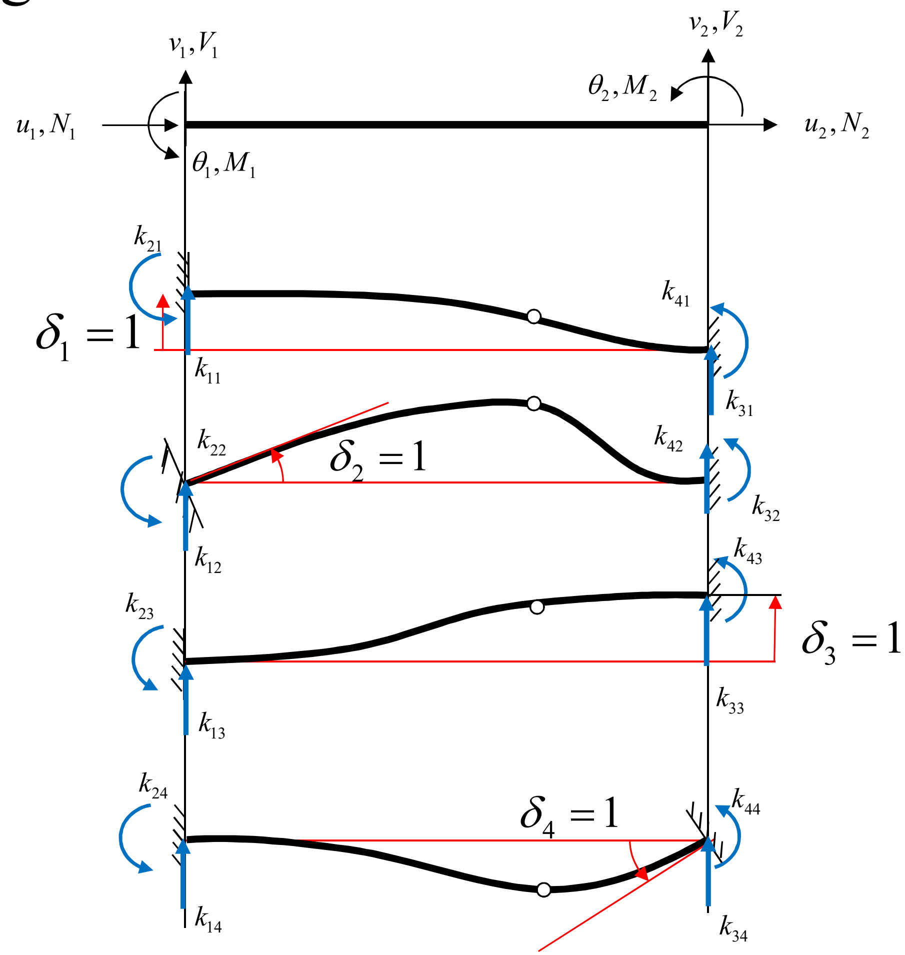
Equilibrium

$$\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/L & 1/L \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1/L & -1/L \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} P \\ M_A \\ M_B \end{Bmatrix}$$

Relationship between node displacement and node forces

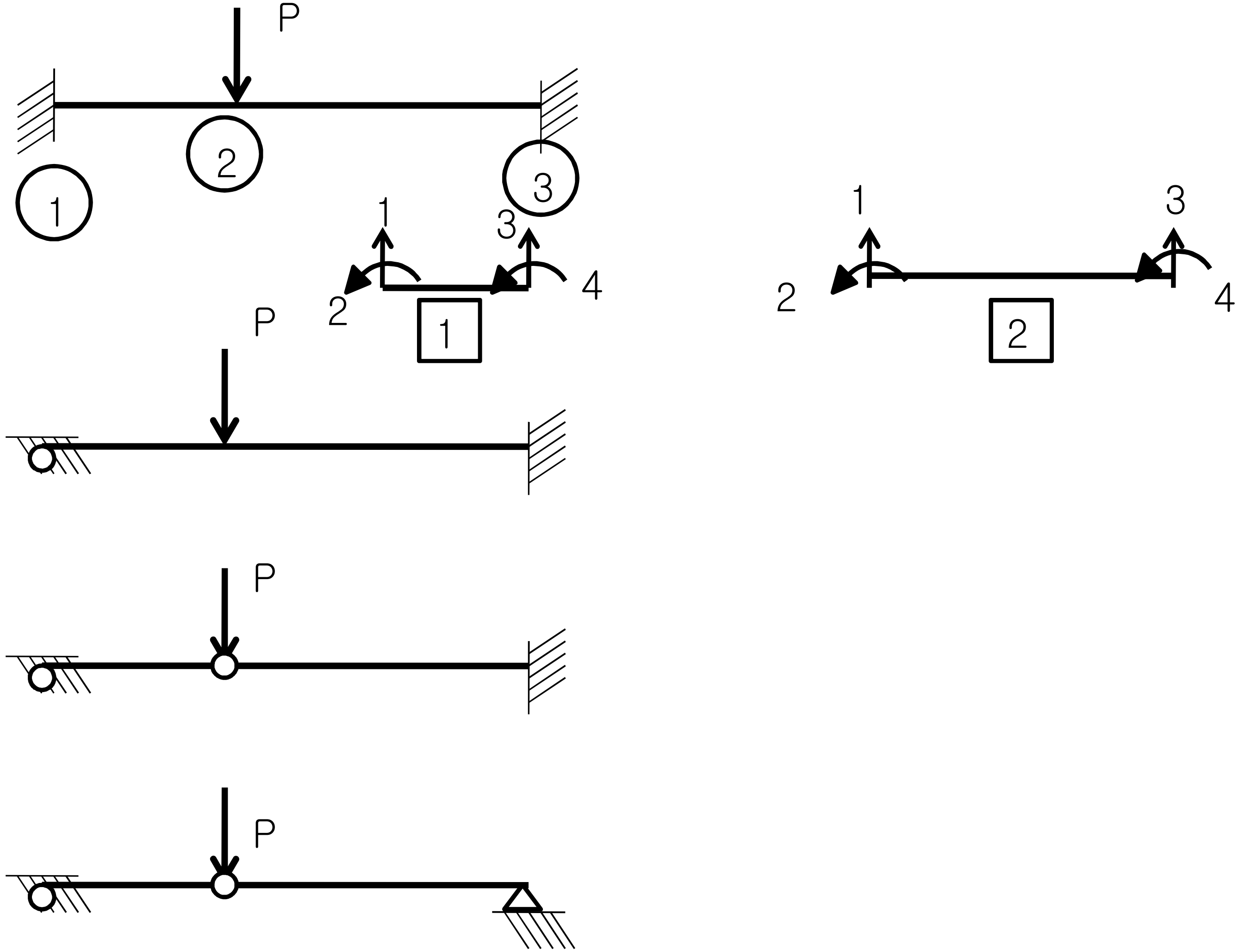
$$\begin{Bmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \\ \dot{r}_4 \\ \dot{r}_5 \\ \dot{r}_6 \end{Bmatrix} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{d}_1 \\ \dot{d}_2 \\ \dot{d}_3 \\ \dot{d}_4 \\ \dot{d}_5 \\ \dot{d}_6 \end{Bmatrix}$$

7.6 Stiffness Matrix for a Beam with an Intermediate Plastic Hinge



$$
\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{Bmatrix} = \frac{3EI}{a^3+b^3}
\begin{bmatrix}
\frac{A\left(a^3+b^3\right)}{3I\left(a+b\right)} & 0 & 0 & -\frac{A\left(a^3+b^3\right)}{3I\left(a+b\right)} & 0 & 0 \\
0 & 1 & a & 0 & -1 & b \\
0 & a & a & 0 & -a & ab \\
-\frac{A\left(a^3+b^3\right)}{3I\left(a+b\right)} & 0 & 0 & \frac{A\left(a^3+b^3\right)}{3I\left(a+b\right)} & 0 & 0 \\
0 & -1 & -a & 0 & 1 & -b \\
0 & b & ab & 0 & -b & b
\end{bmatrix}
\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix}
$$

7.8 Numerical Examples



Step 1

$$k_1 = \begin{bmatrix} 3,147 & 75,521 & -3,147 & 75,521 \\ & 2,416,667 & -75,521 & 1,208,333 \\ & & 3,147 & -75,521 \\ sym & & & 2,416,667 \end{bmatrix} \qquad k_2 = \begin{bmatrix} 393 & 18,880 & -393 & 18,880 \\ & 1,208,333 & -18,880 & 604,167 \\ & & 393 & -18,880 \\ sym & & & 1,208,333 \end{bmatrix}$$

Assembly and elimination by boundary conditions

$$K = \begin{bmatrix} 3,147 & 75,521 & -3,147 & 75,521 & 0 & 0 \\ & 2,416667 & -75,521 & 1,208,333 & 0 & 0 \\ & & 3,147+393 & -75,521+18,880 & -393 & 18880 \\ Sym & & & 2,416,667+1,208,333 & -18,880 & 604,167 \\ 0 & 0 & & & 393 & -18,880 \\ 0 & 0 & & & & 1,208,333 \end{bmatrix}$$

$$\begin{Bmatrix} -1 \\ 0 \end{Bmatrix} = \begin{bmatrix} 3540 & -56,641 \\ & 3,625,000 \end{bmatrix} \begin{Bmatrix} V_3 \\ V_4 \end{Bmatrix}$$

$$\begin{Bmatrix} V_3 \\ V_4 \end{Bmatrix} = \begin{Bmatrix} -3.766*10^{-4} \\ -5.885*10^{-6} \end{Bmatrix}$$

Step 2 Element internal force calculation

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{bmatrix} - & - & -3147 & 75,521 \\ - & - & -75,521 & 1,208,333 \\ - & - & 3147 & -75,521 \\ - & - & 75,521 & 2,416,617 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -3.766*10^{-4} \\ -5.885*10^{-6} \end{Bmatrix} = \begin{Bmatrix} 0.74 \\ 21.33 \\ -0.74 \\ 14.21 \end{Bmatrix}$$

$$\begin{Bmatrix} F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = \begin{bmatrix} 393 & 18,880 & - & - \\ 18,880 & 1,208,333 & - & - \\ -393 & -18,880 & - & - \\ 18,880 & 604,167 & - & - \end{bmatrix} \begin{Bmatrix} -3.766*10^{-4} \\ -5.885*10^{-6} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -0.26 \\ -14.21 \\ 0.26 \\ -10.68 \end{Bmatrix}$$

Load factors corresponding to moments at three nodes

$$\lambda_1 = \frac{5,652}{21.33} = 265$$

$$\lambda_2 = \frac{5,652}{14.21} = 397.7$$

$$\lambda_3 = \frac{5,652}{10.68} = 529.2$$

$$\begin{Bmatrix} V_3 \\ V_4 \end{Bmatrix} = \begin{Bmatrix} -3.766*10^{-4} \\ -5.885*10^{-6} \end{Bmatrix}$$

$$\begin{Bmatrix} V_3 \\ V_4 \end{Bmatrix} = \lambda_1 \begin{Bmatrix} -3.766*10^{-4} \\ -5.885*10^{-6} \end{Bmatrix} = \begin{Bmatrix} -0.0998 \\ -0.00156 \end{Bmatrix}$$

Step 3 Displacement and internal forces

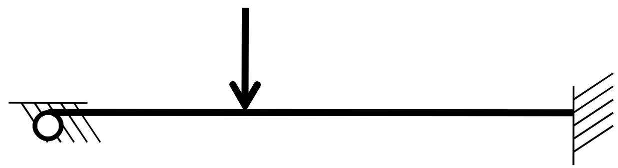
$$\begin{Bmatrix} V_3 \\ V_4 \end{Bmatrix} = \lambda_1 \begin{Bmatrix} -3.766*10^{-4} \\ -5.885*10^{-6} \end{Bmatrix} = \begin{Bmatrix} -0.0998 \\ -0.00156 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \lambda_1 \begin{Bmatrix} 0.74 \\ 21.33 \\ -0.74 \\ 14.21 \end{Bmatrix} = \begin{Bmatrix} 196 \\ 5652 \\ -196 \\ 3766 \end{Bmatrix}$$

$$\begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix} = \begin{Bmatrix} 265 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = \lambda_1 \begin{Bmatrix} -0.26 \\ -14.21 \\ 0.26 \\ -10.68 \end{Bmatrix} = \begin{Bmatrix} -69 \\ -3766 \\ 69 \\ -2830 \end{Bmatrix}$$

Step 4 The plastic hinge at the first node



Step 5 The modified stiffness of the element including the plastic hinge

$$\begin{Bmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \\ \dot{r}_4 \\ \dot{r}_5 \\ \dot{r}_6 \end{Bmatrix} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{3EI}{L^3} & 0 & 0 & -\frac{3EI}{L^3} & \frac{3EI}{L^2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{3EI}{L^3} & 0 & 0 & \frac{3EI}{L^3} & -\frac{3EI}{L^2} \\ 0 & \frac{3EI}{L^2} & 0 & 0 & -\frac{3EI}{L^2} & \frac{3EI}{L} \end{bmatrix} \begin{Bmatrix} \dot{d}_1 \\ \dot{d}_2 \\ \dot{d}_3 \\ \dot{d}_4 \\ \dot{d}_5 \\ \dot{d}_6 \end{Bmatrix}$$

$$k_1 = \begin{bmatrix} 787 & 0 & -787 & 37,760 \\ & 0 & 0 & 0 \\ & & 787 & -37,760 \\ sym & & & 1,812,500 \end{bmatrix}$$

$$k_2 = \begin{bmatrix} 393 & 18,880 & -393 & 18,880 \\ & 1,208,333 & -18,880 & 604,167 \\ & & 393 & -18,880 \\ sym & & & 1,208,333 \end{bmatrix}$$

$$K = \begin{bmatrix} 787 & 0 & -787 & 37,760 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 787+393 & -37,760+18,880 & -393 & 18880 \\ Sym & & & 1,812,500+1,208,333 & -18,880 & 604,167 \\ 0 & 0 & & & 393 & -18,880 \\ 0 & 0 & & & & 1,208,333 \end{bmatrix}$$

$$\begin{Bmatrix} \dot{V}_3 \\ \dot{V}_4 \end{Bmatrix} = -\begin{Bmatrix} 9.416*10^{-4} \\ 5.885*10^{-6} \end{Bmatrix}$$

Step 6 Internal forces in the elements

$$\begin{Bmatrix} \dot{F}_1 \\ \dot{F}_2 \\ \dot{F}_3 \\ \dot{F}_4 \end{Bmatrix} = \begin{bmatrix} - & - & -787 & 37,760 \\ - & - & 0 & 0 \\ - & - & 787 & -37,760 \\ - & - & -37,760 & 1,812,500 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -9.416*10^{-4} \\ -5.885*10^{-6} \end{Bmatrix} = \begin{Bmatrix} 0.519 \\ 0 \\ -0.519 \\ 24.89 \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{F}_3 \\ \dot{F}_4 \\ \dot{F}_5 \\ \dot{F}_6 \end{Bmatrix} = \begin{bmatrix} 393 & 18,880 & - & - \\ 18,880 & 1,208,333 & - & - \\ -393 & -18,880 & - & - \\ 18,880 & 604,167 & - & - \end{bmatrix} \begin{Bmatrix} -9.416*10^{-4} \\ -5.884*10^{-6} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -0.48 \\ -24.89 \\ 0.48 \\ -21.33 \end{Bmatrix}$$

New load factors

$$\lambda_2 = \frac{5,652 - 3766}{24.89} = 75.7$$

$$\lambda_3 = \frac{5,652 - 2830}{21.33} = 132.3$$

The lowest factor at node 2 = 75.7

Step 7 Internal force

$$\begin{Bmatrix} \dot{V}_3 \\ \dot{V}_4 \end{Bmatrix} = -\begin{Bmatrix} 9.416*10^{-4} \\ 5.885*10^{-6} \end{Bmatrix}(75.7) = -\begin{Bmatrix} 0.0712 \\ 0.000445 \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{F}_1 \\ \dot{F}_2 \\ \dot{F}_3 \\ \dot{F}_4 \end{Bmatrix} = (75.7)\begin{Bmatrix} 0.519 \\ 0 \\ -0.519 \\ 24.89 \end{Bmatrix} = \begin{Bmatrix} 39 \\ 0 \\ -39 \\ 1882 \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{F}_3 \\ \dot{F}_4 \\ \dot{F}_5 \\ \dot{F}_6 \end{Bmatrix} = (75.7)\begin{Bmatrix} -0.48 \\ -24.89 \\ 0.48 \\ -21.33 \end{Bmatrix} = \begin{Bmatrix} -36 \\ -1882 \\ 36 \\ -1613 \end{Bmatrix}$$

Step 8 Cumulative displacement and internal forces

$$\begin{Bmatrix} V_3 \\ V_4 \end{Bmatrix} = \begin{Bmatrix} V_3 \\ V_4 \end{Bmatrix}_{elastic} + \begin{Bmatrix} \dot{V}_3 \\ \dot{V}_4 \end{Bmatrix}_{increment} = \begin{Bmatrix} -0.0998 \\ -0.00156 \end{Bmatrix} + \begin{Bmatrix} -0.0712 \\ -0.000445 \end{Bmatrix} = \begin{Bmatrix} -0.1710 \\ -0.0020 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}_{at\_first\_PH} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}_{elastic} + \begin{Bmatrix} \dot{F}_1 \\ \dot{F}_2 \\ \dot{F}_3 \\ \dot{F}_4 \end{Bmatrix}_{increment} = \begin{Bmatrix} 196 \\ 5652 \\ -196 \\ 3766 \end{Bmatrix} + \begin{Bmatrix} 39 \\ 0 \\ -39 \\ 1882 \end{Bmatrix} = \begin{Bmatrix} 235 \\ 5652 \\ -235 \\ 5648 \end{Bmatrix}$$

$$\begin{Bmatrix} F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix}_{at\_first\_PH} = \begin{Bmatrix} F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix}_{elastic} + \begin{Bmatrix} \dot{F}_3 \\ \dot{F}_4 \\ \dot{F}_5 \\ \dot{F}_6 \end{Bmatrix}_{increment} = \begin{Bmatrix} -96 \\ -3766 \\ 69 \\ -2830 \end{Bmatrix} + \begin{Bmatrix} -36 \\ -1882 \\ 36 \\ -1613 \end{Bmatrix} = \begin{Bmatrix} -105 \\ -5648 \\ +105 \\ -4443 \end{Bmatrix}$$

Step 8 Cumulative displacement and internal forces

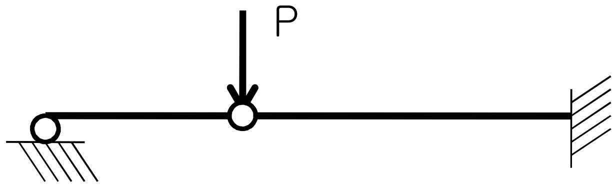
$$\begin{Bmatrix} V_3 \\ V_4 \end{Bmatrix} = \begin{Bmatrix} V_3 \\ V_4 \end{Bmatrix}_{elastic} + \begin{Bmatrix} \dot{V}_3 \\ \dot{V}_4 \end{Bmatrix}_{increment} = \begin{Bmatrix} -0.0998 \\ -0.00156 \end{Bmatrix} + \begin{Bmatrix} -0.0712 \\ -0.000445 \end{Bmatrix} = \begin{Bmatrix} -0.1710 \\ -0.0020 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}_{at\_first\_PH} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}_{elastic} + \begin{Bmatrix} \dot{F}_1 \\ \dot{F}_2 \\ \dot{F}_3 \\ \dot{F}_4 \end{Bmatrix}_{increment} = \begin{Bmatrix} 196 \\ 5652 \\ -196 \\ 3766 \end{Bmatrix} + \begin{Bmatrix} 39 \\ 0 \\ -39 \\ 1882 \end{Bmatrix} = \begin{Bmatrix} 235 \\ 5652 \\ -235 \\ 5648 \end{Bmatrix}$$

$$\begin{Bmatrix} F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix}_{at\_first\_PH} = \begin{Bmatrix} F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix}_{elastic} + \begin{Bmatrix} \dot{F}_3 \\ \dot{F}_4 \\ \dot{F}_5 \\ \dot{F}_6 \end{Bmatrix}_{increment} = \begin{Bmatrix} -96 \\ -3766 \\ 69 \\ -2830 \end{Bmatrix} + \begin{Bmatrix} -36 \\ -1882 \\ 36 \\ -1613 \end{Bmatrix} = \begin{Bmatrix} -105 \\ -5648 \\ +105 \\ -4443 \end{Bmatrix}$$

$$\begin{Bmatrix} P_3 \\ P_4 \end{Bmatrix} = \begin{Bmatrix} 265 \\ 0 \end{Bmatrix} + 75.7 \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 341 \\ 0 \end{Bmatrix}$$

Step 9 The second plastic hinge



Step 10 Stiffness matrix of the modified structure

$$k_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ sym & & & 0 \end{bmatrix} \qquad k_2 = \begin{bmatrix} 98 & 0 & -98 & 9440 \\ & 0 & 0 & 0 \\ & & 98 & -9440 \\ sym & & & 906,250 \end{bmatrix}$$

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0+98 & 0+0 & 0 & -98 \\ Sym & & & 0+0 & 0 & 0 \\ 0 & 0 & & & 98 & -9,440 \\ 0 & 0 & & & & 906,250 \end{bmatrix}$$

$$\dot{V} = -0.01017$$

Incremental forces

$$\begin{Bmatrix} \dot{F}_3 \\ \dot{F}_4 \\ \dot{F}_5 \\ \dot{F}_6 \end{Bmatrix} = \begin{Bmatrix} -1.0 \\ 0 \\ 1.0 \\ -96 \end{Bmatrix}$$

$$\lambda_3 = \frac{5,652-4443}{96} = 12.59 \qquad \dot{V} = (12.59)(-0.01017) = -0.1280$$

$$\begin{Bmatrix} \dot{F}_3 \\ \dot{F}_4 \\ \dot{F}_5 \\ \dot{F}_6 \end{Bmatrix} = (12.59)\begin{Bmatrix} -1.0 \\ 0 \\ 1.0 \\ -96 \end{Bmatrix} = \begin{Bmatrix} -12.6 \\ 0 \\ 12.6 \\ -1209 \end{Bmatrix}$$

$$V_3 = 0.1710 - 0.1280 = -0.2990$$

$$\begin{Bmatrix} F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix}_{at\_collapse} = \begin{Bmatrix} F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix}_{elastic} + \begin{Bmatrix} \dot{F}_3 \\ \dot{F}_4 \\ \dot{F}_5 \\ \dot{F}_6 \end{Bmatrix}_{increment\_1st} + \begin{Bmatrix} \dot{F}_3 \\ \dot{F}_4 \\ \dot{F}_5 \\ \dot{F}_6 \end{Bmatrix}_{increment\_2nd} = \begin{Bmatrix} -96 \\ -3766 \\ 69 \\ -2830 \end{Bmatrix} + \begin{Bmatrix} -36 \\ -1882 \\ 36 \\ -1613 \end{Bmatrix} + \begin{Bmatrix} -12.6 \\ 0 \\ 12.6 \\ -1209 \end{Bmatrix} = \begin{Bmatrix} -118 \\ -5648 \\ +118 \\ -5652 \end{Bmatrix}$$

$$\begin{Bmatrix} P_3 \\ P_4 \end{Bmatrix}_{ultimate} = \begin{Bmatrix} 265 \\ 0 \end{Bmatrix} + 75.7\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} + 12.59\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 353 \\ 0 \end{Bmatrix}$$