Mechanics and Design

Chapter 7. FEM: Plane Stress and Strain

Byeng D. Youn

System Health & Risk Management Laboratory
Department of Mechanical & Aerospace Engineering
Seoul National University

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Plane Stress and Plane Strain

- Finite element in 2-D: Thin plate element required 2 coordinates
- Plane stress and plane strain problems
- Constant-strain triangular element
- Equilibrium equation in 2-D

Plane stress: The stress state when normal stress, which is perpendicular to the plane x-y, and shear stress are both zero.
Plane strain: The strain state when normal strain ε_z, which is perpendicular to the plane x-y, and shear strain γ_{xz}, γ_{yz} are both zero.

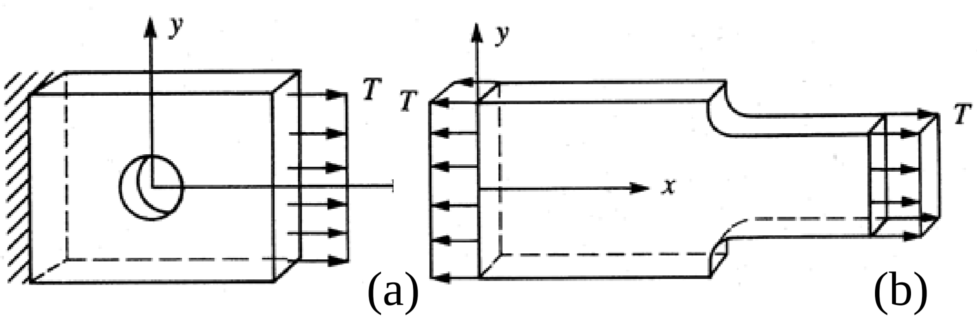


Fig. 7.1 Plane stress: (a), (b)

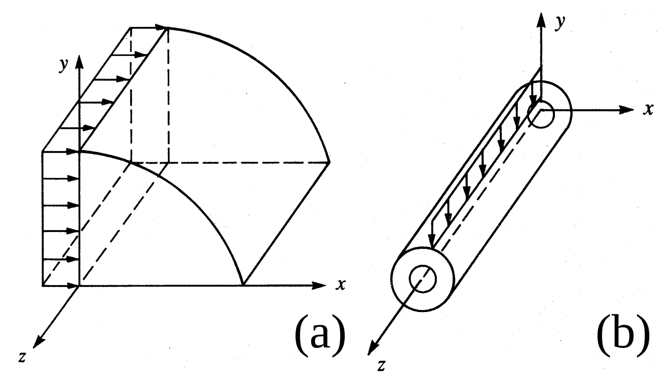
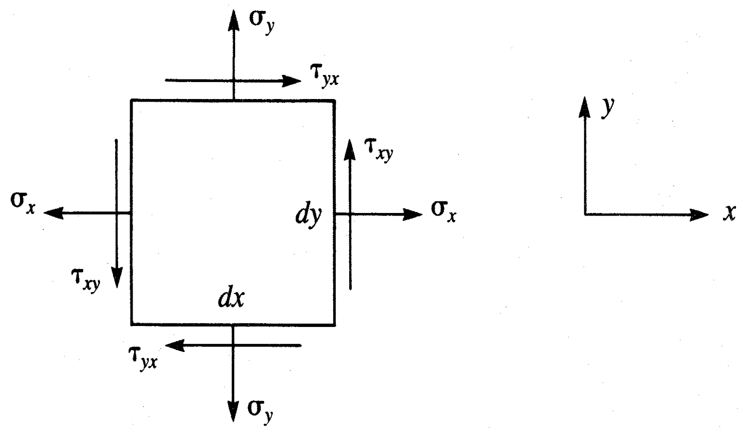


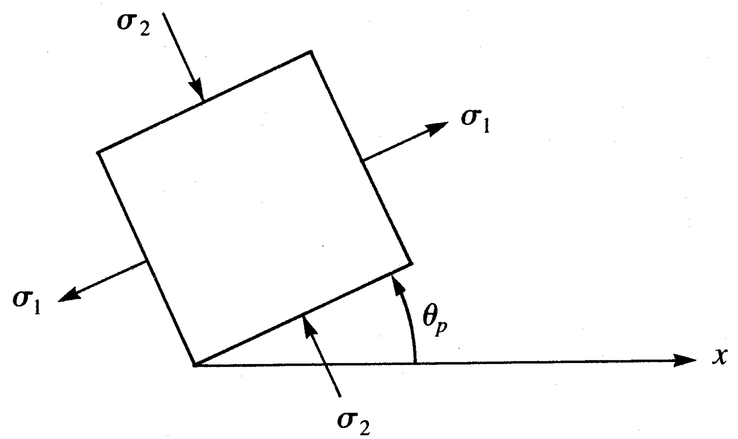
Fig. 7.2 Plane strain: (a), (b)

Plane Stress and Plane Strain

Stress and strain in 2-D



Stresses in 2-D



Principal stress and its direction

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

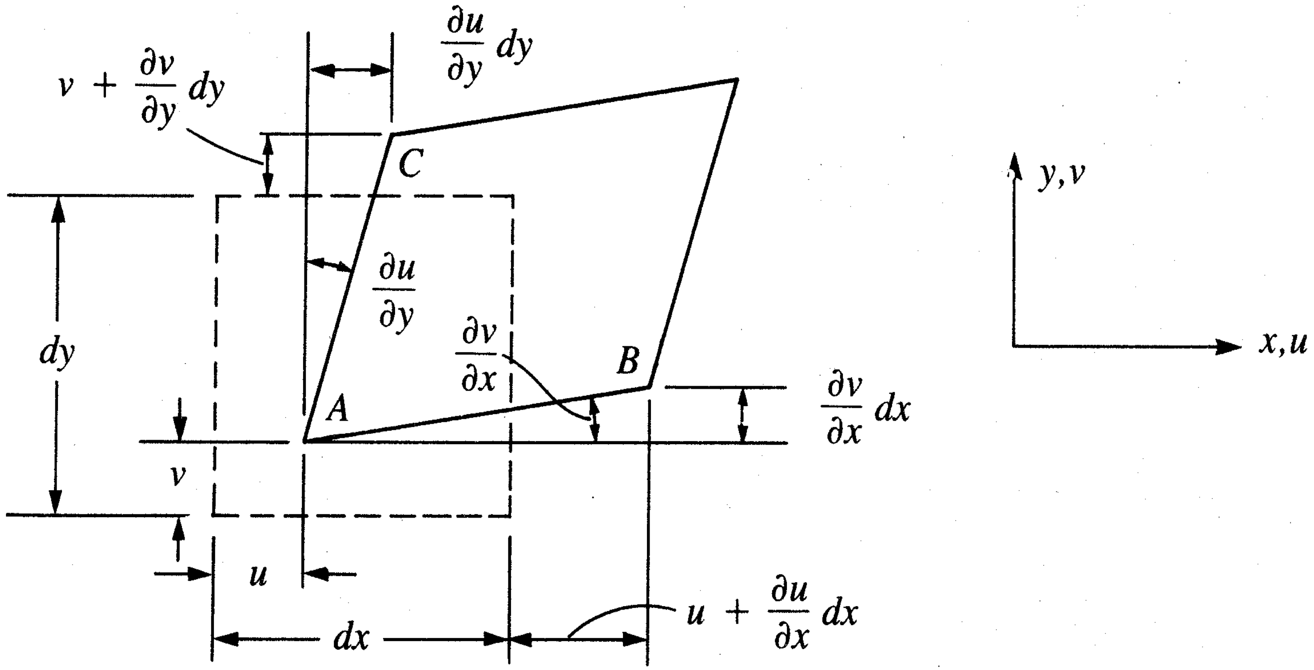
$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_{max}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_{min}$$

Plane Stress and Plane Strain

Stress and strain in 2-D



Displacement and rotation of plane element $x-y$

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Plane Stress and Plane Strain

Stress and strain in 2-D

$$\{\sigma\} = [D]\{\varepsilon\}$$

Stress-strain matrix(or material composed matrix) of isotropic material for plane stress ($\sigma_z = \tau_{xz} = \tau_{yz} = 0$)

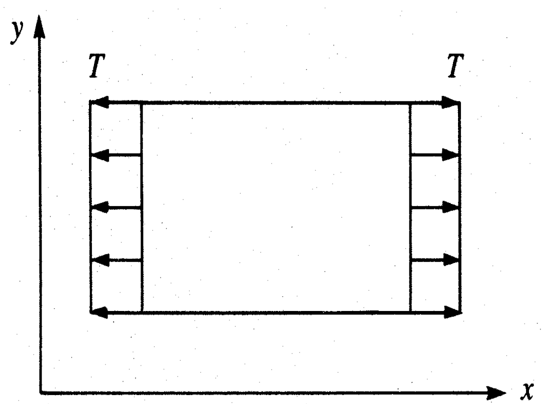
$$[D] = \frac{E}{1-\ \nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

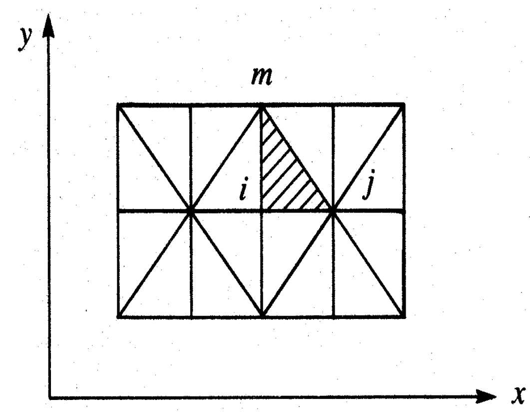
Stress-strain matrix(or material composed matrix) of isotropic material for plane deformation ($\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$)

$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

General Steps of Formulation Process for Plane Triangular Element

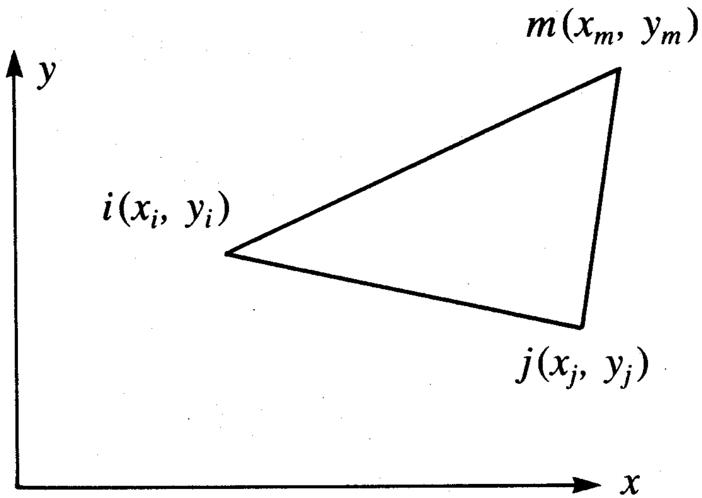
- Step 1: Determination of element type
- Step 2: Determination of displacement function
- Step 3: Relation of deformation rate – strain and stress-strain
- Step 4: Derivation of element stiffness and equation
- Step 5: Construction of global system equations and application of boundary conditions
- Step 6: Calculation of nodal displacement
- Step 7: Calculation of force (stress) in an element





General Steps of Formulation Process for Plane Triangular Element

Step1: Determination of element type



$$\{d\} = \begin{Bmatrix} \underline{d}_i \\ \underline{d}_j \\ \underline{d}_m \end{Bmatrix} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

Considering a triangular element, the nodes i, j, m are notated in the anti-clockwise direction.

The way to name the nodal members in an entire structure must be devised to avoid negative element area.

General Steps of Formulation Process for Plane Triangular Element

Step2: Determination of displacement function

$$u(x,y) = a_1 + a_2 x + a_3 y$$
$$v(x,y) = a_4 + a_5 x + a_6 y$$

Linear function gives a guarantee to satisfy the compatibility.
A general displacement function $\{\psi\}$ containing function u and v can be expressed as below.

$$\{\psi\} = \begin{Bmatrix} a_1 + a_2 x + a_3 y \\ a_4 + a_5 x + a_6 y \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

Substitute nodal coordinates to the equation for obtaining the values of a.

General Steps of Formulation Process for Plane Triangular Element

Step2: Determination of displacement function (Continued)

Calculation of a_1, a_2, a_3:

$$\begin{aligned} u_i &= a_1 + a_2 x_i + a_3 y_i \\ u_j &= a_1 + a_2 x_j + a_3 y_j \\ u_m &= a_1 + a_2 x_m + a_3 y_m \end{aligned} \quad \text{or} \quad \begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Solving a, $\{a\} = [x]^{-1}\{u\}$

Obtaining the inverse matrix of $[x]$, $\quad [x]^{-1} = \dfrac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}$

where, $2A = x_i(y_j - y_m) + x_j(y_m - y_i) + x_m(y_i - y_j)$: 2 times of triangle area.

$$\begin{aligned} \alpha_i &= x_j y_m - y_j x_m \\ \beta_i &= y_j - y_m \\ \gamma_i &= x_m - x_j \end{aligned} \qquad \begin{aligned} \alpha_j &= y_i x_m - x_i y_m \\ \beta_j &= y_m - y_i \\ \gamma_j &= x_i - x_m \end{aligned} \qquad \begin{aligned} \alpha_m &= x_i y_j - y_i x_j \\ \beta_m &= y_i - y_j \\ \gamma_m &= x_j - x_i \end{aligned}$$

General Steps of Formulation Process for Plane Triangular Element

Step2: Determination of displacement function (Continued)

$$\{a\} = [x]^{-1}\{u\}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

Similarly,

$$\begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} v_i \\ v_j \\ v_m \end{Bmatrix}$$

Derivation of displacement function $u(x, y)$ (v can also be derived similarly)

$$\{u\} = \begin{bmatrix} 1 & x & y \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} 1 & x & y \end{bmatrix}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

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General Steps of Formulation Process for Plane Triangular Element

Step2: Determination of displacement function (Continued)

Arranging by deployment:

$$u(x,y) = \frac{1}{2A}\{(\alpha_i + \beta_i x + \gamma_i y)u_i + (\alpha_j + \beta_j x + \gamma_j y)u_j + (\alpha_m + \beta_m x + \gamma_m y)u_m\}$$

As the same way,

$$v(x,y) = \frac{1}{2A}\{(\alpha_i + \beta_i x + \gamma_i y)v_i + (\alpha_j + \beta_j x + \gamma_j y)v_j + (\alpha_m + \beta_m x + \gamma_m y)v_m\}$$

Simple expression of u and v:

$$u(x,y) = N_i u_i + N_j u_j + N_m u_m$$
$$v(x,y) = N_i v_i + N_j v_j + N_m v_m$$

where

$$N_i = \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y)$$
$$N_j = \frac{1}{2A}(\alpha_j + \beta_j x + \gamma_j y)$$
$$N_m = \frac{1}{2A}(\alpha_m + \beta_m x + \gamma_m y)$$

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General Steps of Formulation Process for Plane Triangular Element

Step2: Determination of displacement function (Continued)

Arranging by deployment:

$$\{\psi\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{Bmatrix} N_i u_i + N_j u_j + N_m u_m \\ N_i v_i + N_j v_j + N_m v_m \end{Bmatrix} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & x & N_m \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

Making the equation be simple in a form of matrix, $\{\psi\} = [N]\{d\}$

where $$[N] = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & x & N_m \end{bmatrix}$$

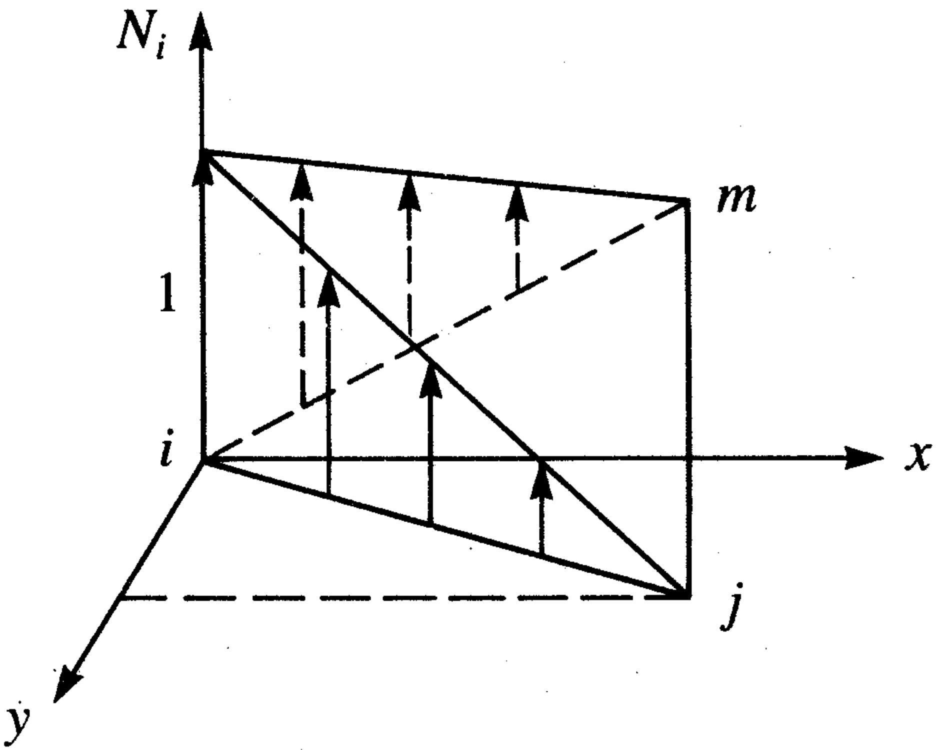
The displacement function $\{\psi\}$ is represented with shape functions N_i, N_j, N_m and nodal displacement $\{d\}$.

General Steps of Formulation Process for Plane Triangular Element

Step2: Determination of displacement function (Continued)

Review of characteristics of shape function:

N_i =1, N_j =0, N_m =0 at nodes (x_i, y_i)



A change of N_i of general elements across the surface x - y

General Steps of Formulation Process for Plane Triangular Element

Step3: Relation of deformation rate – strain and stress-strain

Deformation rate:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

Calculation of partial differential terms

$$\frac{\partial u}{\partial x} = u_{,x} = \frac{\partial}{\partial x}(N_i u_i + N_j u_j + N_m u_m) = N_{i,x} u_i + N_{j,x} u_j + N_{m,x} u_m$$

$$N_{i,x} = \frac{1}{2A}\frac{\partial}{\partial x}(\alpha_i + \beta_i x + \gamma_i y) = \frac{\beta_i}{2A}, \qquad N_{j,x} = \frac{\beta_j}{2A}, \qquad N_{m,x} = \frac{\beta_m}{2A}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{2A}(\beta_i u_i + \beta_j u_j + \beta_m u_m)$$

General Steps of Formulation Process for Plane Triangular Element

Step3: Relation of deformation rate – strain and stress-strain

$$\frac{\partial v}{\partial y} = \frac{1}{2A}(\gamma_i v_i + \gamma_j v_j + \gamma_m v_m)$$

Likewise,

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{2A}(\gamma_i u_i + \beta_i v_i + \gamma_j u_j + \beta_j v_j + \gamma_m u_m + \beta_m v_m$$

Summarizing the deformation rate equation,

$$\{\varepsilon\} = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix} = [B]\{d\} = [\underline{B}_i \quad B_j \quad \underline{B}_m] \begin{Bmatrix} \underline{d}_i \\ \underline{d}_j \\ \underline{d}_m \end{Bmatrix}$$

where,

$$[B_i] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \gamma_i & \beta_i \end{bmatrix} \qquad [B_j] = \frac{1}{2A}\begin{bmatrix} \beta_j & 0 \\ 0 & \gamma_j \\ \gamma_j & \beta_j \end{bmatrix} \qquad [B_m] = \frac{1}{2A}\begin{bmatrix} \beta_m & 0 \\ 0 & \gamma_m \\ \gamma_m & \beta_m \end{bmatrix}$$

General Steps of Formulation Process for Plane Triangular Element

Step3: Relation of deformation rate – strain and stress-strain (Continued)

Strain is constant in an element, for matrix $\underline{B}$ **regardless of x and y coordinates, and is influenced by only nodal coordinates in an element.**

→ CST: Constant – Strain Triangle

Relation of stress - strain

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad \rightarrow \quad \{\sigma\} = [D][B]\{d\}$$

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General Steps of Formulation Process for Plane Triangular Element

Step4: Derivation of element stiffness matrix and equation

Using minimum potential energy principle.

Total potential energy $\pi_p = \pi_p(u_i, v_i, u_j, \dots, v_m) = U + \Omega_b + \Omega_p + \Omega_s$

Strain energy $U = \dfrac{1}{2}\iiint\limits_V \{\varepsilon\}^T\{\sigma\}dV = \iiint\limits_V \{\varepsilon\}^T[D]\{\varepsilon\}dV$

Potential energy due to body force $\Omega_b = -\iiint\limits_V \{\psi\}^T\{X\}dV$

Potential energy due to concentrated load $\Omega_p = -\{d\}^T\{P\}$

Potential energy due to distributed load (or surface force) $\Omega_s = -\iint\limits_S \{\psi\}^T\{T\}dS$

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General Steps of Formulation Process for Plane Triangular Element

Step4: Derivation of element stiffness matrix and equation (Continued)

$\therefore \pi_p$

$$= \frac{1}{2}\iiint_V \{d\}^T[B]^T[D][B]\{d\}dV - \iiint_V \{d\}^T[N]^T\{X\}dV - \{d\}^T\{P\} - \iint_S \{d\}^T[N]^T\{T\}dS$$

$$= \frac{1}{2}\{d\}^T \iiint_V [B]^T[D][B]dV\{d\} - \{d\}^T \iiint_V [N]^T\{X\}dV - \{d\}^T\{P\} - \{d\}^T \iint_S [N]^T\{T\}dS$$

$$= \frac{1}{2}\{d\}^T \iiint_V [B]^T[D][B]dV\{d\} - -\{d\}^T\{f\}$$

where $\{f\} = \iiint_V [N]^T\{X\}dV + \{P\} + \iint_S [N]^T\{T\}dS$

Condition having the minimum is $\dfrac{\partial \pi_p}{\partial\{d\}} = \left[\iiint_V [B]^T[D][B]dV\right]\{d\} - \{f\} = 0$

General Steps of Formulation Process for Plane Triangular Element

Step4: Derivation of element stiffness matrix and equation (Continued)

Condition having the minimum is

$$\frac{\partial \pi_p}{\partial \{d\}} = \left[\iiint_V [B]^T [D][B] dV \right] \{d\} - \{f\} = 0 \quad \rightarrow \quad \iiint_V [B]^T [D][B] dV \{d\} = \{f\}$$

So, the element stiffness matrix is (Case of an element having constant thickness t)

$$[k] = \iiint_V [B]^T [D][B] dV \quad \left(= t \iint_A [B]^T [D][B] dx dy = tA\, [B]^T [D][B] \right)$$

Matrix [k] is a 6x6 matrix, and the element equation is as below

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{16} \\ k_{21} & k_{22} & & k_{26} \\ \vdots & & \ddots & \vdots \\ k_{61} & k_{62} & \cdots & k_{66} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

General Steps of Formulation Process for Plane Triangular Element

Step5: Introduction a combination of element equation and boundary conditions for obtaining a global coordinate system of equation

$$[K]=\sum_{e=1}^{N}[k^{(e)}] \qquad \text{and} \qquad \{F\}=\sum_{e=1}^{N}\{f^{(e)}\}$$

$$\{F\}=[K]\{d\}$$

Step6: Calculation of nodal displacement

Step7: Calculation of force(stress) in an element

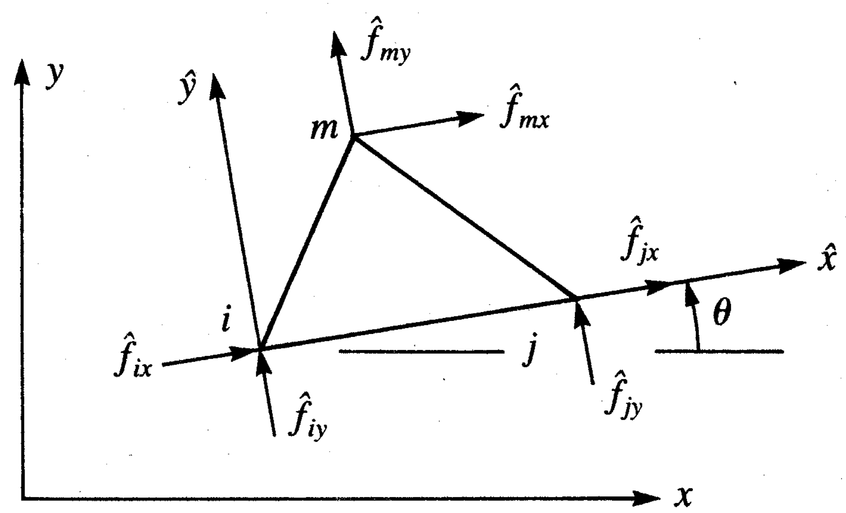
Transformation from the global coordinate system to the local coordinate system: (See Ch. 3)

$$\underline{\hat{d}}=\underline{T}\,\underline{d} \qquad \underline{\hat{f}}=\underline{T}\,\underline{f} \qquad \underline{k}=\underline{T}^{T}\,\underline{\hat{k}}\,\underline{T}$$

Constant-strain triangle(CST) has 6 degrees of freedom.

General Steps of Formulation Process for Plane Triangular Element

$$\underline{T} = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & C & S & 0 & 0 \\ 0 & 0 & -S & C & 0 & 0 \\ 0 & 0 & 0 & 0 & C & S \\ 0 & 0 & 0 & 0 & -S & C \end{bmatrix} \quad \text{where} \quad \begin{aligned} C &= \cos\theta \\ S &= \sin\theta \end{aligned}$$

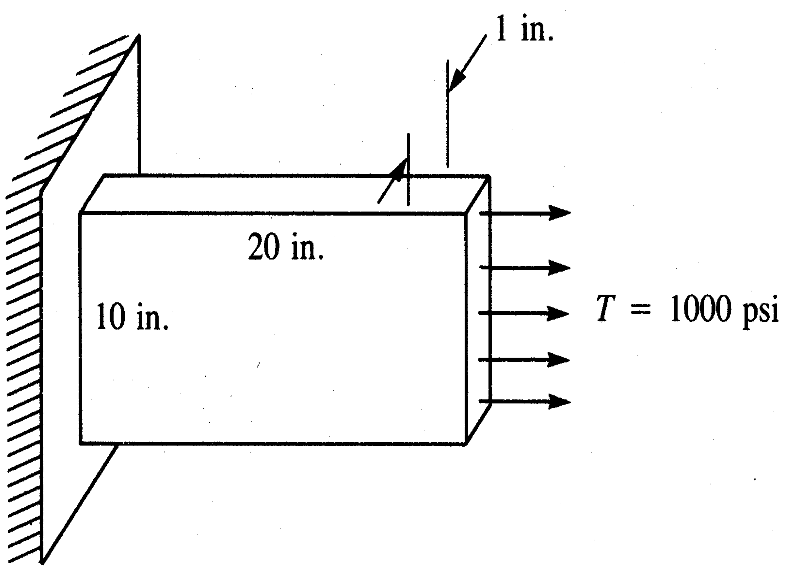


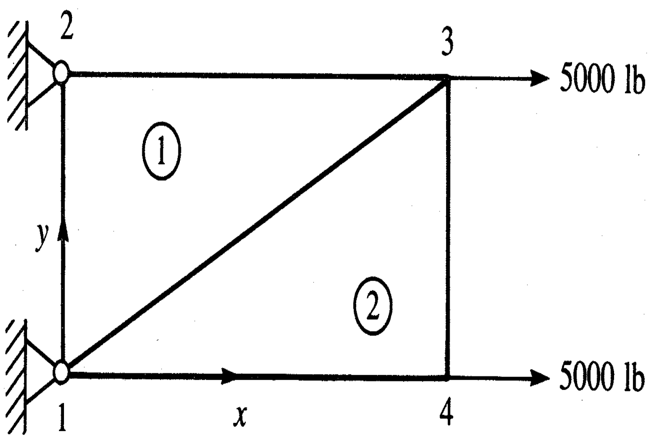
A triangular element with local coordinates system not along to the global coordinate system.

Finite Element Method in a Plane Stress Problem

Find nodal displacements and element stresses in the case of the thin plate(see below figure) under surface force.

thickness t = 1 in, E = 30 x 106 psi, ν =0.30





Finite Element Method in a Plane Stress Problem

(1) Discretization: Surface tension force is replaced by the following nodal loads.

$$F = \frac{1}{2}TA$$

$$F = \frac{1}{2}(1000\,psi)(1\,in. \times 10\,in.)$$

$$F = 5000\,lb$$

The global system of the governing equation is

$$\{F\} = [K]\{d\} \qquad \text{or} \qquad \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{Bmatrix} = \begin{Bmatrix} R_{1x} \\ R_{1y} \\ R_{2x} \\ R_{2y} \\ 5000 \\ 0 \\ 5000 \\ 0 \end{Bmatrix} = [K] \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix} = [K] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix}$$

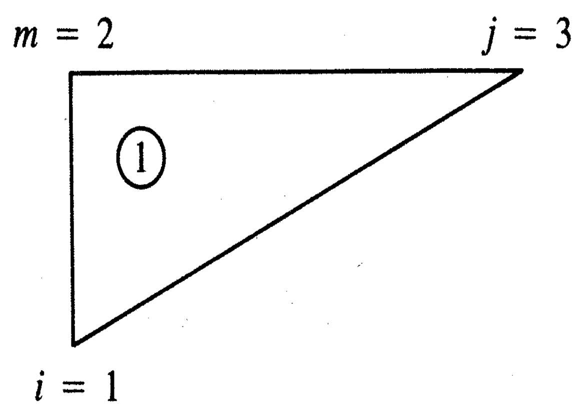
where $[K]$ is a 5x5 matrix.

Finite Element Method in a Plane Stress Problem

(2) A combination of stiffness matrix: $[k] = tA[B]^T[D][B]$

- **Element 1**

• Calculation of matrix [B]



$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}$$

where
$$\beta_i = y_j - y_m = 10 - 10 = 0$$
$$\beta_j = y_m - y_i = 10 - 0 = 10$$
$$\beta_m = y_i - y_j = 0 - 10 = -10$$
$$\gamma_i = x_m - x_j = 0 - 20 = -20$$
$$\gamma_j = x_i - x_m = 0 - 0 = 0$$
$$\gamma_m = x_j - x_i = 20 - 0 = 20$$

and

$$A = \frac{1}{2}bh$$
$$= \left(\frac{1}{2}\right)(20)(10) = 100\ in.^2$$

Finite Element Method in a Plane Stress Problem

Then [B] is

$$[B] = \frac{1}{200}\begin{bmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & -20 & 0 & 0 & 0 & 20 \\ -20 & 0 & 0 & 10 & 20 & -10 \end{bmatrix}$$

- Matrix [D] (Plane stress)

$$[D] = \frac{E}{1-\ \nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} = \frac{30(10^6)}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

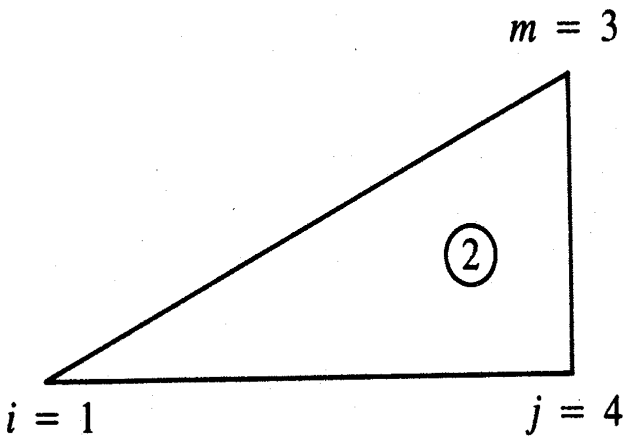
- Calculation of stiffness matrix

i = 1 j = 3 m = 2

$$[k] = tA[B]^T[D\}[B] = \frac{75{,}000}{0.91}\begin{bmatrix} 140 & 0 & 0 & -70 & -140 & 70 \\ 0 & 400 & -60 & 0 & 60 & -400 \\ 0 & -60 & 100 & 0 & -100 & 60 \\ -70 & 0 & 0 & 35 & 70 & -35 \\ -140 & 60 & -100 & 70 & 240 & -130 \\ 70 & -400 & 60 & -35 & -130 & 435 \end{bmatrix}$$

Finite Element Method in a Plane Stress Problem

- **Element 2**
- Calculation of matrix [B]



$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}$$

where

$$\beta_i = y_j - y_m = 0 - 10 = -10$$
$$\beta_j = y_m - y_i = 10 - 0 = 10$$
$$\beta_m = y_i - y_j = 0 - 0 = 0$$
$$\gamma_i = x_m - x_j = 20 - 20 = 0$$
$$\gamma_j = x_i - x_m = 0 - 20 = -20$$
$$\gamma_m = x_j - x_i = 20 - 0 = 20$$

and

$$A = \frac{1}{2}bh$$
$$= \left(\frac{1}{2}\right)(20)(10) = 100\ in.^2$$

Finite Element Method in a Plane Stress Problem

Then [B] is

$$[B] = \frac{1}{200}\begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -10 & -20 & 10 & 20 & 0 \end{bmatrix}$$

- Matrix [D] (Plane stress)

$$[D] = \frac{30(10^6)}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

- Calculation of stiffness matrix

i = 1 j = 4 m = 3

$$[k] = \frac{75{,}000}{0.91}\begin{bmatrix} 100 & 0 & -100 & 60 & 0 & -60 \\ 0 & 35 & 70 & -35 & -70 & 0 \\ -100 & 70 & 240 & -130 & -140 & 60 \\ 60 & -35 & -130 & 435 & 70 & -400 \\ 0 & -70 & -140 & 70 & 140 & 0 \\ -60 & 0 & 60 & -400 & 0 & 400 \end{bmatrix}$$

Finite Element Method in a Plane Stress Problem

Element 1: $$[k] = \frac{375{,}000}{0.91} \begin{array}{c} \begin{array}{cccc} \quad 1 \quad\quad & \quad\quad 2 \quad\quad & \quad\quad 3 \quad\quad & \quad\quad 4 \end{array} \\ \begin{bmatrix} 28 & 0 & -28 & 14 & 0 & -14 & 0 & 0 \\ 0 & 80 & 12 & -80 & -12 & 0 & 0 & 0 \\ -28 & 12 & 48 & -26 & -20 & 14 & 0 & 0 \\ 14 & -80 & -26 & 87 & 12 & -7 & 0 & 0 \\ 0 & -12 & -20 & 12 & 20 & 0 & 0 & 0 \\ -14 & 0 & 14 & -7 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Element 2: $$[k] = \frac{375{,}000}{0.91} \begin{array}{c} \begin{array}{cccc} \quad 1 \quad\quad & \quad\quad 2 \quad\quad & \quad\quad 3 \quad\quad & \quad\quad 4 \end{array} \\ \begin{bmatrix} 20 & 0 & 0 & 0 & 0 & -12 & -20 & 12 \\ 0 & 7 & 0 & 0 & -14 & 0 & 14 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -14 & 0 & 0 & 28 & 0 & -28 & 14 \\ -12 & 0 & 0 & 0 & 0 & 80 & 12 & -80 \\ -20 & 14 & 0 & 0 & -28 & 12 & 48 & -26 \\ 12 & -7 & 0 & 0 & 14 & -80 & -26 & 87 \end{bmatrix} \end{array}$$

Finite Element Method in a Plane Stress Problem

(3) Calculation of displacement: Superpositioning element stiffness matrix, global system of stiffness matrix is obtained as below.

$$
[K] = \frac{375{,}000}{0.91}
\begin{array}{c}
\begin{array}{cccccccc}
\multicolumn{2}{c}{1} & \multicolumn{2}{c}{2} & \multicolumn{2}{c}{3} & \multicolumn{2}{c}{4}
\end{array} \\
\begin{bmatrix}
48 & 0 & -28 & 14 & 0 & -26 & -20 & 12 \\
0 & 87 & 12 & -80 & -26 & 0 & 14 & -7 \\
-28 & 12 & 48 & -26 & -20 & 14 & 0 & 0 \\
14 & -80 & -26 & 87 & 12 & -7 & 0 & 0 \\
0 & -26 & -20 & 12 & 48 & 0 & -28 & 14 \\
-26 & 0 & 14 & -7 & 0 & 87 & 12 & -80 \\
-20 & 14 & 0 & 0 & -28 & 12 & 48 & -26 \\
12 & -7 & 0 & 0 & 14 & -80 & -26 & 87
\end{bmatrix}
\end{array}
$$

Finite Element Method in a Plane Stress Problem

Substituting [K] to $\{F\} = [K]\{d\}$

$$\begin{Bmatrix} R_{1x} \\ R_{1y} \\ R_{2x} \\ R_{2y} \\ 5000 \\ 0 \\ 5000 \\ 0 \end{Bmatrix} = \frac{375{,}000}{0.91}\begin{bmatrix} 48 & 0 & -28 & 14 & 0 & -26 & -20 & 12 \\ 0 & 87 & 12 & -80 & -26 & 0 & 14 & -7 \\ -28 & 12 & 48 & -26 & -20 & 14 & 0 & 0 \\ 14 & -80 & -26 & 87 & 12 & -7 & 0 & 0 \\ 0 & -26 & -20 & 12 & 48 & 0 & -28 & 14 \\ -26 & 0 & 14 & -7 & 0 & 87 & 12 & -80 \\ -20 & 14 & 0 & 0 & -28 & 12 & 48 & -26 \\ 12 & -7 & 0 & 0 & 14 & -80 & -26 & 87 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix}$$

Applying given boundary conditions with elimination of columns and rows.

$$\begin{Bmatrix} 5000 \\ 0 \\ 5000 \\ 0 \end{Bmatrix} = \frac{375{,}000}{0.91}\begin{bmatrix} 48 & 0 & -28 & 14 \\ 0 & 87 & 12 & -80 \\ -28 & 12 & 48 & -26 \\ 14 & -80 & -26 & 87 \end{bmatrix}\begin{Bmatrix} d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix}$$

Finite Element Method in a Plane Stress Problem

Transposing the displacement matrix to the left side

$$\begin{Bmatrix} d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix} = \frac{0.91}{375{,}000} \begin{bmatrix} 48 & 0 & -28 & 14 \\ 0 & 87 & 12 & -80 \\ -28 & 12 & 48 & -26 \\ 14 & -80 & -26 & 87 \end{bmatrix}^{-1} \begin{Bmatrix} 5000 \\ 0 \\ 5000 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 609.6 \\ 4.2 \\ 663.7 \\ 104.1 \end{Bmatrix} \times 10^{-6}\, in.$$

The solution of 1-D beam under tension force is

$$\delta = \frac{PL}{AE} = \frac{(10{,}000)20}{10(30 \times 10^6)} = 670 \times 10^{-6}\, in.$$

Therefore, x-component of the displacement at nodes in the equation $\iiint_V [B]^T[D\}[B]dV\{d\} = \{f\}$ of 2-D plane is quite accurate when considering the coarse grids.

Finite Element Method in a Plane Stress Problem

(4) Stresses at each node: $\{\sigma\} = [D][B]\{d\}$

Element 1

$$\{\sigma\} = \frac{E}{1-\ \nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \times \left(\frac{1}{2A}\right)\begin{bmatrix} \beta_1 & 0 & \beta_4 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_4 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_4 & \beta_4 & \gamma_3 & \beta_3 \end{bmatrix}\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{4x} \\ d_{4y} \\ d_{3x} \\ d_{3y} \end{Bmatrix}$$

Calculating,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 1005 \\ 301 \\ 2.4 \end{Bmatrix} psi$$

THANK YOU
FOR LISTENING