Let (x, y) be a pair of feasible solutions of (3.7) and (3.8). From feasibility and nonnegativity of y, in turn, we get $c^T x = y^T Ax \geq y^T b$.

Theorem 3.2

Weak Duality *(약쌍대정리) Every primal-dual pair of feasible solutions, (x, y), satisfies $c^T x \geq y^T b$.*

Weak duality tells us some useful facts.

Corrolary 3.3

- *If the objective of one problem can be improved arbitrarily (minimized or maximized without a bound), the other problem is infeasible.*
- *If the objective values from a pair of feasible (x, y) are equal, x and y are optimal solutions.*

We actually have a stronger relation between two problems.

Theorem 3.4

Strong Duality *(강쌍대정리) If one of (3.7) and (3.7) has an optimal solution, so does the other and their objective values are equal.*

Proof: We assume (3.7) has an optimal solution with objective value δ and show (3.8) also has an optimal solution with objective value δ. It is, by weak duality, equivalent to the feasibility of the system

$$\begin{aligned} A^T y &\geq c \\ -A^T y &\geq -c \\ I y &\geq 0 \\ b^T y &\geq \delta. \end{aligned} \tag{3.9}$$

Assume on the contrary (3.9) is infeasible. Then, by Farkas Lemma (Theorem 2.1), there are u, v, w, and z satisfying the followings.

Proof(*cont'd*):

$$u, v, w, z \geq 0$$
$$u^T A^T - v^T A^T + w^T + z b^T = 0$$
$$u^T c - v^T c + z\delta > 0.$$

Substituting $x' = v - u$, and eliminating w we get

$$z \geq 0$$
$$Ax' \geq zb$$
$$c^T x' < z\delta.$$

If $z = 0$, then $Ax' \geq 0$, $c^T x' < 0$. For any feasible solution $\bar{x}$ of (3.7), $\bar{x} + \alpha x'$ is also feasible for all $\alpha > 0$. Its objective value can be made arbitrarily negative by taking $\alpha \to +\infty$. A contradiction to that (3.7) has an optimal solution.

If $z > 0$, then $\frac{1}{z}x'$ is feasible and its objective value less than δ. Also a contradiction. The other direction is proved similarly. □

Exercise 3.5

If one problem is infeasible, then the other is infeasible or unbounded (i.e. its objective can be improved unboundedly).

Remark 3.6

We can prove Farkas Lemma by using duality. Therefore, two are equivalent.

Exercise 3.7

(FYI 참고) Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times p}$. *Then polyhedron* $P = \{(x, y) \in \mathbb{R}^{n+p} : Ax + By \geq b,\ x \geq 0,\ y \geq 0\}$ *projected to* y*-space is given by*

$$\pi_y(P) = \{y \in \mathbb{R}^p |\ y \geq 0, u^T(b - By) \leq 0, \ \ \forall\, u \geq 0 \text{ s.t. } u^T A \leq 0\}.$$

Proposition 3.8

Suppose $Ax \geq b$ *is feasible and* $c^T x \geq d$ *is redundant to it. (I.e.* $Ax \geq b$ *implies* $c^T x \geq d$*.) Then there is* $y \geq 0$*:* $y^T A = c^T$, $y^T b \geq d$*. In other words, we can construct an inequality dominating* $c^T x \geq d$ *by a nonnegative combination of inequalities of* $Ax \geq b$*. Then, in particular,* $r\begin{pmatrix} A \\ c^T \end{pmatrix} = r(A)$*.*

증명: From the assumption, $\min\{c^T x : Ax \geq b\}$ is greater than or equal to d. By strong duality, the optimal values of two problem are the same and hence $\max\{b^T y : A^T y = c,\, y \geq 0\}$, is also greater than or equal to d. I.e. there is $y \geq 0$: $y^T A = c^T$, $y^T b \geq d$. $\square$

Exercise 3.9

Suppose $y^T c \geq 0 \;\forall\; y : y^T v_i \geq 0$ *for* $i = 1, \ldots, k$*. Then, there is* $\lambda \geq 0$*:* $c = \lambda_1 v_1 + \cdots + \lambda_k v_k$*.*

Exercise 3.10

Suppose the standard linear program has an optimal solution

$$\begin{array}{llcl} \text{min} & c^T x & & \\ \text{s.t} & Ax & = & b \\ & x & \geq & 0. \end{array}$$

Compute the range of K for which the following system is feasible.

$$\begin{array}{rcl} Ax & = & b \\ x & \geq & 0 \\ A^T y + s & = & c \\ s & \geq & 0 \\ s^T x & = & K \end{array}$$

Exercise 3.11

Write the dual of the linear program.

$$\begin{array}{ll} \min & 0^T x \\ \textit{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

Derive a necessary and sufficient condition for a standard linear system has no feasible solution.

Consider the correspondence between (3.7) and (3.8) in the variables of one problem and the constraints of the other:

$$x_j \longleftrightarrow y^T A_{.j} = c_j,$$
$$A_{i.}x \geq b_i \longleftrightarrow y_i \geq 0.$$

Definition 3.12

(Slackness of a constraint or sign restriction) Given a pair of feasible solutions $\bar{x}$ and $\bar{y}$, the *slackness of a constraint or sign restriction* is the absolute value of the difference between its two sides when $x = \bar{x}$ and $y = \bar{y}$.

For instance, the slack of the i-th constraint of (3.7) is $A_{i.}\bar{x} - b_i$, and the slack of the i-the nonnegativity restriction $y_i \geq 0$ of (3.8) is $\bar{y}_i$. An equality constraint has a slack 0.

Theorem 3.13

Complementary slackness *(상보 여유성) A pair of feasible solutions* (x^*, y^*) *from (3.7) and (3.8) is optimal if and only if the product of the slacks is zero for every pair of corresponding constraint and nonnegativity restriction.*

Proof: From weak and strong duality theorems, a feasible pair (x^*, y^*) is optimal exactly when

$$\begin{aligned} 0 &= c^T x^* - (y^*)^T b = (y^*)^T A x^* - (y^*)^T b = (y^*)^T (A x^* - b) \\ &= y_1^*(A_1.x^* - b_1) + \cdots + y_m^*(A_m.x^* - b_m). \end{aligned} \tag{3.10}$$

Since $A_i.x^* - b_i \geq 0$, $y_i \geq 0$ for every i, the condition is equivalent to $y_1^*(A_1.x^* - b_1) = 0$, …, $y_m^*(A_m.x^* - b_m) = 0$. Hence the theorem. $\square$

Exercise 3.14

Consider the following LP.

$$\begin{array}{llllll} \textit{min} & 18x_1 & +12x_2 & +2x_3 & +6x_4 & & \\ \textit{s.t} & 3x_1 & +x_2 & -2x_3 & +x_4 & = & 2 \\ & x_1 & +3x_2 & & -x_4 & = & 2 \\ & x_1 \geq 0 & x_2 \geq 0 & x_3 \geq 0 & x_4 \geq 0 & & \end{array}$$

1) *Write the dual.*
2) *Find an optimal solution of the dual from graphic method.*
3) *Obtain a primal optimum from the complementary slackness.*

Definition 4.1

A set $H \subseteq \mathbb{R}^n$ is called a *hyperplane* if $H = \{x : a^T x = b\}$ for a nonzero vector $a \in \mathbb{R}^n$ and a real number b; G a *halfspace* if $G = \{x : a^T x \geq b\}$ for a nonzero vector $a \in \mathbb{R}^n$ and a real number b.

Every hyperplane defined by $a \in \mathbb{R}^n$ is a translation of the null space of $a \in \mathbb{R}^n$. (Hyperplanes are a special class of *affine* spaces.

Definition 4.2

For a feasible solution $\bar{x}$ of an LP, y is said to be a *feasible direction*, if we can move into y for a positive distance maintaining feasibility, namely if there is $\bar{\lambda}$ such that $\bar{x} + \lambda y$ is feasible for every $0 < \lambda < \bar{\lambda}$.

For any $\bar{x} \in F = \{x : a^T x \geq b\}$, y is a feasible direction if $c^T y \geq 0$.

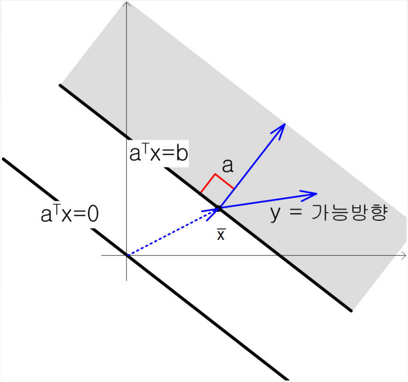

Figure: The null space, hyperplane, and half space defined by a, and feasible direction

Exercise 4.3

For any feasible solution of $Ax \geq b$, then y is a feasible direction if and only if y is a feasible solution of the 'homogeneous' linear system $Ax \geq 0$.

For a simplicity of illustration, consider a 2-dim linear program of minimizing $c^T x$ over the feasible solutions of the linear system $A_1.x \geq b_1$ and $A_2.x \geq b_2$. Let the intersection x^* of the two hyperplanes be an optimal solution. $c^T x$ increases in a direction y if $c^T y < 0$. The points of the blue region (with boundary excluded) indicate the *increasing directions*.